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The maximal operator of the Fejér means of the character system of the *p*-series field in the Kaczmarz rearrangement

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Abstract. The main aim of this paper is to prove that the maximal operator $\sigma^{*\chi}$ of the Fejér means of the character system of the p-series field in the Kaczmarz rearrangement is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$ and is not bounded from the Hardy space $H_{1/2}(G_p)$ to the space $L_{1/2}(G_p)$.

1. Introduction

The first result with respect to the a.e. convergence of the Walsh–Fejér means $\sigma_n f$ is due to Fine [1]. Later, Schipp [5] showed that the maximal operator $\sigma^* f$ is of weak type $(1,1)$, from which the a.e. convergence follows by standard argument. Schipp's result implies by interpolation also the boundedness of $\sigma^* : L_\alpha \to L_\alpha$ $(1 < \alpha \leq \infty)$. This fails to hold for $\alpha = 1$ but Fujii [2] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1 (see also Simon [6]). Fujii's theorem was extened by Weisz [10]. Namely, he proved that the maximal operator of the Fejér means of the one-dimensional Walsh–Fourier series is bounded from the martingale Hardy space $H_\alpha(I)$ to the space $L_\alpha(I)$ for $\alpha > 1/2$. Simon [7] gave a counterexample, which shows that this boundedness does not hold for $0 < \alpha < 1/2$. In the endpoint case $\alpha = 1/2$ Weisz [13] proved that σ^* is bounded from the Hardy space $H_{1/2}(I)$ to the space weak-$L_{1/2}(I)$.

If the Walsh system is taken in the Kaczmarz ordening, the analogue of the statement of Schipp [5] is due to Gát [3]. Moreover he proved an (H_1, L_1)-type estimation. Gát's result was extended to the Hardy space by Simon [8], who

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proved that σ^* is of type (H_α, L_α) for $\alpha > 1/2$. Weisz [13] showed that in the endpoint case $\alpha = 1/2$ the maximal operator is of weak type $\left(H_{1/2}, L_{1/2}\right)$.

Gát and Nagy [4] proved the a.e. convergence $\sigma_n^\chi f \to f$ $(n \to \infty)$ for an integrable function $f \in L_1(G_p)$, where $\sigma_n f$ is the Fejér means of the function f with respect to the character system in the Kaczmarz rearrangement. They also proved that the maximal operator $\sigma^{*\chi}$ is of type (α, α) for all $1 < \alpha \leq +\infty$, of weak type $(1,1)$ and $\|\sigma^* f\|_1 \leq c\, \|f\|_{H_1}$.

The main aim of this paper is to generalize the results of Gát and Nagy [4] and we prove that the maximal operator $\sigma^{*\chi}$ of the Fejér means of the character system of the p-series field in the Kaczmarz rearrangement is bounded from the Hardy space $H_{1/2}(G_p)$ to the space weak-$L_{1/2}(G_p)$ and is not bounded from the Hardy space $H_{1/2}(G_p)$ to the space $L_{1/2}(G_p)$.

2. Definitions and notation

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let $2 \leq p \in \mathbb{N}$ and denote by $\mathbb{Z}_p$ the pth cyclic group, that is, $\mathbb{Z}_p$ can be represented by the set $\{0, 1, \ldots, p-1\}$, where the group operation is mod p addition and every subset is open. The Haar measure on $\mathbb{Z}_p$ is given so that

$$\mu_k(\{j\}) := \frac{1}{j} \quad (j \in \mathbb{Z}).$$

The group operation on G_p is coordinate-wise addition, the normalized Haar measure μ is the product measure. The topology on G_p is the product topology, a base for the neighborhoods of G_p can be given thus:

$$I_0(x) := G_p, \quad I_n(x) := \{y \in G_p : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\},$$

$$(x \in G_p,\ n \in \mathbb{N}).$$

Let $0 = (0 : i \in \mathbb{N}) \in G_p$ denote the null element of G_p, $I_n := I_n(0)$ $(n \in \mathbb{N})$. Let

$$\Delta := \{I_n(x) : x \in G_p,\ n \in \mathbb{N}\}.$$

The elements of Δ are intervals of G_p. Set $e_i := (0, \ldots, 0, 1, 0, \ldots) \in G_p$ the ith coordinate of which is 1, the rest are zeros.

