Publ. Math. Debrecen
71/1-2 (2007), 27–41

On weakly symmetric Riemannian manifolds

By ABSOS ALI SHAIKH (Burdwan) and SANJIB KUMAR JANA (Burdwan)

Abstract. The object of the present paper is to study weakly symmetric Riemannian manifolds. Among others it is shown that a conformally flat weakly symmetric Riemannian manifold is of hyper quasi-constant curvature which generalizes the notion of quasi-constant curvature and also such a manifold is a quasi-Einstein manifold. Finally several examples of weakly symmetric manifolds of both zero and non-zero scalar curvature are obtained.

1. Introduction

In 1989 L. Tamássy and T. Q. Binh [7] introduced the notions of weakly symmetric and weakly projective symmetric manifolds. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called weakly symmetric if the curvature tensor R of type $(0, 4)$ satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) = {} & A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V)\\ & + C(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V)\\ & + E(V)R(Y,Z,U,X) \qquad (1.1)\end{aligned}$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where A, B, C, D and E are 1-forms(non-zero simultaneously) and ∇ is the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold, and an n-dimensional manifold of this kind is denoted by $(WS)_n$. Then in 1994 M. C. Chaki [2] introduced the notion of a generalized

Mathematics Subject Classification: 53B35, 53B05.
Key words and phrases: weakly symmetric manifold, conformally flat manifold, hyper-quasi constant curvature, scalar curvature.

pseudo symmetric manifold, whose defining condition is a little stronger than $(WS)_n$. If in (1.1) the 1-form A is replaced by $2A$, and E is replaced by A, then a $(WS)_n$ will be a generalized pseudo symmetric manifold. Since the notion of $(WS)_n$ is weaker than that of a generalized pseudo symmetric manifold, we confined ourselves to the study of $(WS)_n$. The existence of a $(WS)_n$ is proved by M. Prvanović [6]. Then U. C. De and S. Bandyopadhyay [5] gave an example of a $(WS)_n$ by a suitable metric, and proved that in a $(WS)_n$ the associated 1-forms $B = C$ and $D = E$. Hence the defining condition of a $(WS)_n$ reduces to the following form:

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) = {} & A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) \\ & + B(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) \\ & + D(V)R(Y,Z,U,X). \end{aligned} \tag{1.2}$$

Section 2 is concerned with some fundamental results of $(WS)_n$. In Section 3 we study conformally flat $(WS)_n$, and prove that such a manifold of non-zero constant scalar curvature is of hyper quasi-constant curvature. The last section deals with some examples of $(WS)_n$. In [5] De and Bandyopadhyay obtained an example of $(WS)_n$, which is of zero scalar curvature.

As a natural question arises, whether there exists or not $(WS)_n$ of non-zero scalar curvature? The last section provides the answer of this question by several examples.

2. Fundamental results of a $(WS)_n$ $(n > 2)$

Let $\{e_i : i = 1, \ldots, n\}$ be an orthonormal basis of the tangent spaces in a neighbourhood of a point of the manifold. Then setting $Y = V = e_i$ in (1.2), and taking summation over $i, 1 \leq i \leq n$, we get

$$\begin{aligned}(\nabla_X S)(Z,U) = {} & A(X)S(Z,U) + B(Z)S(X,U) + D(U)S(X,Z) \\ & + B(R(X,Z)U) + D(R(X,U)Z), \end{aligned} \tag{2.1}$$

where S is the Ricci tensor of type $(0,2)$. From (2.1) it follows that a $(WS)_n$ $(n > 2)$ is weakly Ricci symmetric (briefly$(WRS)_n$ $(n > 2)$) [8] if

$$B(R(X,Z)U) + D(R(X,U)Z) = 0 \tag{2.2}$$

for all X, U, Z. This leads to the following:

Theorem 1. *A* $(WS)_n$ $(n > 2)$ *satisfying the condition* (2.2) *is a* $(WRS)_n$.
Again from (2.1) *it follows that*

$$dr(X) = rA(X) + 2B(QX) + 2D(QX), \tag{2.3}$$

where r *is the scalar curvature of the manifold,* Q *is the Ricci-operator i.e.,* $g(QX, Y) = S(X, Y)$.

From (2.3), we can state the following:

Theorem 2. *If a* $(WS)_n$ $(n > 2)$ *is of non-zero constant scalar curvature, then the 1-form* A *can be expressed as*

$$A(X) = -\frac{2}{r}[B(QX) + D(QX)] \tag{2.4}$$

for all X.

Also (2.3) leads to the following:

Corollary. *If a* $(WS)_n (n > 2)$ *is of zero scalar curvature, then the relation* $B(QX) + D(QX) = 0$ *holds for all* X.

