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On the tangency of sets of the class $\tilde{M}_{p,k}$

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Abstract. In the present paper the problem of the tangency of sets of the class $\tilde{M}_{p,k}$ in generalized metric spaces is considered.

Introduction. Let E be an arbitrary non-empty set and E_0 the family of all non-empty subsets of this set.

Let ϱ be any metric of the set E. We shall denote

$$(1)\qquad \varrho(x,A)=\inf_{y\in A}\varrho(x,y)\quad \text{for}\quad x\in E,\ A\in E_0.$$

Let k be an arbitrary but fixed positive real number. By A' we shall denote the set of all cluster points of the set $A\in E_0$.

We assume by definition (see [5])

$$(2)\qquad \tilde{M}_{p,k}=\Big\{A\in E_0 : p\in A' \text{ and there exists } \mu>0 \text{ such that for an arbitrary } \varepsilon>0 \text{ there exists } \delta>0 \text{ such that for every pair of points } (x,y)\in[A,p;\mu,k] \text{ if } \varrho(x,p)<\delta \text{ and } \frac{\varrho(x,A)}{\varrho^k(x,p)}<\delta \text{ then } \frac{\varrho(x,y)}{\varrho^k(x,p)}<\varepsilon\Big\}$$

where

$$(3)\qquad [A,p;\mu,k]= \{(x,y): x\in E,\ y\in A \text{ and } \mu\varrho(x,A)<\varrho^k(x,p)=\varrho^k(y,p)\}.$$

Let ℓ denote any non-negative real function defined on the Cartesian product $E_0\times E_0$. The pair (E,ℓ) we shall call a generalized metric space (see [8]).

We say that the set $A\in E_0$ has the Darboux property at the point p of the space (E,ℓ) if there exists a number $\tau>0$ such that $A\cap S_\ell(p,r)\neq\emptyset$ for

$r \in (0, \tau)$. $S_\ell(p,r)$ denotes here the sphere with the centre at the point p and the radius r in the space (E,ℓ). The set of all sets having the Darboux property at the point p of the space (E,ℓ) will be denoted by $D_p(E,\ell)$.

Let a, b be any non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r\to 0+]{} 0, \quad b(r) \xrightarrow[r\to 0+]{} 0. \tag{4}$$

The pair (A,B) of sets $A, B \in E_0$ we call (a,b)-clustered at the point p of the space (E,ℓ) if 0 is the cluster point of the set of all real numbers $r > 0$ such that the sets $A \cap S_\ell(p,r)_{a(r)}$, $B \cap S_\ell(p,r)_{b(r)}$ are non empty.

The sets $S_\ell(p,r)_{a(r)}$, $S_\ell(p,r)_{b(r)}$ are so-called $a(r)$, $b(r)$-neighbourhoods of the sphere $S_\ell(p,r)$ in the space (E,ℓ) (see [8]). From the above definitions it follows that the pair (A,B) is (a,b)-clustered at the point p of the space (E,ℓ) if the sets $A, B \in D_p(E,\ell)$.

Let $d_\varrho A$ denote the diameter of the set $A \in E_0$ and $\varrho(A,B)$ the distance of the sets $A, B \in E_0$ in the metric space (E,ϱ).

By F_ϱ we shall denote the class of the functions ℓ fulfilling the conditions:

1° $\ell : E_0 \times E_0 \to \langle 0, \infty)$,
2° there exist a numbers m, M such that $0<m\leq M<\infty$ and $m\varrho(A,B) \leq \ell(A,B) \leq M d_\varrho(A \cup B)$ for $A, B \in E_0$,
3° the function ℓ generates on the set E the metric ℓ_0 defined by the formula: $\ell_0(x,y) = \ell(\{x\},\{y\})$ for $x, y \in E$.

From Lemma 1.1 of the paper [5] it follows that for an arbitrary set $A \in \tilde{M}_{p,k} \cap D_p(E,\ell)$

$$\frac{1}{r^k} d_\ell \left(A \cap S_\ell(p,r)_{a(r)}\right) \xrightarrow[r\to 0+]{} 0, \tag{5}$$

when

$$\frac{a(r)}{r^{k+1}} \xrightarrow[r\to 0+]{} \alpha, \quad (\alpha < \infty). \tag{6}$$

According to the definition given by W. Waliszewski in the paper [8], the set $A \in E_0$ is (a,b)-tangent of order k to the set $B \in E_0$ at the point p of the space (E,ℓ), which we shall write in the form $(A,B) \in T_\ell(a,b,k,p)$, iff the pair (A,B) is (a,b)-clustered at the point p of this space and

$$\frac{1}{r^k} \ell\big(A \cap S_\ell(p,r)_{a(r)}, B \cap S_\ell(p,r)_{b(r)}\big) \xrightarrow[r\to 0+]{} 0. \tag{7}$$

We call $T_\ell(a,b,k,p)$ the relation of (a,b)-tangency of sets of order k at the point p of the space (E,ℓ).

