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Integrals of weighted maximal kernels with respect to Vilenkin systems

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Abstract. The integrals of maximal Dirichlet- and Fejér kernels are infinite, so we have to use some weight function to "pull them back" to the finite. In this paper we give necessary and sufficient conditions for weight functions to get finite integrals on arbitrary Vilenkin groups. Especially some equivalence to the finiteness of integral norms of weighted maximal kernels follows in the so-called bounded case. We investigate also the role of the bounded structure of Vilenkin groups in this connection. Similar results are known with respect to the Walsh–Kaczmarz–Dirichlet kernels proved by Gy. Gát [1].

1. Introduction

In this section we introduce the most important definitions and notations and formulate some known results with respect to the Vilenkin systems. For details we refer to the book Schipp–Wade–Simon and Pál [3] and to Vilenkin [5].

If $m = (m_0, m_1, \dots, m_k, \dots)$ is a sequence of natural numbers such that $m_k \geq 2$ $(k \in \mathbb{N} := \{0, 1, \dots\})$ then for all $k \in \mathbb{N}$ we shall denote by Z_{m_k} the m_k-th discrete cyclic group. Let Z_{m_k} be represented by $\{0, 1, \dots, m_k - 1\}$. The group operation in Z_{m_k}, i.e. the addition modulo m_k will be denoted by $\oplus$. G_m will denote the complete direct product of Z_{m_k}'s, then G_m forms a compact Abelian group with Haar measure 1. The usual symbol L^1 denotes the Lebesgue space of complex-valued functions f defined on G_m with the norm $\|f\|_1 := \int_{G_m} |f|$.

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The elements of G_m are sequences of the form $x = (x_k, k \in \mathbb{N})$, where $x_k \in Z_{m_k}$ for every $k \in \mathbb{N}$. If $y = (y_k, k \in \mathbb{N}) \in G_m$, then $x \dotplus y := (x_k \oplus y_k, k \in \mathbb{N})$ is the sum of x, y in G_m. The topology of the group G_m is completely determined by the sets

$$I_n := \{(x_0, x_1, \dots, x_k, \dots) \in G_m : x_j = 0 \quad (j = 0, \dots, n-1)\}$$

$(0 \neq n \in \mathbb{N}, I_0 := G_m)$.

It is well-known that the characters of G_m (the so-called Vilenkin system) form a complete orthonormal system $\widehat{G}_m$ in L^1. If

$$r_n(x) := \exp \frac{2\pi \imath x_n}{m_n}$$

$(n \in \mathbb{N}, x = (x_0, x_1, \dots) \in G_m, \imath := \sqrt{-1}\,)$, then these functions and their finite products are evidently characters. Let these products be ordered in Paley's sense, which means the following enumeration of the elements of $\widehat{G}_m$. We write each $n \in \mathbb{N}$ uniquely in the form

$$n = \sum_{k=0}^{\infty} n_k M_k,$$

where $M_0 := 1$, $M_k := \prod_{j=0}^{k-1} m_j$ $(k \geq 1)$ and $n_k \in Z_{m_k}$ $(k \in \mathbb{N})$. It can easily be seen that the elements of $\widehat{G}_m$ are nothing but the functions

$$\Psi_n := \prod_{k=0}^{\infty} r_k^{n_k}.$$

If $m_n = 2$ for all $n \in \mathbb{N}$, then $\widehat{G}_m$ is the well-known Walsh–Paley system.

Let D_n and K_n be the n-th Dirichlet and Fejér kernel, respectively, defined by

$$D_n := \sum_{k=0}^{n-1} \Psi_k, \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k \qquad (0 < n \in \mathbb{N}).$$

