*Publ. Math. Debrecen*
**71/1-2** (2007), 21–26

# The unit group of $FA_4$

By R. K. SHARMA (Delhi), J. B. SRIVASTAVA (Delhi)
and MANJU KHAN (Delhi)

**Abstract.** A complete characterization of the unit group $\mathscr{U}(FA_4)$ of the group algebra $FA_4$ of the alternating group of degree 4, $A_4$, over a finite field $F$ has been obtained.

## 1. Introduction and result

Let $FG$ be the group algebra of a group $G$ over a field $F$. For a normal subgroup $H$ of $G$, the canonical homomorphism $g \mapsto gH : G \longrightarrow G/H$ can be extended to an algebra homomorphism from $FG$ to $F[G/H]$ defined by

$$\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g gH,$$

for $a_g \in F$. The kernel of this homomorphism, denoted by $\omega(H)$, is the ideal of $FG$ generated by $\{h-1 \mid h \in H\}$. Thus $FG/\omega(H) \cong F[G/H]$. The augmentation ideal $\omega(FG)$ of the group algebra $FG$ is defined by

$$\omega(FG) = \left\{ \sum_{g\in G} a_g g \in FG \;\middle|\; a_g \in F,\ \sum_{g\in G} a_g = 0 \right\}.$$

Clearly $\omega(G) = \omega(FG)$. In general, $\omega(H) = \omega(FH)FG = FG\omega(FH)$. For $H = G$, $FG/\omega(G) \cong F$, showing Jacobson radical of $FG$, $J(FG)$, is contained in $\omega(FG)$. It is known that $J(FG) = \omega(FG)$ when $G$ is a finite $p$-group and the characteristic of $F$, $\mathrm{char}(F)$, is $p$.

*Mathematics Subject Classification:* 16S34, 17B30.
*Key words and phrases:* group algebras, unit groups.

The lower central chain of $G$ is given by

$$G=\gamma_1(G)\supseteq\gamma_2(G)\supseteq\cdots\supseteq\gamma_{m+1}(G)\supseteq\cdots$$

where $\gamma_{m+1}(G)=(\gamma_m(G),G)$, for $m\geq 1$. For $g_1,g_2\in G$ the commutator $(g_1,g_2)=g_1^{-1}g_2^{-1}g_1g_2$. The group $G$ is said to be nilpotent of class $n$ if $\gamma_{n+1}(G)=(1)$ and $\gamma_n(G)\neq(1)$. In this paper, the work is on the alternating group of degree 4, $A_4$, whose presentation is given by

$$A_4=\left\langle\sigma,a\mid\sigma^3=a^2=(\sigma a)^3=1\right\rangle$$

where $\sigma=(1,2,3)$ and $a=(1,2)(3,4)$. Thus, with $b=(1,3)(2,4)$ and $c=(1,4)(2,3)$,

$$A_4=\left\{1,a,b,c,\sigma,\sigma a,\sigma b,\sigma c,\sigma^2,\sigma^2a,\sigma^2b,\sigma^2c\right\}.$$

The distinct conjugacy classes of $A_4$ are $\mathscr{C}_0=\{1\}$, $\mathscr{C}_1=\{a,b,c\}$, $\mathscr{C}_2=\{\sigma,\sigma a,\sigma b,\sigma c\}$, $\mathscr{C}_3=\{\sigma^2,\sigma^2a,\sigma^2b,\sigma^2c\}$. Hence $\left\{\widehat{\mathscr{C}_0},\widehat{\mathscr{C}_1},\widehat{\mathscr{C}_2},\widehat{\mathscr{C}_3}\right\}$ form a basis of the center $Z(FA_4)$ (cf. Lemma 4.1.1 of [4]), where $\widehat{\mathscr{C}_i}$ denotes the class sum. We obtain the relations between $\widehat{\mathscr{C}_0}$, $\widehat{\mathscr{C}_1}$, $\widehat{\mathscr{C}_2}$, $\widehat{\mathscr{C}_3}$ given by

$$\widehat{\mathscr{C}_2}^2=4\widehat{\mathscr{C}_3},\quad\widehat{\mathscr{C}_2}^4=4^3\widehat{\mathscr{C}_2},\quad\widehat{\mathscr{C}_3}^2=4\widehat{\mathscr{C}_2},\quad\widehat{\mathscr{C}_3}^4=4^3\widehat{\mathscr{C}_3},$$
$$\text{and}\quad\widehat{\mathscr{C}_2}^3=4^2(\widehat{\mathscr{C}_0}+\widehat{\mathscr{C}_1})=\widehat{\mathscr{C}_3}^3.$$