In this paper the problem of the tangency of sets of the class $\tilde{M}_{p,k}$ is considered, our considerations being based on W. Waliszewski's definition.

For $k = 1$ the class $\tilde{M}_{p,k}$ contains among other things such classes of sets as H_p, A_p^* (see [1]) and the class I_p of all simple arcs with an origin at the point $p \in E$.

Moreover W. Waliszewski's definition essentialy generalizes known earlier definitions of the tangency of sets in a metric space (E, ϱ).

One of these definitions (the most general) says that the set $A \in E_0$ is tangent to the set $B \in E_0$ at the point $p \in E$ if $p \in A'$ and

$$\frac{\varrho(x,B)}{\varrho(p,x)} \xrightarrow[A \ni x \to p]{} 0. \tag{8}$$

The condition (8) is equivalent to the condition

$$\frac{1}{r} \sup\{\varrho(x,B) : x \in A \text{ and } \varrho(p,x) = r\} \xrightarrow[r \to 0+]{} 0. \tag{9}$$

If we put

$$\varrho_0(A,B) = \sup_{x \in A} \varrho(x,B) \text{ for } A, B \in E_0, \tag{10}$$

then the condition (9) can be written in the form

$$\frac{1}{r} \varrho_0(A \cap S_\varrho(p,r), B) \xrightarrow[r \to 0+]{} 0, \tag{11}$$

where $S_\varrho(p,r)$ denotes the sphere with centre at the point p and radius r in a metric space (E, ϱ).

Setting $a(r) = 0$, $b(r) = r$ for $r > 0$, condition (11) can be written

$$\frac{1}{r} \varrho_0\big(A \cap S_\varrho(p,r)_{a(r)},\ B \cap S_\varrho(p,r)_{b(r)}\big) \xrightarrow[r \to 0+]{} 0, \tag{12}$$

which, according to the definition of the tangency relation $T_\ell(a,b,k,p)$, means that $(A,B) \in T_{\varrho_0}(a,b,1,p)$.

This implies that if we replace the function ϱ_0 in condition (12) by an arbitrary non-negative real function ℓ defined on the Cartesian product $E \times E$, then we get the definition of the tangency of sets in the sense of W. Waliszewski.

Because the function ϱ_0 is a particular case of the function ℓ, W. Waliszewski's definition does in reality generalize the above mentioned definition of the tangency of sets in a metric space (E, ϱ).

In this paper (by hypothesis) the function ℓ generates on the set E the metric ℓ_0, so one can also speak about considering the problem of the tangency of sets of the class $\tilde{M}_{p,k}$ in the ordinary metric space (E, ℓ_0).

In Section 1 of the present paper the problem of the tangency of sets of the class $\tilde{M}_{p,k}$ for the functions ℓ belonging to the class F_ϱ is considered.

In Section 2 this problem is examined for the functions ℓ belonging to a certain class contained in the class F_ϱ.

1. Let a, b be arbitrary non-negative real functions defined in the right-hand side neighbourhood of 0 such that

$$\frac{a(r)}{r^{k+1}} \xrightarrow[r\to 0+]{} \alpha, \quad \frac{b(r)}{r^{k+1}} \xrightarrow[r\to 0+]{} \beta \tag{1.1}$$

where $\alpha, \beta \in \langle 0, \infty)$.