The norm (or quasinorm) of the space $L_\alpha(G_p)$ is defined by

$$\|f\|_\alpha := \left(\int_{G_p} |f(x)|^\alpha d\mu(x)\right)^{1/\alpha} \quad (0 < \alpha < +\infty).$$

Let $\Gamma(p)$ denote the character group of G_p. We arrange the elements of $\Gamma(p)$ as follows: For $k \in \mathbb{N}$ and $x \in G_p$ denote by r_k the k-th generalized Rademacher function:

$$r_k(x) := \exp\left(\frac{2\pi i x_k}{p}\right) \quad (i := \sqrt{-1},\ x \in G_p,\ k \in \mathbb{N}).$$

Let $n \in \mathbb{N}$. Then

$$n = \sum_{i=0}^{\infty} n_i p^i, \quad \text{where } 0 \leq n_i < p \qquad (n_i,\ i \in \mathbb{N}),$$

where n is expressed in the number system with base p. Put

$$|n| := \max(j \in \mathbb{N} : n_j \neq 0) \quad \text{i.e., } p^{|n|} \leq n < p^{|n|+1}.$$

Now we define the sequence of functions $\psi := (\psi_n : n \in \mathbb{N})$ by

$$\psi_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} \quad (x \in G_p,\ n \in \mathbb{N}).$$

We remark that $\Gamma(p) = \{\psi_n : n \in \mathbb{N}\}$ is a complete orthogonal system with respect to the normalized Haar measure on G_p.

The character group $\Gamma(p)$ can be given in the Kaczmarz rearrangement as follows: $\Gamma(p) = \{\chi_n : n \in \mathbb{N}\}$, where

$$\chi_n(x) := r_{|n|}^{n_{|n|}}(x) \prod_{k=0}^{|n|-1} \left(r_{|n|-1-k}(x)\right)^{n_k} \quad (x \in G_p,\ n \in \mathbb{P}),$$

$$\chi_0(x) = 1 \quad (x \in G_p).$$

Let the transformation $\tau_A : G_p \to G_p$ be defined as follows:

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \ldots, x_0, x_A, x_{A+1}, \ldots).$$

The transformation is measure-preserving and $\tau_A(\tau_A(x)) = x$. By the definition of τ_A, we have

$$\chi_n(x) = r_{|n|}^{n_{|n|}}(x)\psi_{n-n_{|n|}p^n}(\tau_{|n|}(x)) \quad (n \in \mathbb{N},\ x \in G_p).$$

For a function f in $L_1(G_p)$the Fourier coefficients, the partial sums of Fourier series, the Dirichlet kernels, the Fejér means and the Fejér kernels are defined as follows:

$$\hat{f}^{\gamma}(n) := \int_{G_p} f\gamma_n, \quad S_n^{\gamma}(f, x) := \sum_{k=0}^{n-1} \hat{f}^{\gamma}(k)\gamma_k(x), \quad D_n^{\gamma} := \sum_{k=0}^{n-1} \gamma_k,$$

$$\sigma_n^{\gamma}(f) := \frac{1}{n}\sum_{k=1}^{n} S_k^{\gamma}(f), \quad K_n^{\gamma} := \frac{1}{n}\sum_{k=1}^{n} D_k^{\gamma\alpha}(x),$$

where $\gamma_n = \psi_n$ or χ_n.

Let

$$K_{a,b} := \sum_{j=a}^{a+b-1} D_j^{\gamma} \quad (a, b \in \mathbb{N}),$$

and

$$n^{(s)} := \sum_{i=s}^{\infty} n_i p^i \quad (n, s \in \mathbb{N}).$$

By a simple calculation we get

$$nK_n^{\gamma} = \sum_{s=0}^{|n|} \sum_{l=0}^{n_s-1} K^{\gamma}_{n^{(s+1)}+lp^s, p^s} + D_n^{\gamma}. \tag{1}$$

The p^nth Dirichlet kernels have a closed form:

$$D_{p^n}^{\psi}(x) = D_{p^n}^{\chi}(x) = \begin{cases} p^n & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n, \end{cases} \quad \text{where } x \in G_p. \tag{2}$$

We define the maximal operator

$$\sigma^{*\gamma} f := \sup_{n \in \mathbb{P}} |\sigma_n^{\gamma} f| \quad (f \in L_1(G_p).$$

The space weak-$L_{\alpha}(G_p)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak}-L_{\alpha}(G_p)} := \sup_{\rho>0} \rho \mu (|f| > \rho)^{1/\alpha} < +\infty.$$

The σ-algebra generated by the intervals I_k of length p^{-k} will be denoted by F_k $(k \in N)$.