Using these relations one may prove, by induction on $r$, that for $i=2,3$,

$$(\widehat{\mathscr{C}_i})^{3r+1}=4^{3r}\widehat{\mathscr{C}_i},\quad\text{for}\quad r\geq 0. \tag{1}$$

We define a matrix representation of $A_4$,

$$\theta: A_4\longrightarrow\mathscr{U}(F\oplus\mathbb{M}(3,F))$$

by the assignment

$$\sigma\mapsto\left(1,\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\right)\quad\text{and}\quad a\mapsto\left(1,\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\right)$$

and can be extended to an algebra homomorphism

$$\theta^*: FA_4\longrightarrow F\oplus\mathbb{M}(3,F),$$

where $\mathbb{M}(n, F)$ denotes the algebra of all $n \times n$ matrices over $F$. We use $V_1$ to denote $1+J(FA_4)$, the kernel of the epimorphism from $\mathscr{U}(FA_4)$ to $\mathscr{U}(FA_4/J(FA_4))$ which is induced by the canonical homomorphism: $FA_4 \longrightarrow FA_4/J(FA_4)$.

Allen and Hobby in [2], also Al-Sohebany in [1] have worked on the characterization of the unit group $\mathscr{U}(\mathbb{Z}A_4)$, and obtained that the group $A_4$ has a torsion free normal complement in $V(\mathbb{Z}A_4)$, the subgroup of the unit group of augmentation 1. Subsequently, Sharma and Gongopadhaya has given presentations of the torsion free normal complement of $A_4$ in $V(\mathbb{Z}A_4)$ and of $V(\mathbb{Z}A_4)$ in [5], [6]. Conjugacy classes of all elements of finite order in $V(\mathbb{Z}A_4)$ have been studied by Allen and Hobby in [3] . However, so far, the structure of the unit group $\mathscr{U}(FA_4)$, for $\mathrm{char}(F) = p > 0$ is not known.

This paper gives a complete characterization of the unit group $\mathscr{U}(FA_4)$, for $\mathrm{char}(F) = p > 0$ by proving the following:

**Theorem.** *Let $\mathscr{U}(FA_4)$ be the group of units of the group algebra $FA_4$ of the alternating group of degree 4 over a finite field $F$ of positive characteristic $p$. Let $F_2$ be a quadratic extension of the field $F$ and $V_1 = 1 + J(FA_4)$, where $J(FA_4)$ denotes the Jacobson radical of the group algebra $FA_4$.*

(1) *If $p = 2$, then $V_1$ is a nilpotent group of class 2 and $\mathscr{U}(FA_4)$ is centrally metabelian, but not metabelian.*

(2) *If $p = 3$, then $V_1$ is a central subgroup of exponent $|F|$ and*

$$\mathscr{U}(FA_4)/V_1 \cong F^* \times GL(3, F)$$

(3) *If $p > 3$ and $|F| = p^n$, then*

$$\mathscr{U}(FA_4) \cong \begin{cases} GL(3,F) \times F^* \times F^* \times F^* & \text{if } \ 3 \mid (p-1) \text{ or } n \text{ is even;} \\ GL(3,F) \times F_2^* \times F^* & \text{if } n \text{ is odd.} \end{cases}$$

*Here $F^* = F \setminus \{0\}$ and $GL(3, F)$ is the general linear group of degree 3 over $F$.*

## 2. Proof of the theorem

(1) Let $\mathrm{char}(F) = 2$ with $|F| = 2^n$. Set $K_4 = \{1, a, b, c\}$ so that $K_4$ is a normal subgroup of $A_4$ and $[A_4 : K_4] = 3$. Then, by Theorem 7.2.7 of [4],

$$J(FA_4) = J(FK_4)FA_4 = \omega(FK_4)FA_4 = \omega(K_4).$$

Hence $FA_4/J(FA_4) \cong F[A_4/K_4]$. Note that $A_4/K_4 = \langle \bar{\sigma} = \sigma K_4 \rangle$, is a cyclic group of order 3, say $C_3$. Suppose $x = \alpha_0 + \alpha_1\bar{\sigma} + \alpha_2\bar{\sigma}^2 \in FC_3$, for $\alpha_i \in F$. If $n$ is even, then $3 \mid (2^n - 1)$ and consequently,

$$x^{2^n} = {\alpha_0}^{2^n} + {\alpha_1}^{2^n}(\bar{\sigma})^{2^n} + {\alpha_2}^{2^n}(\bar{\sigma}^2)^{2^n} = x;$$

so that $o(x) \mid (2^n - 1)$, when $x \in \mathscr{U}(FC_3)$. Thus $FC_3 \cong F \oplus F \oplus F$, if $n$ is even. When $n$ is odd, $3 \nmid (2^n - 1)$; but $3 \mid (2^{2n} - 1)$ and as above, $x^{2^{2n}} = x, \forall x \in FC_3$. Hence if $n$ is odd, $FC_3 \cong F_2 \oplus F$.