Interchanging Z and U in (2.1), and then subtracting the resultant from (2.1), we obtain by virtue of the Bianchi identity

$$\begin{aligned} &[B(Z) - D(Z)]S(X, U) - [B(U) - D(U)]S(X, Z) \\ &\qquad - [B(R(Z, U)X) - D(R(Z, U)X)] = 0. \end{aligned} \tag{2.5}$$

Replacing X and U by e_i, and taking summation over i, $1 \le i \le n$, we get

$$r[B(Z) - D(Z)] = 2[B(QZ) - D(QZ)]. \tag{2.6}$$

We define the vector field ρ by $T(X) = g(X, \rho) = B(X) - D(X)$ for all X. Then (2.6) yields

$$T(QX) = \frac{r}{2}T(X). \tag{2.7}$$

Hence we can state the following:

Theorem 3. *In a* $(WS)_n (n > 2)$, $r/2$ *is an eigenvalue of the Ricci tensor corresponding to the eigenvector* ρ *defined by* $T(X) = g(X, \rho) = B(X) - D(X) \neq 0$ *for all* X.

Again from (2.5) we can state the following:

Theorem 4. *In a* $(WS)_n$ $(n > 2)$ *the relation*

$$T(Z)S(X, U) - T(U)S(X, Z) - T(R(Z, U)X) = 0 \tag{2.8}$$

holds for all vector fields X, Z, U, *and* T *is a 1-form defined by* $T(X) = B(X) - D(X) \neq 0$ *for all* X.

3. Conformally flat $(WS)_n$

Let (M^n, g) $(n \geq 3)$ be a conformally flat $(WS)_n$. It is known that in a conformally flat Riemannian manifold (M^n, g) $(n \geq 3)$ the following relations hold:

$$\begin{aligned}&(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X)\\ &= \frac{1}{2(n-1)}[g(Y,Z)dr(X) - g(X,Y)dr(Z)].\end{aligned} \tag{3.1}$$

Interchanging X and U in (2.1), and then subtracting the resultant from (2.1), we obtain by virtue of (3.1) that

$$\begin{aligned}&[A(X) - D(X)]S(U,Z) - [A(U) - D(U)]S(X,Z) + B(R(X,U)Z)\\ &+ 2D(R(X,U)Z) = \frac{1}{2(n-1)}[g(Z,U)dr(X) - g(X,Z)dr(U)].\end{aligned} \tag{3.2}$$

Let ρ_1, ρ_2, ρ_3 be the associated vector fields corresponding to the 1-forms A, B, D respectively, i.e., $g(X,\rho_1) = A(X)$, $g(X,\rho_2) = B(X)$ and $D(X) = g(X,\rho_3)$. Substituting U by ρ_2 in (3.2), and then using (2.3), we get

$$\begin{aligned}&[A(X) - D(X)]B(QZ) - [A(\rho_2) - D(\rho_2)]S(X,Z) + R(X,\rho_2,Z,\rho_2)\\ &+ 2R(X,\rho_2,Z,\rho_3) = B(Z)[rA(X) + 2B(QX) + 2D(QX)]\\ &- g(X,Z)[rA(\rho_2) + 2B(Q\rho_2) + 2D(Q\rho_2)].\end{aligned} \tag{3.3}$$

If the manifold has non-zero constant scalar curvature, then (3.3) yields by virtue of (2.4) that

$$\begin{aligned}&[A(\rho_2) - D(\rho_2)]S(X,Z) - [A(X) - D(X)]B(QZ)\\ &+ R(\rho_2,X,Z,\rho_2) + 2R(\rho_2,X,Z,\rho_3) = 0.\end{aligned} \tag{3.4}$$

Again, since the manifold under consideration is conformally flat, we have

$$\begin{aligned}R(X,Y,Z,W) = &\frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\\ &+ S(X,W)g(Y,Z) - S(Y,W)g(X,Z)]\\ &+ \frac{r}{(n-1)(n-2)}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)].\end{aligned} \tag{3.5}$$

From (3.5), it follows that

$$\begin{aligned}&R(\rho_2,X,Z,\rho_2) + 2R(\rho_2,X,Z,\rho_3) = \frac{1}{n-2}[S(X,Z)\{B(\rho_2) + 2B(\rho_3)\}\\ &- B(QZ)\{B(X) + 2D(X)\} + g(X,Z)\{B(Q\rho_2) + 2B(Q\rho_3)\}\end{aligned}$$

$$- B(Z)\{B(QX) + 2D(QX)\}] + \frac{r}{(n-1)(n-2)}$$
$$\times [B(Z)\{B(X) + 2D(X)\} - g(X,Z)\{B(\rho_2) + 2B(\rho_3)\}]. \tag{3.6}$$