Denote by $f = \left(f^{(n)},\ n \in \mathbb{N}\right)$ the one-parameter martingale with respect to $(F_n,\ n \in N)$ (for details see, e.g., [9]–[12]) The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in N} \left|f^{(n)}\right|.$$

In case $f \in L_1(G_p)$, the maximal function can also be given by

$$f^*(x) = \sup_{n \geq 1} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u) d\mu(u) \right|, \quad x \in G_p.$$

For $0 < \alpha \leq \infty$ the Hardy martingale space $H_p(G_p)$ consists all martingales for which

$$\|f\|_{H_{\alpha}} := \|f^*\|_{\alpha} < \infty.$$

If $f \in L_1(G_p)$ then it is easy to show that the sequence $(S_{p^n}(f) : n \in \mathbb{N})$ is a martingale. If f is a martingale, that is $f = (f^{(0)}, f^{(1)}, \dots)$, then the Fourier coefficients must be defined in a slightly different way:

$$\widehat{f}(j) = \lim_{k\to\infty} \int_{G_p} f^{(k)}(x)\, \gamma_j(x)\, dx.$$

The Fourier coefficients of $f \in L_1(G_p)$ are the same as those of the martingale $(S_{p^n}(f) : n \in \mathbb{N})$ obtained from f.

A bounded measurable function a is an α-atom, if there exists an interval I, such that

a) $\int_I a d\mu = 0$;

b) $\|a\|_\infty \le \mu(I)^{-1/\alpha}$;

c) $\operatorname{supp} a \subset I$.

3. Formulation of the main results

Theorem 1. *The maximal operator $\sigma^{*\chi}$ is bounded from the Hardy space $H_{1/2}(G_p)$ to the space weak-$L_{1/2}(G_p)$.*

Theorem 2. *The maximal operator $\sigma^{*\chi}$ is not bounded from the Hardy space $H_{1/2}(G_p)$ to the space $L_{1/2}(G_p)$.*

4. Auxiliary propositions

We shall need the following lemmas (see [4], [13]).

Lemma 1 (Weisz)**.** *Suppose that an operator V is sublinear, and for some $0<\alpha<1$*

$$\sup_{\rho>0} \rho^\alpha \, \mu\{x \in C_p \backslash I : |Va(x)| > \rho\} \le c_\alpha < \infty$$

for every α-atom a, where I denotes the support of the atom. If V is bounded from L_{α_1} to L_{α_1} for a fixed $1 < \alpha_1 \le \infty$, then

$$\|Vf\|_{\text{weak-}L_\alpha(G_p)} \le c_\alpha \|f\|_{H_\alpha}.$$

Lemma 2 (Gát, Nagy). *Suppose that $s,b,n \in \mathbb{N}$ and $x \in I_b \backslash I_{b+1}$. If $s \le b \le |n|$, then*

$$\left|K^{\psi}_{n^{(s+1)}+lp^s,p^s}(x)\right| \le cp^{s+b},$$

while if $b < s \le |n|$, then

$$K^{\psi}_{n^{(s+1)}+lp^s,p^s}(x) = \begin{cases} 0 & \text{if } x - x_b e_b \notin I_s, \\ \omega_{n^{(s+1)}}(x)p^{s+b-1} & \text{if } x - x_b e_b \in I_s. \end{cases}$$

Lemma 3 (Gát, Nagy). *Let $A \in \mathbb{N}$ and $n := n_A p^A + n_{A-1}p^{A-1} + \dots + n_0 p^0$. Then*

$$\begin{aligned} nK^{\chi}_n(x) = 1 &+ \sum_{j=0}^{A-1}\sum_{i=1}^{p-1} r^i_j(x)p^j K^{\psi}_{p^j}(\tau_j(x)) + \sum_{j=0}^{A-1} p^j D^{\psi}_{p^j}(x)\sum_{l=1}^{p-1}\sum_{i=0}^{l-1} r^l_j(x) \\ &+ p^A \sum_{l=1}^{n_A-1} r^l_A(x)K^{\psi}_{p^A}(\tau_A(x)) + r^{n_A}_A(x)(n-n_A p^A)K^{\psi}_{n-n_A p^A}(\tau_A(x)) \\ &+ \left(n - n_A p^A\right)\sum_{i=0}^{n_A-1} r^i_A(x)D^{\psi}_{p^A}(x) + p^A \sum_{j=1}^{n_A-1}\sum_{i=0}^{j-1} r^i_A(x)D^{\psi}_{p^A}(x). \end{aligned}$$