Now, observe that $x(a-1)y(b-1) \in Z(FA_4)$, $\forall x, y \in A_4$, so that $\omega(K_4)^2 \subseteq Z(FA_4)$, and consequently, $\omega(K_4)^3 = 0$. For $\xi, \eta \in \omega(K_4)$, we have

$$(1 + \xi, 1 + \eta) \equiv (1 - \xi - \eta)(1 + \xi + \eta) \equiv 1 \mod Z(FA_4).$$

Thus $\gamma_2(V_1) \subseteq Z(FA_4)$ and hence $\gamma_3(V_1) = (1)$, so that $V_1$ is a nilpotent group of class 2.

Since $\mathscr{U}(FA_4)/V_1$ is an Abelian group, we have $\mathscr{U}(FA_4)' \subseteq V_1$, therefore $\mathscr{U}(FA_4)'' \subseteq V_1' \subseteq Z(FA_4)$. Hence $\mathscr{U}(FA_4)$ is centrally metabelian. But it is not metabelian because $((u_1, u_2), (u_1, u_3)) \neq 1$, where

$$\begin{aligned} u_1 &= 1 + \sigma + \sigma^2 a + \sigma^2 b + b \\ u_2 &= 1 + \sigma + \sigma^2 a + \sigma^2 c + b \\ u_3 &= 1 + \sigma^2 a + \sigma^2 b + \sigma^2 c + c. \end{aligned}$$

(2) Let char$(F) = 3$ and $|F| = 3^n$. Assume $x \in \text{Ker } \theta^*$ (cf. Introduction for $\theta^*$) with

$$x = \sum_{(i,t) \in I} \alpha_i t + \alpha_{i+1}\sigma t + \alpha_{i+2}\sigma^2 t,$$

for $\alpha_i \in F$ and $I = \{(0, 1), (3, a), (6, b), (9, c)\}$. Then $\theta^*(x) = 0$ gives the following systems of equations:

$$\sum_{i=0}^{11} \alpha_i = 0 \tag{2}$$

$$\alpha_i + \alpha_j - \alpha_k - \alpha_l = 0, \tag{3}$$

where $(i, j, k, l) \in \{(0, 6, 3, 9), (1, 10, 4, 7), (2, 5, 8, 11), (2, 8, 5, 11), (0, 9, 3, 6), (1, 4, 7, 10), (1, 7, 4, 10), (2, 11, 5, 8), (0, 3, 6, 9)\}$. Solving the system over $F$ we get

$$\alpha_i = \alpha_{i+3j}, \quad \text{for } i = 0, 1, 2 \quad \text{and} \quad j = 1, 2, 3.$$

Further, from equation (2), $\alpha_0+\alpha_1+\alpha_2=0$. Hence

$$\text{Ker }\ \theta^* = \{\alpha_0(\widehat{\mathscr{C}_0}+\widehat{\mathscr{C}_1})+\alpha_1\widehat{\mathscr{C}_2}+\alpha_2\widehat{\mathscr{C}_3} \mid \alpha_0+\alpha_1+\alpha_2=0\}.$$

Thus $x^{3^n}=0$, $\forall x\in$ Ker $\theta^*$, so that Ker $\theta^*\subseteq J(FA_4)$. Also, since $\theta^*$ is onto, $\theta^*(J(FA_4))\subseteq J(F\oplus\mathbb{M}(3,F))=0$ and so $J(FA_4)\subseteq$ Ker $\theta^*$. Hence Ker $\theta^*=J(FA_4)$, therefore

$$(FA_4)/J(FA_4)\cong F\oplus\mathbb{M}(3,F).$$

Now, since $\mathscr{U}(FA_4)/V_1\cong\mathscr{U}(FA_4/J(FA_4))$, we have

$$\mathscr{U}(FA_4)/V_1\cong F^*\times GL(3,F).$$

Since $x^{3^n}=0$, for all $x\in J(FA_4)$, we have $V_1$ is a central subgroup of exponent $3^n$.