We need the following well-known results with respect to the kernels from the Vilenkin–Fourier analysis (see e.g. Pál–Simon [2], Simon [4]):

$$D_n = \Psi_n \sum_{k=0}^{\infty} \sum_{j=m_k-n_k}^{m_k-1} r_k^j D_{M_k} \qquad \left(n = \sum_{k=0}^{\infty} n_k M_k \in \mathbb{N}\right); \tag{1}$$

$$D_{M_n} = \chi_{I_n} M_n \qquad (n \in \mathbb{N}), \tag{2}$$

where χ_A denotes the characteristic function of a set A; if $n \in \mathbb{N}$ and for some $s \in \mathbb{N}$ we have $M_{s-1} \le n < M_s$, then

$$|K_n(x)| = \frac{1}{n}\left|\sum_{\nu=0}^{s-1}\sum_{i=\nu}^{s-1}\sum_{j=m_i-n_i}^{m_i-1} r_i(x)^j c_{ij}^{\nu}(x)\right| \qquad (x \in G_m), \tag{3}$$

where

$$c_{ij}^{\nu}(x) := n_\nu D_{M_i}(x) - \sum_{k=0}^{m_\nu-1} k m_\nu^{-1} \sum_{l=0}^{m_\nu-1} r_\nu(le_\nu)^{n_\nu-k} r_i(le_\nu)^j D_{M_i}(x \dot{+} le_\nu),$$

$le_\nu := (0, 0, \dots, 0, l, 0, \dots) \in G_m$ and l is the $(\nu+1)$-th coordinate of the element in question.

From now on C will denote positive absolute constant not always the same at different occurences.

2. Theorems

Let $\alpha : [0, +\infty) \to (0, +\infty)$ be a monotone increasing function and define the weighted maximal functions D_α^*, K_α^* as follows:

$$D_\alpha^* := \sup_n \frac{|D_n|}{\alpha(n)}, \quad K_\alpha^* := \sup_n \frac{|K_n|}{\alpha(n)}.$$

Then the next statements are true.

Theorem 2.1. *There are positive absolute contsants C_1, C_2 such that*

$$C_1 \sum_{k=0}^{\infty} \frac{\log m_k}{\alpha(M_{k+1})} \le \|R_\alpha^*\|_1 \le C_2 \sum_{k=0}^{\infty} \frac{\log m_k}{\alpha(M_k)},$$

where $R_\alpha^* := D_\alpha^*$ or $R_\alpha^* := K_\alpha^*$.

Proof. First we deal with D_α^* and with the right hand side inequality of the theorem. To this end we write $\|D_\alpha^*\|_1$ as $\|D_\alpha^*\|_1 = \sum_{A=0}^{\infty} \int_{I_A \setminus I_{A+1}} D_{\alpha^*}$. If $A \in \mathbb{N}$ and $x \in I_A \setminus I_{A+1}$, then we get by (1) and (2) that

$$|D_n(x)| = \left|\sum_{k=0}^{A-1} n_k M_k + M_A \sum_{j=m_A-n_A}^{m_A-1} r_A(x)^j\right| = |D_{\tilde{n}}(x)|,$$

if $\tilde{n} := \sum_{k=0}^{A} n_k M_k$. Therefore

$$D^*_\alpha(x) = \sup_{n<M_{A+1}} \frac{|D_n(x)|}{\alpha(n)} \le \sum_{k=0}^{A} \sup_{M_k\le n<M_{k+1}} \frac{|D_n(x)|}{\alpha(n)}$$
$$\le \sum_{k=0}^{A} \sup_{M_k\le n<M_{k+1}} \frac{|D_n(x)|}{\alpha(M_k)}.$$

When here $n < M_{k+1} \le M_A$ (i.e. $k = 0, \dots, A-1$), then (see above) $|D_n(x)| \le \sum_{l=0}^{k} n_l M_l < M_{k+1}$. Furthermore, for $M_A \le n < M_{A+1}$ we get

$$|D_n(x)| = \sum_{k=0}^{A-1} n_k M_k + M_A \left|\sum_{j=1}^{n_A} \exp\frac{2\pi\imath j x_A}{m_A}\right| \le M_A + \left|M_A \frac{\exp\frac{2\pi\imath n_A x_A}{m_A} - 1}{\exp\frac{2\pi\imath x_A}{m_A} - 1}\right|$$
$$= M_A\left(1 + \frac{|\sin\frac{\pi n_A x_A}{m_A}|}{\sin\frac{\pi x_A}{m_A}}\right) \le C M_A\left(1 + \frac{m_A}{\tilde{x}_A}\right),$$