Corollary 1. *We have*

$$\sup_n \int_{G_p} |K^{\chi}_n(x)|\, d\mu(x) < +\infty.$$

Lemma 4. *Let $n < p^{A+1}$, $A > N$ and $x \in I_N\ (x_0,\dots,x_m \ne 0,0,\dots,0,x_l \ne 0,0,\dots,0)$, $m = -1,0,\dots,l-1$, $l = 0,\dots,N-1$. Then*

$$\int_{I_N} n\,|K^w_n\,(\tau_A(x-t))|\, d\mu(t) \le c\frac{p^A}{p^{m+l}},$$

where

$$\begin{aligned} &I_N\,(x_0,\dots,x_m \ne 0,0,\dots,0,x_l \ne 0,0,\dots,0) \\ &:= I_N\,(0,\dots,0,x_l \ne 0,0,\dots,0)\,, \quad \text{for } m = -1. \end{aligned}$$

Proof. It is evident that for $x \in I_N\ (x_0,\dots,x_m \ne 0,0,\dots,0,x_l \ne 0,0,\dots,0)$ we have

$$\int_{I_N} |D^{\chi}_n(x-t)|\, d\mu(t) \le c\sum_{j=0}^{A}\int_{I_N}\left|D^{\psi}_{p^j}(\tau_A(x-t))\right| d\mu(t) \le c\sum_{j=0}^{A-l}\frac{p^j}{p^A} \le \frac{c}{p^l}. \quad (3)$$

From Lemma 2 we obtain that $K^{w}_{n^{(s+1)}+lp^s,p^s}(\tau_A(x-t))=0$ for $s\geq A-m$. Hence we can suppose that $s<A-m$.

Using Lemma 2 $K^{w}_{n^{(s+1)}+lp^s,p^s}(\tau_A(x-t))\neq 0$ implies that

1) $t\in I_N(0,\dots,0,x_N,\dots,x_{A-1})$ if $0\leq s<A-m$;

2) $t\in I_A(0,\dots,0,x_N,\dots,x_{q-1},\ t_q\neq x_q,x_{q+1},\dots,x_{A-1})$ if $A-N<s<A-l$;

3) $t\in I_A(0,\dots,0,t_N,\dots,t_{A-s-1},x_{A-s},\dots,x_{q-1},\ t_q\neq x_q,x_{q+1},\dots,x_{A-1})$ if $1\leq s\leq A-N$;

4) $t\in I_A(0,\dots,0,t_N,\dots,t_{q-1},\ t_q\neq x_q,x_{q+1},\dots,x_{A-s},\dots,x_{A-1})$ if $1\leq s<A-N$;

consequently, from (1) and (3) we can write

$$
\begin{aligned}
&\int_{I_N} n\left|K^{w}_{n}(\tau_A(x-t))\right|d\mu(t)\\
&\leq \sum_{s=0}^{A-m}\sum_{l=0}^{n_s-1}\int_{I_N}\left|K^{w}_{n^{(s+1)}+lp^s,p^s}(\tau_A(x-t))\right|d\mu(t)+\int_{I_N}\left|D^{\chi}_{n}(x-t)\right|d\mu(t)\\
&\leq c\Bigg\{\sum_{s=0}^{A-m}\frac{p^{s+A-l}}{p^A}+\sum_{s=A-N}^{A-l}\sum_{q=N}^{A}\frac{p^{s+A-q}}{p^A}\\
&\quad+\sum_{s=0}^{A-N}\sum_{q=A-s}^{A}\frac{p^{s+A-q}p^{A-s-N}}{p^A}+\sum_{s=0}^{A-N}\sum_{q=N}^{A-s}\frac{p^{s+A-q}p^{q-N}}{p^A}\Bigg\}\\
&\leq c\left\{\frac{p^A}{p^{m+l}}+\frac{p^A}{p^{N+l}}+\frac{p^A}{p^{2N}}+\sum_{s=0}^{A-N}\frac{p^s\,(A-s-N+1)}{p^{2N}}\right\}\leq c\frac{p^A}{p^{m+l}}.
\end{aligned}
$$