Theorem 1.1. *If the functions a, b fulfil the condition* (1.1), *and the function $\ell \in F_\varrho$, then $(A, B) \in T_\ell(a, b, k, p)$ for arbitrary sets $A, B \in \tilde{M}_{p,k} \cap D_p(E, \ell)$ such that $A \subset B$ or $A \supset B$.*

Proof. Let us suppose that $A \subset B$ for $A, B \in \tilde{M}_{p,k} \cap D_p(E, \ell)$. Then for an arbitrary function $\ell \in F_\varrho$ we have

$$\begin{aligned} 0 &\le \frac{1}{M}\ell\left(A \cap S_\ell(p,r)_{a(r)},\ B \cap S_\ell(p,r)_{b(r)}\right) \le \\ &\le d_\varrho\left((A \cap S_\ell(p,r)_{a(r)}) \cup (B \cap S_\ell(p,r)_{b(r)})\right) \le \\ &\le d_\varrho\left((B \cap S_\ell(p,r)_{a(r)}) \cup (B \cap S_\ell(p,r)_{b(r)})\right) \le \\ &\le d_\varrho\left(B \cap S_\ell(p,r)_{\max\{a(r),b(r)\}}\right). \end{aligned} \tag{1.2}$$

From the fact that the set $B \in \tilde{M}_{p,k} \cap D_p(E, \ell)$, the function $\ell \in F_\varrho$, from (1.1) and from Lemma 1.1 of the paper [5] it follows

$$\frac{1}{r^k} d_\varrho\left(B \cap S_\ell(p,r)_{\max\{a(r),b(r)\}}\right) \xrightarrow[r\to 0+]{} 0. \tag{1.3}$$

Hence and from (1.2) we get

$$\frac{1}{r^k}\ell\left(A \cap S_\ell(p,r)_{a(r)},\ B \cap S_\ell(p,r)_{b(r)}\right) \xrightarrow[r\to 0+]{} 0. \tag{1.4}$$

In view of $A, B \in D_p(E, \ell)$ the pair (A, B) is (a, b)-clustered at the point p of the space (E, ℓ).
From here and from (1.4) it follows that $(A, B) \in T_\ell(a, b, k, p)$.

If $A \supset B$ for $A, B \in \tilde{M}_{p,k} \cap D_p(E, \ell)$, then the proof of this theorem is analogous.

Corollary 1.1. *If the sequence of sets* $A_1, A_2, \ldots, A_n \in \tilde{M}_{p,k} \cap D_p(E,\ell)$ *is a monotonically increasing or a monotonically decreasing sequence, and the functions* a, b *fulfil condition* (1.1), $\ell \in F_\varrho$, *then* $(A_i, A_j) \in T_\ell(a,b,k,p)$ *for* $i, j = 1, 2, \ldots, n$.

Corollary 1.2. *If* $A, B \in \tilde{M}_{p,k} \cap D_p(E,\ell)$, *the functions* a, b *fulfil condition* (1.1), *and* $\ell \in F_\varrho$, *then* $(A, A) \in T_\ell(a,b,k,p)$, $(A, A\cup B) \in T_\ell(a,b,k,p)$ *and* $(A, A\cap B) \in T_\ell(a,b,k,p)$, *for* $A \cap B \in D_p(E,\ell)$.

From Corollary 1.2 it follows that the tangency relation $T_\ell(a,b,k,p)$ is reflexive in the class of sets $\tilde{M}_{p,k} \cap D_p(E,\ell)$ for a function $\ell \in F_\varrho$ and functions a, b fulfilling the condition (1.1).

Theorem 1.2. *If the functions* a, b *fulfil the condition* (1.1) *and* $\ell \in F_\varrho$, *then for arbitrary sets* $A, B, C \in \tilde{M}_{p,k} \cap D_p(E,\ell)$ *such that* $B \subset C$ *or* $B \supset C$, *if* $(A,B) \in T_\ell(a,b,k,p)$ *then* $(A,C) \in T_\ell(a,b,k,p)$.

Proof. Let us assume that $(A,B) \in T_\ell(a,b,k,p)$. This implies

$$\frac{1}{r^k}\ell(A \cap S_\ell(p,r)_{a(r)},\ B \cap S_\ell(p,r)_{b(r)}) \xrightarrow[r\to 0+]{} 0. \tag{1.5}$$

Hence and from the fact that $\ell \in F_\varrho$ we get

$$\frac{1}{r^k}\varrho(A \cap S_\ell(p,r)_{a(r)},\ B \cap S_\ell(p,r)_{b(r)}) \xrightarrow[r\to 0+]{} 0. \tag{1.6}$$

Let us suppose that $B \subset C$ for $B, C \in \tilde{M}_{p,k} \cap D_p(E,\ell)$. Hence and from (1.6) we have

$$\frac{1}{r^k}\varrho(A \cap S_\ell(p,r)_{a(r)},\ C \cap S_\ell(p,r)_{b(r)}) \xrightarrow[r\to 0+]{} 0. \tag{1.7}$$