Using (3.6) in (3.4) we obtain

$$\begin{aligned} S(X,Z) = \alpha g(X,Z) &+ \alpha_1 B(X)B(Z) + \alpha_2 B(Z)D(X) \\ &+ \alpha_3 B(X)\tilde{B}(Z) + \alpha_4 B(Z)\tilde{B}(X) + \alpha_5 B(Z)\tilde{D}(X) \\ &+ \alpha_6 \tilde{B}(Z)D(X) + \alpha_7 A(X)\tilde{B}(Z), \end{aligned} \tag{3.7}$$

where $\alpha, \alpha_1, \dots, \alpha_7$ are scalars in terms of r, $B(\rho_2)$ and $B(\rho_3)$, and $\tilde{B}(X) = B(QX), \tilde{D}(X) = D(QX)$ for all X. This leads to the following:

Theorem 5. *In a conformally flat $(WS)_n$ $(n \geq 3)$ of non-zero constant scalar curvature the Ricci tensor S has the form* (3.7).

Again, using (2.4) in (3.7), we have

$$\begin{aligned} S(X,Z) = \alpha g(X,Z) &+ \alpha_1 B(X)B(Z) + \alpha_2 B(Z)D(X) + \alpha_3 B(X)\tilde{B}(Z) \\ &+ \alpha_4 B(Z)\tilde{B}(X) + \alpha_5 B(Z)\tilde{D}(X) + \alpha_6 \tilde{B}(Z)D(X) \\ &+ \alpha_7 \left(-\frac{2}{r}\right)[\tilde{B}(X) + \tilde{D}(X)]\tilde{B}(Z). \end{aligned} \tag{3.8}$$

According to Chen and Yano [4], a Riemannian manifold (M^n, g) $(n > 3)$ is said to be of quasi-constant curvature if it is conformally flat, and its curvature tensor R of type (0, 4) has the form

$$\begin{aligned} R(X,Y,Z,W) = a[g(Y,Z)g(X,W) &- g(X,Z)g(Y,W)] \\ &+ b[g(X,W)A(Y)A(Z) - g(X,Z)A(Y)A(W) \\ &+ g(Y,Z)A(X)A(W) - g(Y,W)A(X)A(Z)], \end{aligned} \tag{3.9}$$

where A is a 1-form, and a, b are scalars of which $b \neq 0$.

Generalizing this notion we define the manifold of hyper quasi-constant curvature as follows:

A Riemannian manifold (M^n, g) $(n > 3)$ is said to be of hyper quasi-constant curvature if it is conformally flat, and its curvature tensor R of type (0, 4) satisfies the condition

$$\begin{aligned} R(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + g(X,W)P(Y,Z) \\ - g(X,Z)P(Y,W) + g(Y,Z)P(X,W) - g(Y,W)P(X,Z), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned}P(Y,Z) = (\beta BD)(Y,Z) = \beta_1 B(Z)D(Y) + \beta_2 B(Z)B(Y) + \beta_3 \tilde{B}(Z)B(Y)\\ + \beta_4 B(Z)\tilde{B}(Y) + \beta_5 B(Z)\tilde{D}(Y) + \beta_6 \tilde{B}(Z)D(Y)\\ + \beta_7 \tilde{B}(Z)\tilde{B}(Y) + \beta_8 \tilde{B}(Z)\tilde{D}(Y),\end{aligned}$$

and $\beta_1, \beta_2, \dots, \beta_8$ are non-zero scalars.

Now in view of (3.8) we obtain from (3.5) that

$$\begin{aligned}R(X,Y,Z,W) = a_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + g(X,W)P'(Y,Z)\\ - g(X,Z)P'(Y,W) + g(Y,Z)P'(X,W)\\ - g(Y,W)P'(X,Z), \qquad (3.11)\end{aligned}$$

where

$$\begin{aligned}P'(Y,Z) = (\beta' BD)(Y,Z) = \beta'_1 B(Z)D(Y) + \beta'_2 B(Z)B(Y) + \beta'_3 \tilde{B}(Z)B(Y)\\ + \beta'_4 B(Z)\tilde{B}(Y) + \beta'_5 B(Z)\tilde{D}(Y) + \beta'_6 \tilde{B}(Z)D(Y)\\ + \beta'_7 \tilde{B}(Z)\tilde{B}(Y) + \beta'_8 \tilde{B}(Z)\tilde{B}(Y),\end{aligned}$$

and $\beta'_1, \beta'_2, \dots, \beta'_8$ are non-zero scalars. Comparing (3.10) and (3.11), it follows that the manifold is of hyper quasi-constant curvature . This leads to the following:

Theorem 6. *A conformally flat $(WS)_n$ $(n > 3)$ of non-zero constant scalar curvature is a manifold of hyper quasi-constant curvature.*