(3) Assume $p>3$. Since $p\nmid|A_4|$, by Artin–Wedderburn theorem we have

$$FA_4\cong\mathbb{M}(n_1,D_1)\oplus\mathbb{M}(n_2,D_2)\oplus\cdots\oplus\mathbb{M}(n_r,D_r),$$

where $D_i$'s are finite dimensional division algebras over $F$. Thus $D_i$' s are finite fields, as $F$ is finite. Since $FA_4$ is noncommutative, there exists a $k$ such that $n_k>1$, so that $n_k$ will be either 2 or 3. Further, since $\dim_F(Z(FA_4))=4$, we will get either of the following two possibilities only.

$$FA_4\cong\mathbb{M}(3,F)\oplus F\oplus F\oplus F$$
$$FA_4\cong\mathbb{M}(3,F)\oplus F_2\oplus F$$

If $3\mid(p-1)$, then $p^n\equiv 1 \mod 3$, for all $n$. Using the equation (1) in Section 1, we compute that

$$\widehat{\mathscr{C}_2}^{p^n}=\widehat{\mathscr{C}_2}^{3r+1}=(4^{3r})\widehat{\mathscr{C}_2}=(4^{p^n-1})\widehat{\mathscr{C}_2}=\widehat{\mathscr{C}_2}.$$

Similarly, we have $\widehat{\mathscr{C}_3}^{p^n}=\widehat{\mathscr{C}_3}$. Also, note that $(\widehat{\mathscr{C}_0}+\widehat{\mathscr{C}_1})^{p^n}=\widehat{\mathscr{C}_0}+\widehat{\mathscr{C}_1}$. Thus $x^{p^n}=x$, for all $x\in Z(FA_4)$. In particular, if $x\in\mathscr{U}(Z(FA_4))$, then $o(x)\mid(p^n-1)$. Hence $FA_4\cong\mathbb{M}(3,F)\oplus F\oplus F\oplus F$. Also, when $3\nmid(p-1)$ but $n$ is even, then $3\mid(p^n-1)$, so that

$$FA_4\cong\mathbb{M}(3,F)\oplus F\oplus F\oplus F.$$

If $3\nmid(p-1)$ and $n$ is odd, we get $3\nmid(p^n-1)$ and so $3\mid(p^n+1)$. Then $3\mid(p^{2n}-1)$, which implies $x^{p^{2n}}=x$ for all $x\in Z(FA_4)$. Thus

$$FA_4\cong\mathbb{M}(3,F)\oplus F_2\oplus F.$$

Hence

$$\mathscr{U}(FA_4) \cong \begin{cases} GL(3,F) \times F^* \times F^* \times F^* & \text{if } 3 \mid (p-1) \text{ or } n \text{ is even;} \\ GL(3,F) \times F_2^* \times F^* & \text{if } n \text{ is odd.} \end{cases}$$

This completes the proof of the Theorem.

Acknowledgements. We would like to thank the referee for the valuable suggestions which improved the quality of the presentation.

## References

[1] A. A. Al-Sohebany, On the group of units of $\mathbf{Z}[A_4]$, *Arab Gulf J. Sci. Res.* **3**(1) (1985), 227–235.

[2] P. J. Allen and C. Hobby, A characterization of units in $\mathbf{Z}[A_4]$, *J. Algebra* **66**(2) (1980), 534–543.

[3] P. J. Allen and C. Hobby, Elements of finite order in $V(\mathbf{Z}A_4)$, *Pacific J. Math.* **138**(1) (1989), 1–8.

[4] D. S. Passman, The algebraic structure of group rings, *Wiley-Interscience [John Wiley & Sons], New York*, 1977.

[5] R. K. Sharma and S. Gangopadhyay, On chains in units of $\mathbf{Z}A_4$, *Math. Sci. Res. Hot-Line* **4**(9) (2000), 1–33.

[6] R. K. Sharma and S. Gangopadhyay, On units in $\mathbf{Z}A_4$, *Math. Sci. Res. Hot-Line* **4**(8) (2000), 13–29.

R. K. SHARMA
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, DELHI
NEW DELHI-110016
INDIA

*E-mail:* rksharma@maths.iitd.ac.in

J. B. SRIVASTAVA
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, DELHI
NEW DELHI-110016
INDIA

*E-mail:* jbsrivas@maths.iitd.ac.in

MANJU KHAN
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, DELHI
NEW DELHI-110016
INDIA

*E-mail:* manjukhan.iitd@gmail.com

*(Received September 6, 2005; revised June 26, 2006)*