Lemma 4 is proved. □

Lemma 5. *Let $n\in\mathbb{N}$. Then*

$$\int_{G_p}\max_{1\leq N\leq 2^n}\left(N\left|K^{\psi}_{N}(\tau_n(x))\right|\right)^{1/2}d\mu(x)\geq c\frac{n+1}{\log(n+2)}.$$

Proof. It is evident that

$$\int_{G_p} D^{\psi}_{j}(\tau_n(x))\,D^{\psi}_{i}(\tau_n(x))\,d\mu(x)=\int_{G_p} D^{\psi}_{j}(x)D^{\psi}_{i}(x)d\mu(x)=\min\{i,j\}.$$

Then we can write

$$\int_{G_p}\left(\sum_{j=1}^{N} D_j^{\psi}\left(\tau_n(x)\right)\right)^2 d\mu(x)=\sum_{j=1}^{N}\sum_{i=1}^{N}\int_{G_p} D_j^{\psi}\left(\tau_n(x)\right) D_i^{\psi}\left(\tau_n(x)\right) d\mu(x)$$
$$=\sum_{j=1}^{N}\sum_{i=1}^{N}\min\{i,j\}\geq c_0N^3. \tag{4}$$

It is well-known that

$$\int_{G_p}\left|K_N^{\psi}\left(\tau_n(x)\right)\right| d\mu(x)\leq c_1<\infty,\quad N=1,2,\ldots,p^n,\ n=0,1,\ldots. \tag{5}$$

Denote

$$A_{N_i}:=\left\{x\in G_p:\left|K_{N_i}^{\psi}\left(\tau_n(x)\right)\right|\leq\frac{c_0}{2c_1}N_i\right\}$$

and

$$B_{N_i}:=G_p\backslash A_{N_i},$$

where

$$N_i:=\frac{p^n}{n^{3i}},\quad i=1,2,\ldots,\left[\frac{n}{3\log_2 n}\right],\ n\geq 2.$$

By (5) and from the fact that $|K_{N_i}^{\psi}(\tau_n(x))|=O(N_i)$ we can write

$$c_0N^3\leq\int_{G_p}\left(N_i\left|K_{N_i}^{\psi}(\tau_n(x))\right|\right)^2 d\mu(x)=\int_{A_{N_i}}\left(N_i\left|K_{N_i}^{\psi}(\tau_n(x))\right|\right)^2 d\mu(x)$$
$$+\int_{B_{N_i}}\left(N_i\left|K_{N_i}^{\psi}(\tau_n(x))\right|\right)^2 d\mu(x)\leq\frac{c_0}{2c_1}N_i^3\int_{A_{N_i}}\left|K_{N_i}^{\psi}(\tau_n(x))\right|d\mu(x)$$
$$+\int_{B_{N_i}}\left(N_i\left|K_{N_i}^{\psi}(\tau_n(x))\right|\right)^{3/2}\left(N_i\left|K_{N_i}^{\psi}(\tau_n(x))\right|\right)^{1/2} d\mu(x)$$
$$\leq\frac{c_0}{2}N_i^3+c_2N_i^3\int_{B_{N_i}}\left(N_i\left|K_{N_i}^{\psi}(\tau_n(x))\right|\right)^{1/2} d\mu(x),$$

consequently

$$\int_{B_{N_i}}\left(N_i\left|K_{N_i}^{\psi}(\tau_n(x))\right|\right)^{1/2} d\mu(x)\geq c_3>0. \tag{6}$$