Now putting $U = \rho$ in (2.8), and then using (2.7), we get

$$(r/2)T(X)T(Z) - T(\rho)S(X,Z) + R(\rho,Z,X,\rho) = 0. \qquad (3.12)$$

Let us now suppose that a $(WS)_n$ $(n > 3)$ is conformally flat, and of non-zero scalar curvature. Then (3.5) yields

$$\begin{aligned}R(\rho,Z,X,\rho) = \frac{1}{n-2}\Big[T(\rho)S(X,Z) - rT(X)T(Z) + \frac{r}{2}T(\rho)g(X,Z)\Big]\\ + \frac{r}{(n-1)(n-2)}[T(Z)T(X) - T(\rho)g(X,Z)]. \qquad (3.13)\end{aligned}$$

Using (3.13) in (3.12), it follows that

$$2(n-1)T(\rho)S(X,Z) = rT(\rho)g(X,Z) + r(n-2)T(X)T(Z). \qquad (3.14)$$

We shall now show that $T(\rho) \neq 0$. For if $T(\rho) = 0$, then (3.14) implies that

$$r(n-2)T(X)T(Z) = 0.$$

Since $T(X) \neq 0$ for all X, and $n > 3$, the above relation yields $r = 0$, a contradiction to the assumption that the manifold is of non-zero scalar curvature. Thus we have $T(\rho) \neq 0$. Consequently (3.14) yields

$$S(X,Z) = \alpha g(X,Z) + \beta T(X)T(Z), \tag{3.15}$$

where α, β are non-zero scalars.

Again, according to Chaki and Maity [3], a Riemannian manifold is said to be quasi-Einstein, if its Ricci tensor is of the form

$$S = pg + q\omega \otimes \omega,$$

where p, q are scalars of which $q \neq 0$ and ω is a 1-form. This leads to the following:

Theorem 7. *A conformally flat $(WS)_n$ $(n > 3)$ of non-vanishing scalar curvature is a quasi-Einstein manifold with respect to the 1-form T defined by $T(X) = B(X) - D(X) \neq 0$ for all X.*

Again, using (3.15) in (3.5), it follows that

$$\begin{aligned} R(X,Y,Z,W) &= \gamma[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad + \delta[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ &\quad + g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)], \end{aligned} \tag{3.16}$$

where γ and δ are non-zero scalars. Comparing (3.16) and (3.9), we can state the following:

Theorem 8. *A conformally flat $(WS)_n$ $(n > 3)$ of non-vanishing scalar curvature is a manifold of quasi-constant curvature with respect to the 1-form T defined by $T(X) = B(X) - D(X) \neq 0$ for all X.*

Using the expression of T in (3.16), it can be easily seen that

$$\begin{aligned} R(X,Y,Z,W) &= \gamma[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad + g(X,W)\{\delta BD\}(Y,Z) - g(X,Z)\{\delta BD\}(Y,W) \\ &\quad + g(Y,Z)\{\delta BD\}(X,W) - g(Y,W)\{\delta BD\}(X,Z), \end{aligned}$$

where $\{\delta BD\} = \delta(BB - BD - DB + DD)$.

Comparing the above relation with (3.10), we can state the following:

Theorem 9. *A conformally flat $(WS)_n$ $(n > 3)$ of non-zero scalar carvature is a manifold of hyper quasi-constant curvature.*

4. Some examples of $(WS)_n$

This section deals with several examples of $(WS)_n$. On the real number space R^n (with co-ordinates $x^1, x^2, \ldots, x^n$) we define a suitable Riemannian metric g such that R^n becomes a Riemannian manifold (M^n, g). We calculate the components of the curvature tensor and its covariant derivative, and then we verify the defining relation (1.2).

Example 1. We define a Riemannian metric on the 4-dimensional real number space R^4 by the formula

$$ds^2 = g_{ij}dx^i dx^j = f(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (x^1)^2(dx^4)^2, \quad (i, j = 1, 2, \ldots, 4), \tag{4.1}$$

where $f = a_0 + a_1x^3 + a_2(x^3)^2$, a_0, a_1, a_2 are non-constant functions of x^1 only. Then the only non-vanishing components of the Christoffel symbols and the curvature tensor are

$$\Gamma^2_{11} = \frac{1}{2}f_{.1}, \quad \Gamma^2_{13} = -\Gamma^3_{11} = \frac{1}{2}f_{.3}, \quad \Gamma^4_{14} = \frac{1}{x^1}, \quad \Gamma^2_{44} = -x^1,$$
$$R_{1331} = \frac{1}{2}f_{.33} = a_2 \neq 0, \tag{4.2}$$

and the components which can be obtained from these by the symmetric properties. Here '.' denotes the partial differentiation with respect to the coordinates. Using the above relations, it can be easily shown that the scalar curvature of the manifold is zero. Therefore R^4 with the considered metric is a Riemannian manifold M^4 whose scalar curvature is zero. The only non-zero covariant derivatives of R are