where $\tilde{x}_A := x_A$ if $x_A \le m_A/2$, while $x_A := m_A - x_A$ in the case $x_A > m_A/2$. (We recall that $x \in I_A \setminus I_{A+1}$, i.e. $x_A \ne 0$.) Summarizing the above facts we have

$$\|D^*_\alpha\|_1 \le \sum_{A=0}^{\infty} \int_{I_A\setminus I_{A+1}} \sum_{k=0}^{A} \frac{\sup_{M_k\le n<M_{k+1}} |D_n(x)|}{\alpha(M_k)}$$
$$\le \sum_{A=0}^{\infty} \int_{I_A\setminus I_{A+1}} \sum_{k=0}^{A-1} \frac{M_{k+1}}{\alpha(M_k)} + C\sum_{A=0}^{\infty} \frac{M_A}{\alpha(M_A)} \int_{I_A\setminus I_{A+1}} \left(1 + \frac{m_A}{\tilde{x}_A}\right) dx$$
$$\le \sum_{k=0}^{\infty} \frac{M_{k+1}}{\alpha(M_k)} \sum_{A=k+1}^{\infty} \left(\frac{1}{M_A} - \frac{1}{M_{A+1}}\right) + C\sum_{A=0}^{\infty} \frac{M_A}{\alpha(M_A)} \sum_{x_A=1}^{m_A-1} \frac{1}{M_{A+1}} \left(1 + \frac{m_A}{\tilde{x}_A}\right)$$
$$\le \sum_{k=0}^{\infty} \frac{1}{\alpha(M_k)} + C\sum_{A=0}^{\infty} \frac{M_A}{\alpha(M_A)} \sum_{1\le l\le m_A/2} \frac{1}{M_{A+1}} \left(1 + \frac{m_A}{l}\right)$$
$$\le \sum_{k=0}^{\infty} \frac{1}{\alpha(M_k)} + C\sum_{A=0}^{\infty} \frac{M_A}{\alpha(M_A)} \frac{m_A \log m_A}{M_{A+1}} \le C\sum_{k=0}^{\infty} \frac{\log m_k}{\alpha(M_k)}.$$

The right hand side inequality for $\|K^*_\alpha\|_1$ follows trivially from the case $R^*_\alpha = D^*_\alpha$, since

$$K^*_\alpha = \sup_n \frac{|\sum_{k=1}^{n} D_k|}{n\alpha(n)} \le \sup_n \frac{\sum_{k=1}^{n} |D_k|}{n\alpha(n)} \le \sup_n \frac{1}{n} \sum_{k=1}^{n} \frac{|D_k|}{\alpha(k)}$$

$$\leq \sup_n \frac{\sum_{k=1}^n D_\alpha^*}{n} = D_\alpha^*.$$

For the proof of the estimate of $\|K_\alpha^*\|_1$ from below we compute first $|K_{qM_p}(x)|$ ($x \in G_m$), if $p \in \mathbb{N}$ and $q :=$ the entire part of $\frac{m_p}{2}$. It is clear that $(qM_p)_i = 0$ ($\mathbb{N} \ni i \neq p$) and $(qM_p)_p = q$. Applying (3) we get

$$\begin{aligned}
\left|K_{qM_p}(x)\right| &= \frac{1}{qM_p}\Big|\sum_{\nu=0}^{p} M_\nu \sum_{j=m_p-q}^{m_p-1} r_p^j(x)(qM_p)_\nu D_{M_p}(x) \\
&- \sum_{\nu=0}^{p} M_\nu \sum_{j=m_p-q}^{m_p-1} r_p^j(x) \sum_{k=0}^{m_\nu-1} \frac{k}{m_\nu} \sum_{l=0}^{m_\nu-1} (r_\nu(le_\nu))^{(qM_p)_\nu-k} r_p^j(le_\nu) D_{M_p}(x\dot{+}le_\nu)\Big| \\
&= \frac{1}{qM_p}\Big|M_p \sum_{j=m_p-q}^{m_p-1} r_p^j(x) q D_{M_p}(x) \\
&- \sum_{\nu=0}^{p} M_\nu \sum_{j=m_p-q}^{m_p-1} r_p^j(x) \sum_{k=0}^{m_\nu-1} \frac{k}{m_\nu} \sum_{l=0}^{m_\nu-1} (r_\nu(le_\nu))^{(qM_p)_\nu-k} r_p^j(le_\nu) D_{M_p}(x\dot{+}le_\nu)\Big|.
\end{aligned}$$