Denote

$$C_{N_i}:=B_{N_i}\backslash\bigcup_{j=1}^{i-1}B_{N_j}.$$

From the definition of the set B_{N_i} we obtain

$$\frac{c_0}{2c_1} N_i \mu\left(B_{N_j}\right) < \int_{B_{N_i}} \left|K_{N_i}^{\psi}(\tau_n(x))\right| d\mu(x) \leq \int_{G_p} \left|K_{N_i}^{\psi}(\tau_n(x))\right| d\mu(x) \leq c_1,$$

hence

$$\mu\left(B_{N_j}\right) \leq \frac{c_4}{N_i}. \tag{7}$$

Combining (6) and (7) we get

$$\begin{aligned}\int_{C_{N_i}} \left(N_i \left|K_{N_i}^{\psi}(\tau_n(x))\right|\right)^{1/2} d\mu(x) &\geq \int_{B_{N_i}} \left(N_i \left|K_{N_i}^{\psi}(\tau_n(x))\right|\right)^{1/2} d\mu(x) \\ - \sum_{j=1}^{i-1} \int_{C_{N_j}} \left(N_i \left|K_{N_i}^{\psi}(\tau_n(x))\right|\right)^{1/2} d\mu(x) &\geq c_3 - c_4 N_i \sum_{j=1}^{i-1} \operatorname{mes}\left(C_{N_j}\right) \\ \geq c_3 - c_5 N_i \sum_{j=1}^{i-1} \frac{1}{N_j} \geq c_3 - \frac{c_6}{n^3} \geq c_7, \quad \text{for } n \geq n_0.\end{aligned}$$

Consequently we can write

$$\begin{aligned}&\int_{G_p} \max_{1\leq N \leq 2^n} \left(N \left|K_N^{\psi}(\tau_n(x))\right|\right)^{1/2} d\mu(x) \\ &\geq \sum_{j=1}^{[n/(3\log n)]} \int_{C_{N_i}} \max_{1\leq N \leq 2^n} \left(N \left|K_N^{\psi}(\tau_n(x))\right|\right)^{1/2} d\mu(x) \\ &\geq \sum_{j=1}^{[n/(3\log n)]} \left(N_i \left|K_{N_i}^{\psi}(\tau_n(x))\right|\right)^{1/2} d\mu(x) \geq c_8 \frac{n}{\log n},\end{aligned}$$

which completes the proof of Lemma 5. □

5. Proofs of the main results

Proof of Theorem 1. Let a be an arbitrary atom with support I and $\mu(I) = p^{-N}$. We may assume that $I = I_N$. It is easy to see that $\sigma_n(a) = 0$ if $n \leq p^N$. Therefore we can suppose that $n > p^N$.

From Lemma 3 and (2) we write

$$\sigma_n^{\chi} a(x) = \int_{G_p} a(t) K_n^{\chi}(x-t) d\mu(t) = \int_{I_N} a(t) K_n^{\chi}(x-t) d\mu(t)$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{j=N+1}^{A-1} p^j \sum_{l=1}^{p-1} \int_{I_N} a(t) r_j^l (x-t) K_{p^j}^{\psi}(\tau_j(x-t)) d\mu(t) \\
&\quad + \frac{p^A}{n}\sum_{l=1}^{n_A-1} \int_{I_N} a(t) r_j^l (x-t) K_{p^A}^{\psi}(\tau_A(x-t)) d\mu(t) \\
&\quad + \frac{1}{n}\int_{I_N} a(t) r_A^{n_A}(x-t) n^{(A-1)} K_{n^{(A-1)}}^{\psi}(\tau_A(x-t) d\mu(tt) \\
&= \sigma_n^{1,\chi} a(x) + \sigma_n^{2,\chi} a(x) + \sigma_n^{3,\chi} a(x). \qquad (8)
\end{aligned}
$$

Since $|a| \leq cp^{N/\alpha}$, we have

$$
\left|\sigma_n^{1,\chi} a(x) + \sigma_n^{2,\chi} a(x)\right| \leq \frac{cp^{N/\alpha}}{n}\sum_{j=N+1}^{A} p^j \int_{I_N} \left|K_{p^j}^{\psi}(\tau_j(x-t))\right| d\mu(t). \qquad (9)
$$

Let

$$
x \in I_N \left(x_0, \ldots, x_m \neq 0, 0, \ldots, 0, x_l \neq 0, 0, \ldots, 0\right)
$$

$$
\text{for some } m = -1, 0, \ldots, l-1,\ l = 0, \ldots, N-1.
$$

Then using Lemma 2 $K_{p^j}^{\psi}\left(\tau_j\left(x-t\right)\right) \neq 0$ implies that

$$
t \in I_j\left(0, \ldots, 0, x_N, \ldots, x_{j-1}\right), \quad m = l,\ x_0 = \cdots = x_{m-1} = 0.
$$