$$R_{1331,1} = \frac{1}{2}f_{.331} = (a_2)_{.1} \neq 0, \tag{4.3}$$

and the components which can be obtained from (4.3) by the symmetric properties, where ',' denotes the covariant derivative with respect to the metric tensor. Hence our (M^4, g) is neither flat nor locally symmetric. We shall now show that this M^4 is a $(WS)_4$, i.e. it satisfies (1.2). Let us now consider the 1-forms

$$\begin{aligned} A_i(x) &= -d(x^2x^3) \quad \text{for } i = 1 & B_i(x) &= d(x^2x^3) \quad \text{for } i = 1 \\ &= 0, \quad \text{otherwise} & &= 0, \quad \text{otherwise} \\ D_i(x) &= d(\log a_2) \quad \text{for } i = 1 \\ &= 0, \quad \text{otherwise} \end{aligned} \tag{4.4}$$

at any point $x \in M$. In our M^4 (1.2) reduces with these 1-forms to the following equations

$$\begin{aligned}
&\text{(i)} && R_{1331,1} = A_1 R_{1331} + B_1 R_{1331} + B_3 R_{1131} + D_3 R_{1311} + D_1 R_{1331}\\
&\text{(ii)} && R_{1131,3} = A_3 R_{1131} + B_1 R_{3131} + B_1 R_{1331} + D_3 R_{1131} + D_1 R_{1133}\\
&\text{(iii)} && R_{1311,3} = A_3 R_{1311} + B_1 R_{3311} + B_3 R_{1311} + D_1 R_{1331} + D_1 R_{1313},
\end{aligned}$$

since for the cases other than (i), (ii) and (iii) the components of each term of (1.2) vanish identically, and the relation (1.2) holds trivially. Now, from (4.2), (4.3) and (4.4) we get the following relations for the right hand side (R.H.S.) and left hand side (L.H.S.) of (i):
R.H.S. of (i) $= (A_1 + B_1 + D_1)R_{1331} = d(\log a_2)R_{1331} = (a_2)_{.1} =$ L.H.S. of (i). Also R.H.S. of

$$\begin{aligned}
\text{(ii)} &= -d(x^2x^3)(R_{3131} + R_{1331}),\\
&= 0 \text{ (by the skew symmetric property of} R)\\
&= \text{L.H.S. of (ii)}.
\end{aligned}$$

By a similar argument as in (ii) it can be shown that the relation (iii) is true.

Hence we can state the following:

Theorem 10. *Let (M^4, g) be a Riemannian manifold endowed with the metric*

$$ds^2 = g_{ij}dx^i dx^j = f(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (x^1)^2(dx^4)^2$$
$$(i, j = 1, 2, \ldots, 4),$$

where $f = a_0 + a_1x^3 + a_2(x^3)^2$, a_0, a_1, a_2 are non-constant functions of x^1 only. Then (M^4, g) is a weakly symmetric manifold of vanishing scalar curvature which is not locally symmetric.

In particular, if we take $a_2 = e^{x^1}$, then (4.2) and (4.3) respectively reduce to the following:

$$R_{1331} = e^{x^1} \neq 0 \tag{4.5}$$

$$R_{1331,1} = e^{x^1} \neq 0, \tag{4.6}$$

and hence the manifold under consideration is not locally symmetric. If we consider the 1-forms

$$\begin{aligned}
A_i(x) &= \frac{1}{2} \quad \text{for } i = 1 & \qquad B_i(x) &= -\frac{1}{4} \quad \text{for } i = 1\\
&= 0, \quad \text{otherwise} & &= 0, \qquad \text{otherwise}
\end{aligned}$$

$$\begin{aligned} D_i(x) &= \frac{3}{4} \quad \text{for } i = 1 \\ &= 0, \quad \text{otherwise}, \end{aligned} \tag{4.7}$$

then proceeding similarly as in the previous case, it can easily be shown that the manifold under consideration satisfies (i)–(iii), and hence is a $(WS)_4$. Thus we have the following:

Theorem 11. *Let (M^4, g) be a Riemannian manifold endowed with the metric*

$$ds^2 = g_{ij}dx^i dx^j = f(dx^1)^2 + 2dx^1 dx^2 + (dx^3)^2 + (x^1)^2(dx^4)^2$$
$$(i, j = 1, 2, \ldots, 4),$$

where $f = a_0 + a_1x^3 + e^{x^1}(x^3)^2$, and a_0, a_1 are non-constant functions of x^1 only. Then (M^4, g) is a weakly symmetric manifold with vanishing scalar curvature, and is not locally symmetric.