From $A, C \in \tilde{M}_{p,k} \cap D_p(E,\ell)$, $\ell \in F_\varrho$ and from Lemma 1.1 of the paper [5] it follows

$$\frac{1}{r^k}d_\varrho(A \cap S_\ell(p,r)_{a(r)}) \xrightarrow[r\to 0+]{} 0,\ \frac{1}{r^k}d_\varrho(C \cap S_\ell(p,r)_{b(r)}) \xrightarrow[r\to 0+]{} 0. \tag{1.8}$$

Moreover

$$\begin{aligned} 0 \le \frac{1}{M}\ell\left(A \cap S_\ell(p,r)_{a(r)},\ C \cap S_\ell(p,r)_{b(r)}\right) \le \\ \le d_\varrho\left((A \cap S_\ell(p,r)_{a(r)}) \cup (C \cap S_\ell(p,r)_{b(r)})\right) \le \\ \le d_\varrho\left((A \cap S_\ell(p,r)_{a(r)}) + d_\varrho(C \cap S_\ell(p,r)_{b(r)})\right) + \\ + \varrho\left(A \cap S_\ell(p,r)_{a(r)}, C \cap S_\ell(p,r)_{b(r)}\right). \end{aligned}$$

Hence from (1.7) and (1.8) we get

$$\frac{1}{r^k}\ell\left(A\cap S_\ell(p,r)_{a(r)},\ C\cap S_\ell(p,r)_{b(r)}\right)\xrightarrow[r\to 0+]{} 0. \tag{1.9}$$

Owing to $A, C\in D_p(E,\ell)$, the pair (A,C) is (a,b)-clustered at the point p of the space (E,ℓ).

Hence and from (1.9) it follows that $(A,C)\in T_\ell(a,b,k,p)$.

Let us now suppose that $B\supset C$ for $B,C\in \tilde{M}_{p,k}\cap D_p(E,\ell)$. From here and from the assumption that $\ell\in F_\varrho$ we get

$$\begin{aligned}0 &\le \frac{1}{M}\ell\left(A\cap S_\ell(p,r)_{a(r)},\ C\cap S_\ell(p,r)_{b(r)}\right)\le\\ &\le d_\varrho\left((A\cap S_\ell(p,r)_{a(r)})\cup(C\cap S_\ell(p,r)_{b(r)})\right)\le\\ &\le d_\varrho\left((A\cap S_\ell(p,r)_{a(r)})\cup(B\cap S_\ell(p,r)_{b(r)})\right)\le\\ &\le d_\varrho\left((A\cap S_\ell(p,r)_{a(r)})+d_\varrho(B\cap S_\ell(p,r)_{b(r)})\right)+\\ &\quad+\varrho\left(A\cap S_\ell(p,r)_{a(r)},B\cap S_\ell(p,r)_{b(r)}\right).\end{aligned} \tag{1.10}$$

Now by Lemma 1.1 of [5]

$$\frac{1}{r^k}d_\varrho(A\cap S_\ell(p,r)_{a(r)})\xrightarrow[r\to 0+]{} 0,\quad \frac{1}{r^k}d_\varrho(B\cap S_\ell(p,r)_{b(r)})\xrightarrow[r\to 0+]{} 0,$$

and from here, from (1.6) and from (1.10) we obtain

$$\frac{1}{r^k}\ell(A\cap S_\ell(p,r)_{a(r)},\ C\cap S_\ell(p,r)_{b(r)})\xrightarrow[r\to 0+]{} 0.$$

Hence and from the fact that the pair of sets (A,C) is (a,b)-clustered at the point p of the space (E,ℓ) it follows that $(A,C)\in T_\ell(a,b,k,p)$. This ends the proof.

This theorem implies the

Corollary 1.3. *If the functions a,b fulfil the condition* (1.1) *and* $\ell\in F_\varrho$, *then for arbitrary sets $A,B_1,B_2,\dots,B_n\in\tilde{M}_{p,k}\cap D_p(E,\ell)$ such that the sequence $B_1,B_2,\dots,B_n$ is monotonically increasing or monotonically decreasing, if $(A,B_i)\in T_\ell(a,b,k,p)$ then $(A,B_j)\in T_\ell(a,b,k,p)$ for $i,j=1,2,\dots,n$.*

2. Let F^*_ϱ be the class of functions ℓ fulfilling the conditions:

1° $\ell: E_0\times E_0\to\langle 0,\infty)$,

2° $\varrho(A,B)\le\ell(A,B)\le d_\varrho(A\cup B)$ for $A,B\in E_0$.