Consequently we can write

$$
\begin{aligned}
\left|\sigma_n^{1,\chi} a(x) + \sigma_n^{2,\chi} a(x)\right| &\leq \frac{cp^{N/\alpha}}{p^A}\sum_{j=N+1}^{A} p^j \frac{p^{j-l}}{p^j} \mathbf{1}_{I_N(0,\ldots,0,x_l\neq 0,0,\ldots,0)}(x) \\
&\leq \frac{cp^{N/\alpha}}{p^l} \mathbf{1}_{I_N(0,\ldots,0,x_l\neq 0,0,\ldots,0)}(x). \qquad (10)
\end{aligned}
$$

From Lemma 4 we have

$$
\begin{aligned}
\left|\sigma_n^{3,\chi} a(x)\right| &\leq \frac{cp^{N/\alpha}}{p^A}\int_{I_N}\left(n - n_A p^A\right)\left|K_{n-n_A p^A}^{\psi}\left(\tau_A(x-t)\right)\right| d\mu(t) \\
&\leq \frac{cp^{N/\alpha}}{p^A}\frac{p^A}{p^{m+l}} \leq \frac{cp^{N/\alpha}}{p^{m+l}}. \qquad (11)
\end{aligned}
$$

Combining (8)–(11) we get

$$
\sigma^{*\chi} a(x) \leq \frac{cp^{N/\alpha}}{p^l}\mathbf{1}_{I_N(0,\ldots,0,x_l\neq 0,0,\ldots,0)}\left(x\right) + \frac{cp^{N/\alpha}}{p^{m+l}} \qquad (12)
$$

for

$$x \in I_N\left(x_0, \ldots, x_m \neq 0, 0, \ldots, 0, x_l \neq 0, 0, \ldots, 0\right),$$

$$m = -1, 0, \ldots, l-1,\ l = 0, \ldots, N-1.$$

Now we apply Lemma 1. We may suppose that $a \in L_\infty(G_p)$ is a 1/2-atom with respect to $I_N (n \in \mathbb{N})$. Denote

$$I_N^{m,l} := I_N\left(x_0, \ldots, x_m \neq 0, 0, \ldots, 0, x_l \neq 0, 0, \ldots, 0\right),$$

$$m = -1, 0, \ldots, l-1,\ l = 0, \ldots, N-1.$$

Then it is evident that

$$G_p \backslash I_N = \bigcup_{l=0}^{N-1} \bigcup_{m=-1}^{l-1} \bigcup_{x_0=0}^{p-1} \cdots \bigcup_{x_{m-1}=0}^{p-1} \bigcup_{x_m=1}^{p-1} \bigcup_{x_l=1}^{p-1} I_N^{m,l}.$$

Suppose that $\rho = cp^\lambda$ for some $\lambda \in \mathbb{N}$. Then from (10) we have

$$p^{\lambda/2} \mu \left\{ x \in G_p \backslash I_N : \sup_n \left| \sigma_n^{1,\chi} a(x) + \sigma_n^{2,\chi} a(x) \right| > cp^\lambda \right\} = 0$$

for $\lambda > 2N - l$. Hence we can suppose that $\lambda \leq 2N - l$ and $x \in I_N\ (0, \ldots, x_l \neq 0, 0, \ldots, 0)$ for some $l = 0, \ldots, N-1$. Now we get

$$p^{\lambda/2} \mu \left\{ x \in G_p \backslash I_N : \sup_n \left| \sigma_n^{1,\chi} a(x) + \sigma_n^{2,\chi} a(x) \right| > p^\lambda \right\}$$

$$\leq cp^{\lambda/2} \sum_{l=0}^{2N-\lambda} \sum_{x_l=1}^{p-1} \frac{1}{p^N} \leq c \frac{N - \lambda/2}{p^{N-\lambda/2}} \leq c < \infty.$$