Example 2. Let M be an open subset of $R^n (n \geq 4)$ endowed with the metric

$$\begin{aligned} ds^2 = g_{ij}dx^i dx^j = f(dx^1)^2 + 2dx^1 dx^2 + \sum_{k=3}^{n}(dx^k)^2, \\ (i, j = 1, 2, \ldots, n), \end{aligned} \tag{4.8}$$

where $f = a_0 + a_1x^3 + e^{x^1}\left\{\frac{1}{2}(x^3)^2 + \frac{1}{6}(x^3)^3 + \cdots + \frac{1}{(n-2)(n-3)}(x^3)^{n-2}\right\}$, a_0, a_1, are non-constant functions of x^1 only, and $0 < x^3 < 1$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor and their covariant derivatives are

$$\Gamma^2_{11} = \frac{1}{2}f_{.1}, \quad \Gamma^2_{13} = -\Gamma^3_{11} = \frac{1}{2}f_{.3}, \quad S_{11} = \frac{1}{2}f_{.33},$$

$$R_{1331} = \frac{1}{2}f_{.33} = \frac{1}{2}e^{x^1}\left[\frac{1-(x^3)^{n-3}}{1-x^3}\right] \neq 0,$$

$$R_{1331,1} = \frac{1}{2}f_{.331} = \frac{1}{2}e^{x^1}\left[\frac{1-(x^3)^{n-3}}{1-x^3}\right] \neq 0$$

$$R_{1331,3} = \frac{1}{2}f_{.333} = \frac{1}{2}e^{x^1}\left[\frac{1-(n-3)(x^3)^{n-4}+(n-4)(x^3)^{n-3}}{(1-x^3)^2}\right]$$

and the components which can be obtained from these by the symmetric properties, where '.' denotes the partial differentiation, and S_{ij} denotes the components of the Ricci tensor. Using the above relations, it can be easily shown that the scalar curvature of the manifold is zero. Therefore our M^n with the considered metric is a Riemannian manifold, which is neither locally symmetric nor recurrent.

We shall now show that this M^n is a $(WS)_n$ i.e., it satisfies (1.2). If we consider the 1-forms

$$
\begin{aligned}
A_i(x) &= \frac{n-3}{n} && \text{for } i=1 \qquad & B_i(x) &= \frac{2}{n} && \text{for } i=1\\
&= 0, && \text{otherwise} & &= \frac{1}{1-x^2} && \text{for } i=3\\
D_i(x) &= \frac{1}{n} && \text{for } i=1 & &= 0, && \text{otherwise}\\
&= -\frac{(n-3)(x^3)^{n-4}}{1-(x^3)^{n-3}} && \text{for } i=3 & & && \\
&= 0, && \text{otherwise} & & &&
\end{aligned} \tag{4.9}
$$

at any point $x \in M$, then proceeding similarly as in Example 1, it can be shown that the manifold under consideration is a $(WS)_n$.

Thus we can state the following:

Theorem 12. *Let (M^n, g) $(n \geq 4)$ be a Riemannian manifold endowed with the metric given in (4.8). Then (M^n, g) is a weakly symmetric manifold with vanishing scalar curvature, which is neither locally symmetric nor recurrent.*

Example 3. Let M be an open subset of R^4 endowed with the metric

$$
ds^2 = g_{ij}dx^i dx^j = e^{x^1}(dx^1)^2 + e^{x^1}(dx^2)^2 + e^{x^1}\sin^2 x^2 (dx^3)^2 + e^{x^4}(dx^4)^2 \\
(i,j = 1,2,\ldots,4), \tag{4.10}
$$

where $0 < x^2 < \pi/2$.

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor and their covariant derivatives are

$$
\Gamma^1_{22} = -\frac{1}{2}, \qquad \Gamma^1_{33} = -\frac{1}{2}(\sin x^2)^2, \quad \Gamma^2_{33} = -\sin x^2 \cos x^2,
$$

$$
\Gamma^3_{23} = \cot x^2, \quad \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{44} = \frac{1}{2},
$$

$$
R_{2332} = -\frac{3}{4}e^{x^1}(\sin x^2)^2, \tag{4.11}
$$

$$
R_{2332,1} = \frac{3}{4}e^{x^1}(\sin x^2)^2 \neq 0, \tag{4.12}
$$

and the components which can be obtained from these by the symmetric properties. Here ',' denotes the covariant differentiation. Using the above relations, it can be easily shown that the scalar curvature r of the manifold is given by

$$r = -\frac{3}{2}e^{-x^1} \neq 0.$$

Thus the scalar curvature of the manifold is negative, non-vanishing and non-constant. Therefore our M^4 with the considered metric is a Riemannian manifold, which is neither locally symmetric nor of vanishing scalar curvature.