If $x \in I_p \setminus I_{p+1}$, then by (1) $D_{M_p}(x\dot{+}le_\nu) = 0$ for all $\nu = 0, \ldots, p-1$ and $l = 1, \ldots, m_\nu - 1$. Furthermore, $D_{M_p}(x\dot{+}le_p) = D_{M_p}(x) = M_p$ ($l = 0, \ldots, m_p - 1$). Therefore $\left|K_{qM_p}(x)\right| =$

$$\begin{aligned}
\frac{1}{qM_p}\Big|qM_p^2 \sum_{j=m_p-q}^{m_p-1} r_p^j(x) - M_p^2 \sum_{j=m_p-q}^{m_p-1} r_p^j(x) \sum_{k=0}^{m_p-1} \frac{k}{m_p} \sum_{l=0}^{m_p-1} (r_p(le_p))^{j+q-k} \\
- M_p \sum_{\nu=0}^{p-1} M_\nu \sum_{j=m_p-q}^{m_p-1} r_p^j(x) \sum_{k=0}^{m_\nu-1} \frac{k}{m_\nu}\Big|.
\end{aligned}$$

Here the next equalities hold:

$$\begin{aligned}
M_p \sum_{\nu=0}^{p-1} M_\nu \sum_{j=m_p-q}^{m_p-1} r_p^j(x) \sum_{k=0}^{m_\nu-1} \frac{k}{m_\nu} &= M_p \sum_{\nu=0}^{p-1} M_\nu \sum_{j=m_p-q}^{m_p-1} r_p^j(x) \frac{m_\nu - 1}{2} \\
&= \frac{M_p(M_p-1)}{2} \sum_{j=m_p-q}^{m_p-1} r_p^j(x),
\end{aligned}$$

and

$$M_p^2 \sum_{j=m_p-q}^{m_p-1} r_p^j(x) \sum_{k=0}^{m_p-1} \frac{k}{m_p} \sum_{l=0}^{m_p-1} (r_p(le_p))^{j+q-k}$$

$$= M_p^2 \sum_{s=0}^{q-1} r_p^{s-q}(x) \sum_{k=0}^{m_p-1} \frac{k}{m_p} \sum_{l=0}^{m_p-1} (r_p(le_p))^{s-k} = M_p^2 \sum_{s=0}^{q-1} s r_p^{s-q}(x).$$

(We recall that $\sum_{l=0}^{m_p-1}(r_p(le_p))^{s-k} = 0$, when $s \neq k$.) Thus it follows that

$$\begin{aligned} \left|K_{qM_p}(x)\right| &= \frac{1}{qM_p}\left|\left(qM_p^2 - \frac{M_p(M_p-1)}{2}\right)\sum_{s=0}^{q-1} r_p^s(x) - M_p^2 \sum_{s=0}^{q-1} s r_p^s(x)\right| \\ &= \frac{M_p}{q}\left|\left(q - \frac{1}{2} + \frac{1}{2M_p}\right)\frac{r_p^q(x)-1}{r_p(x)-1} - \frac{q r_p^q(x)}{r_p(x)-1} - \frac{r_p(x)(r_p^q(x)-1)}{(r_p(x)-1)^2}\right| \\ &\geq \frac{M_p}{|r_p(x)-1|}\left(1 - \left(1 - \frac{1}{2q} + \frac{1}{2qM_p}\right)\left|r_p^q(x)-1\right| - \frac{\left|r_p^q(x)-1\right|}{q|r_p(x)-1|}\right). \end{aligned}$$