Using the estimation (11) we have

$$p^{\lambda/2} \mu \left\{ x \in G_p \backslash I_N : \sup_n \left| \sigma_n^{3,\chi} a(x) \right| > cp^\lambda \right\} = 0$$

for $\lambda > 2N - m - l$. Therefore we can suppose that $\lambda \leq 2N - m - l$. Then we obtain

$$p^{\lambda/2} \mu \left\{ x \in G_p \backslash I_N : \sup_n \left| \sigma_n^{3,\chi} a(x) \right| > cp^\lambda \right\}$$

$$\leq cp^{\lambda/2} \sum_{l=0}^{N-1} \sum_{m=-1}^{l-1} \sum_{x_0=0}^{p-1} \cdots \sum_{x_{m-1}=0}^{p-1} \sum_{x_m=1}^{p-1} \sum_{x_l=1}^{p-1} \mu \left\{ x \in I_N^{m,l} : \sup_n \left| \sigma_n^{3,\chi} a(x) \right| > cp^\lambda \right\}$$

$$\leq cp^{\lambda/2} \left\{ \sum_{l=0}^{N-\lambda/2} \sum_{m=0}^{l} \frac{p^m}{p^N} + \sum_{l=N-\lambda/2}^{2N-\lambda} \sum_{m=0}^{2N-\lambda-l} \frac{p^m}{p^N} \right\} \leq c < \infty.$$

Theorem 1 is proved. □

Proof of Theorem 2. Let $n \in \mathbb{P}$ and

$$f_n(x) := D^{\chi}_{p^{n+1}}(x) - D^{\chi}_{p^n}(x) = D^{\psi}_{p^{n+1}}(x) - D^{\psi}_{p^n}(x).$$

It is evident that

$$\widehat{f}^{\chi}_n(v) = \begin{cases} 1 & \text{if } v = p^n, \dots, p^{n+1}-1, \\ 0 & \text{otherwise.} \end{cases}$$

Now we can write that

$$S^{\chi}_k(f_n; x) = \begin{cases} 0, & \text{if } k = 0, \dots, p^n, \\ D^{\chi}_k(x) - D^{\chi}_{p^n}(x), & \text{if } k = p^n+1, \dots, p^{n+1}-1, \\ f_n(x), & \text{if } k \geq p^{n+1}. \end{cases} \tag{13}$$

We have

$$f^{*\chi}_n(x) = \sup_k \left|S^{\chi}_{p^k}(f_n; x)\right| = \left|f_n(x)\right|,$$

$$\left\|f_n\right\|_{H_\alpha} = \left\|f^*_n\right\|_\alpha = \left\|D^{\chi}_{p^n}(x)\right\|_\alpha = p^{n(1-1/\alpha)}. \tag{14}$$

Since

$$D^{\chi}_{k+p^n}(x) - D^{\chi}_{p^n}(x) = w_{p^n}(x)\, D^{\psi}_k(\tau_n(x)), \quad k = 1, 2, \dots, p^n,$$

from (13) we obtain

$$\begin{aligned} \sigma^{*\chi} f_n(x) &\geq \max_{1\leq N\leq p^n} \left|\sigma^{\chi}_{p^n+N}(f_n; x)\right| \\ &= \max_{1\leq N\leq p^n} \frac{1}{p^n+N} \left|\sum_{k=p^n+1}^{p^n+N} S^{\chi}_k(f_n; x)\right| \\ &\geq \frac{1}{2p^n} \max_{1\leq N\leq p^n} \left|\sum_{k=p^n+1}^{p^n+N} (D^{\chi}_k(x) - D^{\chi}_{p^n}(x))\right| \\ &= \frac{1}{2p^n} \max_{1\leq N\leq p^n} \left|\sum_{k=1}^{N} \left(D^{\chi}_{k+p^n}(x) - D^{\chi}_{p^n}(x)\right)\right| \\ &= \frac{1}{2p^n} \max_{1\leq N\leq p^n} \left|\sum_{k=1}^{N} D^{\psi}_k(\tau_n(x))\right|. \end{aligned}$$

From Lemma 5 we get

$$\begin{aligned}\frac{\|\sigma^{*\chi} f_n\|_{1/2}}{\|f_n\|_{1/2}} &\geq \frac{1}{2p^n p^{-n}} \left(\int_{I^d} \max_{1 \leq N \leq 2^n} \left(N \left| K_N(x) \right| \right)^{1/2} d\mu(x) \right)^2 \\ &\geq c \left(\frac{n+1}{\log(n+2)} \right)^2 \to \infty \text{ as } n \to \infty. \end{aligned} \tag{15}$$

Combining (14) and (15) we complete the proof of Theorem 2. □

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