We shall now show that this M^4 is a $(WS)_4$, i.e. it satisfies (1.2). We consider the 1-forms

$$\begin{aligned} A_i(x) &= -x^2 d(\log x^2) && \text{for } i = 1 \\ &= d(\cos x^2 + \sin x^2) && \text{for } i = 2 \\ &= 0, && \text{otherwise} \\ B_i(x) &= d(\cos x^2 - \sin x^2) && \text{for } i = 2 \\ &= 0, && \text{otherwise} \\ D_i(x) &= -d(2\cos x^2) && \text{for } i = 2 \\ &= 0, && \text{otherwise} \end{aligned} \tag{4.13}$$

at any point $x \in M$. In our M^4 (1.2) reduces with these 1-forms to the following equations

$$\begin{aligned} &\text{(i)} && R_{2332,1} = A_1 R_{2332} + B_2 R_{1332} + B_3 R_{2132} + D_3 R_{2312} + D_2 R_{2331} \\ &\text{(ii)} && R_{2232,3} = A_3 R_{2232} + B_2 R_{3232} + B_2 R_{2332} + D_3 R_{2232} + D_2 R_{2233} \\ &\text{(iii)} && R_{2322,3} = A_3 R_{2322} + B_2 R_{3322} + B_3 R_{2322} + D_2 R_{2332} + D_2 R_{2323}, \end{aligned}$$

since for the cases other than (i), (ii) and (iii) the components of each term of (1.2) vanish identically and the relation (1.2) holds trivially. Now from (4.11)–(4.13) we get the following relations for the right hand side (R.H.S.) and left hand side (L.H.S.) of (i):

R.H.S. of (i)= $A_1 R_{2332} = -x^2 d(\log x^2) R_{2332} = +\frac{3}{4}e^{x^1}\sin^2 x^2 =$ L.H.S. of (i). Also R.H.S. of

$$\begin{aligned} \text{(ii)} &= d(\cos x^2 - \sin x^2)(R_{3232} + R_{2332}) \\ &= 0, \text{ by the skew symmetric property of } R \\ &= \text{L.H.S. of (ii)}. \end{aligned}$$

Similarly it can be shown that the relation (iii) is true. Hence we can state the following:

Theorem 13. *Let (M^4, g) be a Riemannian manifold endowed with the metric*

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1}(dx^1)^2 + e^{x^1}(dx^2)^2 + e^{x^1}(\sin x^2)^2(dx^3)^2 + e^{x^4}(dx^4)^2$$
$$(i, j = 1, 2, \ldots, 4),$$

where $0 < x^2 < \pi/2$ and x^1 is finite. Then (M^4, g) is a weakly symmetric manifold with non-vanishing and non-constant scalar curvature, which is neither locally symmetric nor recurrent.

Example 4. Let M be an open subset of $R^n (n \geq 4)$ equipped with the metric

$$\begin{aligned} ds^2 = g_{ij}dx^i dx^j &= (e^{x^1} - 1)[(dx^1)^2 + (dx^2)^2] + (e^{x^1+x^2} - 1)(dx^3)^2 \\ &+ (e^{x^4} - 1)(dx^4)^2 + \delta_{ij}dx^i dx^j, \quad (i, j = 1, 2, \ldots, n), \end{aligned}$$

where x^1 is finite and δ_{ij} denotes the Kronecker delta. Then the only non-vanishing components of the Christoffel symbols, curvature tensor and its covariant derivatives are given by

$$\begin{aligned} \Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^3_{13} = \Gamma^3_{23} = \Gamma^4_{44} = \Gamma^2_{12} = \frac{1}{2}, \quad \Gamma^1_{33} = \Gamma^2_{33} = -\frac{1}{2}e^{x^2}, \\ R_{2332} = \frac{1}{2}e^{x^1+x^2}, \\ R_{2332,1} = -\frac{1}{2}e^{x^1+x^2} \neq 0 \end{aligned}$$

and the components which can be obtained from these by the symmetric properties, where ',' denotes the covariant differentiation. Using the above relations, it can be easily shown that the scalar curvature r of the manifold is given by

$$r = e^{-x^1} \neq 0 \quad \text{for } x^1 \text{ is finite.}$$

Hence the manifold under the considered metric is of non-zero scalar curvature, and it is a Riemannian manifold. If we consider the 1-forms

$$\begin{aligned} A_i(x) &= -x^1 d(\log x^1) && \text{for } i = 1, & B_i(x) &= d(e^{x^1} - e^{x^2}) && \text{for } i = 2 \\ &= d(e^{x^1} + e^{x^2}) && \text{for } i = 2, & &= 0, && \text{otherwise} \\ &= 0, && \text{otherwise} \end{aligned}$$

$$\begin{aligned} D_i(x) &= -d(2e^{x^1}) && \text{for } i = 2, \\ &= 0, && \text{otherwise} \end{aligned}$$

at any point $x \in M$, then proceeding similarly as in Example 3, it can be easily shown that the manifold under consideration is a $(WS)_n$.