It is not hard to see that

$$\left(1 - \frac{1}{2q} + \frac{1}{2qM_p}\right)\left|r_p^q(x)-1\right| \leq 1/4 \quad \text{and} \quad \frac{\left|r_p^q(x)-1\right|}{q|r_p(x)-1|} \leq 1/4,$$

if $x_p \leq \frac{m_p}{3\pi}$ is even and m_p is large enough, say $m_p > 6\pi$. Indeed,

$$|r_p^q(x) - 1| = \left|\exp\frac{2\pi\imath q x_p}{m_p} - 1\right| = 2\left|\sin\frac{\pi q x_p}{m_p}\right| = 0$$

for all $x_p = 1, \ldots, m_p - 1$, if m_p and x_p are even. Assume that $m_p = 2l + 1$ for some $0 < l \in \mathbb{N}$ which implies $q = l$ and $|r_p^q(x) - 1| = 2\left|\sin\frac{\pi l x_p}{2l+1}\right|$. Let $x_p = 2k$ $(k = 1, \ldots, l)$, then

$$|r_p^q(x) - 1| = 2\left|\sin\frac{2\pi l k}{2l+1}\right| = 2\sin\frac{\pi k}{2l+1} \leq \frac{2\pi k}{2l+1},$$

i.e.

$$\left(1 - \frac{1}{2q} + \frac{1}{2qM_p}\right)|r_p^q(x) - 1| \leq \frac{3}{2}|r_p^q(x) - 1| \leq \frac{3\pi k}{2l+1} \leq \frac{1}{2}$$

for all $k \leq \frac{2l+1}{6\pi}$. This last inequality is equivalent to $x_p \leq \frac{m_p}{3\pi}$. Here $x_p \geq 2$, therefore we assume that $m_p > 6\pi$.

On the other hand (see above)

$$\frac{|r_p^q(x) - 1|}{q|r_p(x) - 1|} = 0,$$

when $x_p = 1, \dots, m_p - 1$ and m_p, x_p are even. For $m_p = 2l + 1$ ($0 < l \in \mathbb{N}$), $x_p = 2k$ ($1 \le k \le l/2$) we get

$$\frac{|r_p^q(x) - 1|}{q|r_p(x) - 1|} = \frac{\left|\sin \frac{\pi l x_p}{2l+1}\right|}{l \sin \frac{\pi x_p}{2l+1}} = \frac{\left|\sin \frac{\pi k}{2l+1}\right|}{l \sin \frac{\pi 2k}{2l+1}} \le \frac{\pi}{2} \frac{\pi k/(2l+1)}{l\pi 2k/(2l+1)} = \frac{\pi}{4l} \le \frac{1}{4},$$

if $l \ge \pi$, i.e. if $m_p \ge 2\pi + 1$. Hence in this case

$$\left|K_{qM_p}(x)\right| \ge \frac{1}{2} \frac{M_p}{|r_p(x) - 1|}$$

and so

$$\int_{I_p \setminus I_{p+1}} \left|K_{qM_p}\right| \ge \frac{M_p}{2} \int_{I_p \setminus I_{p+1}} \frac{dx}{|r_p(x) - 1|} \ge \frac{1}{2m_p} \sum_{1 \le x_p \le m_p/(3\pi), x_p \text{ is even}} \frac{1}{\sin \frac{\pi x_p}{m_p}}$$
$$\ge C \sum_{1 \le x_p \le m_p/(3\pi), x_p \text{ is even}} \frac{1}{x_p} \ge C \log m_p.$$

If m_p is "small", i.e. $m_p < 6\pi$ and $p > 0$, then

$$\int_{I_p \setminus I_{p+1}} |K_{M_p}| = \left(\frac{1}{M_p} - \frac{1}{M_{p+1}}\right) \frac{M_p - 1}{2} \ge \frac{1}{8} \ge C \log m_p,$$

since by (1) and (2) $K_{M_p}(x) = (M_p - 1)/2$ ($x \in I_p \setminus I_{p+1}$) follows immediately. Thus