From the definition of the classes of functions F_ϱ and F_ϱ^* it follows that $F_\varrho^* \subset F_\varrho$. Moreover, any function $\ell \in F_\varrho^*$ generates on the set E the metric ϱ because

$$\varrho(x,y) = \varrho(\{x\},\{y\}) \leq \ell(\{x\},\{y\}) \leq d_\varrho(\{x\} \cup \{y\}) = \varrho(x,y). \tag{2.1}$$

Let a_i, b_i $(i = 1,2)$ be arbitrary non-negative real functions defined in a right-hand side neighbourhood of 0 such that

$$\frac{a_i(r)}{r^{k+1}} \xrightarrow[r\to 0+]{} \alpha_i, \quad \frac{b_i(r)}{r^{k+1}} \xrightarrow[r\to 0+]{} \beta_i, \tag{2.2}$$

where $\alpha_i, \beta_i \in \langle 0, \infty)$.

Hence and from Corollary 1.1 of the paper [5] there follows the equivalence

$$(A,B) \in T_{\ell_1}(a_1,b_1,k,p) \equiv (A,B) \in T_{\ell_2}(a_2,b_2,k,p), \tag{2.3}$$

for any sets $A, B \in \tilde{M}_{p,k} \cap D_p(E,\varrho)$ and functions $\ell_1, \ell_2 \in F_\varrho^*$.

Let C be an arbitrary set of the class $\tilde{M}_{p,k} \cap D(E,\varrho)$ such that $C \subset B$ or $C \supset B$.

From here and from Theorem 1.2 the present paper there follows the implication

$$(A,B) \in T_{\ell_2}(a_2,b_2,k,p) \implies (A,C) \in T_{\ell_2}(a_2,b_2,k,p) \tag{2.4}$$

for $\ell_2 \in F_\varrho^*$ and $A, B, C \in \tilde{M}_{p,k} \cap D_p(E,\varrho)$.

Hence and from (2.3) we get

Theorem 2.1. *If the functions a_i, b_i $(i = 1,2)$ fulfil the condition* (2.2) *and ℓ_1, $\ell_2 \in F_\varrho^*$, then for arbitrary sets $A, B, C \in \tilde{M}_{p,k} \cap D_p(E,\varrho)$ such that $B \subset C$ or $B \supset C$, if $(A,B) \in T_{\ell_1}(a_1,b_1,k,p)$ then $(A,C) \in T_{\ell_2}(a_2,b_2,k,p)$.*

From this theorem we get

Corollary 2.1. *If the functions a_k, b_k $(k = 1,2)$ fulfil the condition* (2.2), *and $\ell_1, \ell_2 \in F_\varrho^*$, then for arbitrary sets $A, B_1, B_2, \dots B_n \in \tilde{M}_{p,k} \cap D_p(E,\varrho)$ such that the sequence $B_1, B_2, \dots, B_n$ is monotonically incresing or monotonically decreasing if $(A, B_i) \in T_{\ell_1}(a_1,b_1,k,p)$ then $(A, B_j) \in T_{\ell_2}(a_2,b_2,k,p)$ for $i, j = 1, 2, \dots, n$.*

References

[1] A. Chądzyńska, On some classes of sets related to the symmetry of the tangency relation in a metric space, *Ann. Soc. Math. Polon. Comm. Math.* **16** (1972), 219–228.

[2] S. Gołąb et Z. Moszner, Sur le contact des courbes dans les espaces metriques generaux, *Colloq. Math.* **10** (1963), 305–311.

[3] J. Grochulski, T. Konik and M. Tkacz, On the tangency of sets in metric spaces, *Ann. Polon. Math.* **38** (1980), 121–131.
[4] J. Haanties and Nottrot, Distance geometry. Directions in metric spaces, Torsion, *Indag. Math.* **17** (1955), 405–410.
[5] T. Konik, On the tangency of sets of some class in generalized metric spaces, *Demonstratio Math.* **22,4** (1989), 1093–1107.
[6] T. Konik, On the equivalence of the tangency relation of sets, *Ann. Soc. Math. Polon., Comm. Math.* **31** (1991), 73–78.
[7] Gy. Soós, Richtungsbegriff in metrischen Raumen, Doctoral thesis, *Budapest*, 1970, (*unpublished*).
[8] W. Waliszewski, On the tangency of sets in generalized metric spaces, *Ann. Polon. Math.* **28** (1973), 275–284.

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