Thus we can state the following:

Theorem 14. *Let* (M^n, g) $(n \geq 4)$*be a Riemannian manifold equipped with the metric*

$$ds^2 = g_{ij}dx^i dx^j = (e^{x^1} - 1)[(dx^1)^2 + (dx^2)^2] + (e^{x^1+x^2} - 1)(dx^3)^2 + (e^{x^4} - 1)(dx^4)^2 + \delta_{ij}dx^i dx^j \qquad (i, j = 1, 2, \ldots, n).$$

Then (M^n, g) *is a weakly symmetric manifold of non-zero and non-constant scalar curvature, which is neither locally symmetric nor recurrent.*

Example 5. Let M be an open subset of R^4 equipped with the metric

$$ds^2 = g_{ij}dx^i dx^j = x^3 e^{x^1}(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + x^4 e^{x^1}(dx^4)^2 \quad (i, j = 1, 2, \ldots, 4),$$

where $x^3 \neq 0, x^4 \neq 0$.

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor and their covariant derivatives are given by

$$\Gamma^1_{11} = \Gamma^4_{14} = \frac{1}{2}, \quad \Gamma^3_{11} = -\frac{1}{2}e^{x^1}, \quad \Gamma^1_{13} = \frac{1}{2x^3}, \quad \Gamma^1_{44} = -\frac{x^4}{2x^3}, \quad \Gamma^4_{44} = \frac{1}{2x^4},$$
$$R_{1331} = -\frac{1}{4x^3}e^{x^1} \neq 0, \quad R_{1331,3} = \frac{1}{2(x^3)^2}e^{x^1} \neq 0$$

and the components which can be obtained from these by the symmetric properties. Using the above relations, it can be easily shown that the scalar curvature r of the manifold is given by

$$r = -\big(2x^3\big)^{-2} \neq 0.$$

Hence the manifold with the considered metric is a Riemannian manifold of non-constant negative scalar curvature, which is neither locally symmetric nor recurrent.

If we consider the 1-forms

$$\begin{aligned} A_i(x) &= -\frac{2}{3} \quad \text{for } i = 1, & B_i(x) &= \frac{2}{9} \quad \text{for } i = 1, \\ &= \frac{1}{x^3} \quad \text{for } i = 3, & &= \frac{2}{x^3} \quad \text{for } i = 3, \\ &= 0, \quad \text{otherwise} & &= 0, \quad \text{otherwise} \end{aligned}$$

$$\begin{aligned} D_i(x) &= \frac{4}{9} \quad \text{for } i = 1, \\ &= -\frac{5}{x^3} \quad \text{for } i = 3 \\ &= 0, \quad \text{otherwise} \end{aligned}$$

at any point $x \in M$, then proceeding similarly as in the previous examples, it can easily be shown that the manifold under consideration is weakly symmetric.

Hence we can state the following:

Theorem 15. *Let* (M^4, g) *be a Riemannian manifold equipped with the metric*

$$ds^2 = g_{ij}dx^i dx^j = x^3 e^{x^1}(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + x^4 e^{x^1}(dx^4)^2$$

$$(i, j = 1, 2, \dots, 4),$$

where $x^3 x^4 \neq 0$.

Then (M^4, g) is a weakly symmetric manifold of non-constant and negative scalar curvature, which is neither locally symmetric nor recurrent.

References

[1] T. Q. Binh, On weakly symmetric Riemannian spaces, *Publ. Math. Debrecen* **42** (1993), 103–107.

[2] M. C. Chaki, On generalized pseudo-symmetric manifolds, *Publ. Math. Debrecen* **45** (1994), 305–312.

[3] M. C. Chaki and R. K. Maity, On quasi-Einstein manifold, *Publ. Math. Debrecen* **51**/1–2 (1997), 35–42.

[4] B. Y. Chen and K. Yano, Hypersurfaces of a conformally flat space, *Tensor N. S.* **26** (1972), 318–322.

[5] U. C. De and S. Bandyopadhyay, On weakly symmetric Riemannian spaces, *Publ. Math. Debrecen* **54** (1999), 377–381.

[6] M. Prvanović, On totally umbilical submanifolds immersed in a weakly symmetric Riemannian manifold, *Yzves. Vuz. Mathematika (Kazan)* **6** (1998), 54–64.

[7] L. Tamássy and T. Q. Binh, On weakly symmetric and weakly projective symmetric Riemannian manifolds, *Coll. Math. Soc. J. Bolyai* **50** (1989), 663–670.

[8] L. Tamássy and T. Q. Binh, On weakly symmetrics of Einstein and Sasakian manifolds, *Tensor, N. S.* **53** (1993), 140–148.

ABSOS ALI SHAIKH
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BURDWAN
BURDWAN-713104, WEST BENGAL
INDIA

E-mail: aask2003@yahoo.co.in, aask@epatra.com

SANJIB KUMAR JANA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BURDWAN
BURDWAN-713104, WEST BENGAL
INDIA

E-mail: sanjib_skj@yahoo.com

(Received September 26, 2005; revised May 25, 2006)