$$\|K_\alpha^*\|_1 \ge \sum_{p=1,\, m_p < 6\pi}^{\infty} \int_{I_p \setminus I_{p+1}} \frac{|K_{M_p}|}{\alpha(M_p)} + \sum_{p=0,\, m_p > 6\pi}^{\infty} \int_{I_p \setminus I_{p+1}} \frac{|K_{\Delta_p M_p}|}{\alpha(\Delta_p M_p)}$$
$$\ge C \sum_{p=1}^{\infty} \frac{\log m_p}{\alpha(M_p)} + C \sum_{p=1}^{\infty} \frac{\log m_p}{\alpha(M_{p+1})} \ge C \sum_{p=0}^{\infty} \frac{\log m_p}{\alpha(M_{p+1})}.$$

Therefore the inequality $K_\alpha^* \le D_\alpha^*$ implies the analogous estimation also for $\|D_\alpha^*\|_1$ from below. □

This proves Theorem 2.1.

From Theorem 2.1 some corollaries follow immediately. Namely,

Corollary 2.1. *If* $\sum_{k=0}^{\infty} \frac{\log m_k}{\alpha(M_k)} < +\infty$, *then* $R_\alpha^* \in L^1(G_m)$. *Furthermore, if* $R_\alpha^* \in L^1(G_m)$, *then* $\sum_{k=0}^{\infty} \frac{\log m_k}{\alpha(M_{k+1})} < +\infty$.

In particular, if the sequence m does not grow to rapidly then we can give an equivalent condition for the integrability of D^*_α and K^*_α. Thus the following corollary is easy to derive.

Corollary 2.2. *Assume that there exists a constant $c \geq 1$ such that $m_{k+1} \leq m_k^c$ $(k \in \mathbb{N})$. Then $R^*_\alpha \in L^1$ if and only if $\sum_{k=0}^\infty \frac{\log m_k}{\alpha(M_k)} < +\infty$.*

It is clear that the same equivalence holds if α satisfies the next assumption: there exists a constant $q > 0$ sucht that $\alpha(M_{k+1}) \leq q\alpha(M_k)$ $(k \in \mathbb{N})$. For example if $\delta > 1$ and $\alpha(M_k) = (k+1)^\delta$ $(k \in \mathbb{N})$.

Corollary 2.3. *Assume that the generating sequence m is bounded. Then $R^*_\alpha \in L^1(G_m)$ if and only if $\sum_{k=0}^\infty \frac{1}{\alpha(M_k)} < +\infty$.*

Simple examples show that the boundedness of m in the previous corollary cannot be omitted, although this boundedness is also not necessary. Namely, the next theorem will be proved.

Theorem 2.2. *There exist m and α such that $\sum_{k=0}^\infty \frac{1}{\alpha(M_k)} < +\infty$ and $R^*_\alpha \notin L^1(G_m)$. Furthermore, for some unbounded m the equivalence in Corollary 2.3 holds.*

Proof. We give details only for D^*_α. Let $m_k := 2^{(k+1)^2}$ and $\alpha(M_k) := (k+1)^2$ $(k \in \mathbb{N})$. Then $\sum_{k=0}^\infty 1/\alpha(M_k) < +\infty$ holds trivially. Furthermore,

$$\sum_{k=0}^\infty \frac{\log m_k}{\alpha(M_{k+1})} \geq C \sum_{k=0}^\infty \frac{(k+1)^2}{(k+2)^2} = +\infty,$$

i.e. by Theorem 2.1 we get $\|D^*_\alpha\|_1 = +\infty$.

Now, let $m_{2^l} := 2^{2^l}$ for $l \in \mathbb{N}$ and $m_k := 2$, when $\mathbb{N} \ni k \neq 2^l \quad (l \in \mathbb{N})$. Then by means of simple considerations the next equivalences follow for all α:

$$\sum_{k=0}^\infty \frac{\log m_k}{\alpha(M_k)} < +\infty \iff \sum_{k=0}^\infty \frac{\log m_k}{\alpha(M_{k+1})} < +\infty \iff \sum_{k=0}^\infty \frac{1}{\alpha(M_k)} < +\infty,$$

which completes the proof of Theorem 2.2. □

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