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On a parameterized family of relative Thue equations

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Abstract. Let $k := \mathbb{Q}(\sqrt{-D}\,)$ be an imaginary quadratic number field and $\mathbb{Z}_k$ be the corresponding ring of integers. We consider the family of relative Thue equations

$$F_t(x, y) = x^3 - (t - 1)x^2y - (t + 2)xy^2 - y^3 = \ell$$

with $t, \ell \in \mathbb{Z}_k, t \notin \mathbb{Z}$ and $|\ell| \leq |2t + 1|$. Let $k(\alpha)$ be the cubic extension of k generated by a root α of the polynomial $f_t(x) = F_t(x, 1)$, and let $\mathbb{Z}_{k(\alpha)}$ be its ring of integers. A pair (x, y) with $x, y \in \mathbb{Z}_k$ is a solution of the Thue equation if and only if the element $\gamma = x - \alpha y \in \mathbb{Z}_{k(\alpha)}$ has a norm satisfying $|N_{k(\alpha)/k}(\gamma)| \leq |2t + 1|$. We determine all elements of $\mathbb{Z}_{k(\alpha)}$ having norms less than or equal to $|2t+1|$. Further we solve the above Thue equation for all $t \in \mathbb{Z}_k, t \notin \mathbb{Z}$ with $\Re t = -\frac{1}{2}$ and all $|\ell| \leq |2t + 1|$.

1. Introduction

For squarefree $D \in \mathbb{N}$ let $k := \mathbb{Q}(\sqrt{-D}\,)$ be an imaginary quadratic number field and $\mathbb{Z}_k$ its ring of integers. We consider the family of relative Thue equations

$$F_t(x, y) := x^3 - (t - 1)x^2y - (t + 2)xy^2 - y^3 = \ell \tag{1.1}$$

with $t, \ell \in \mathbb{Z}_k$, $t \notin \mathbb{Z}$ and $|\ell| \leq |2t + 1|$. We are interested in solving this "diophantine" equation, i.e., in determining all pairs (x, y) with $x, y \in \mathbb{Z}_k$ satisfying the equation.

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In Heuberger et al. [1], [2], [3] the instance $|\ell| = 1$ of (1.1) has been treated and a complete list of solutions has been achieved for $\Re t = -\frac{1}{2}$ in [2] resp. [3], and for general t in [1]. We note that in the rational integer case Thomas [11] and Mignotte [8] solved the equations

$$F_t(x, y) = \pm 1$$

completely, whereas the Diophantine inequalities

$$|F_t(x, y)| \leq 2t + 1$$

were treated by Mignotte et al. [9].

Let

$$f_t(x) := F_t(x, 1) = x^3 - (t-1)x^2 - (t+2)x - 1$$

and $\alpha = \alpha^{(1)}$ be a root of f_t. Then the other roots of f_t are given by (cf. [2])

$$\alpha^{(2)} = -1 - \frac{1}{\alpha}, \qquad \alpha^{(3)} = -\frac{1}{\alpha + 1}. \tag{1.2}$$

Let $k(\alpha)$ be the cubic extension of k generated by the polynomial $f_t(x)$ (these extensions are referred to as Shanks' simplest cubic fields in the rational integer instance, compare [10]).

Let (x, y) with $x, y \in \mathbb{Z}_k$ be a solution of (1.1). Denoting by $N_{k(\alpha)/k}(\gamma)$ the relative norm of γ over k we have

$$N_{k(\alpha)/k}(x - \alpha y) = F_t(x, y) = \ell. \tag{1.3}$$

Therefore, solving the Thue equation (1.1) for $|\ell| \leq |2t + 1|$ is equivalent to determining all elements $\gamma = x - \alpha y$ whose norm is bounded by $|2t+1|$ in absolute value.

The aim of this paper is twofold:

In the first place we want to extend the result of [5] where all elements $\gamma \in \mathbb{Z}_k[\alpha]$ with norms satisfying $|N_{k(\alpha)/k}(\gamma)| \leq |2t + 1|$ for $|t| \geq 14$ were determined. In the present paper we give the corresponding result for all values of t (cf. Theorem 2.1 for all t with $\Re t \leq -\frac{1}{2}$ and Corollary 2.3 for all t with $\Re t = -\frac{1}{2}$). It turns out that this extension reveals a variety of exceptional cases.

Secondly we will apply Corollary 2.3 for all t with $\Re t = -\frac{1}{2}$ to gain all solutions of the Thue equation (1.1) in this instance (cf. Theorem 9.2).

The paper is organized as follows.

In Section 2 we will use an improved version of a result from [5] in order to show that $|N_{k(\alpha)/k}(\gamma)| \leq |2t + 1|$ implies that γ is associated to an element of $\mathbb{Z}_k$ or

$$\gamma = \mu\beta\alpha^{b_1}(\alpha + 1)^{b_2} \tag{1.4}$$

where μ is a unit of $\mathbb{Z}_k$, b_1, b_2 are rational integers and β can attain only finitely many values (see Theorem 2.1 and Corollary 2.3).

All solutions of the Thue equation (1.1) for $\Re t = -\frac{1}{2}$ will be determined in Section 3 to Section 9. In Theorem 9.2 all these solutions are provided. In the following we give a short outline of the proof of Theorem 9.2.

Suppose that (x, y) is a solution of (1.1), let $\gamma^{(l)} = x - \alpha^{(l)} y$ for $l \in \{1, 2, 3\}$ and choose $j \in \{1, 2, 3\}$ such that

$$|\gamma^{(j)}| = \min_{l \in \{1,2,3\}} |\gamma^{(l)}| = \min_{l \in \{1,2,3\}} |x - \alpha^{(l)} y|. \tag{1.5}$$

In Section 3 we will prove that

$$\left| \frac{x}{y} - \alpha^{(j)} \right| = \left| \frac{\gamma^{(j)}}{y} \right| \leq \frac{3.95297}{|y|^3} \tag{1.6}$$

for $|t| \geq 6$ and $|y| \geq 4$. If, in addition, $\Re t = -\frac{1}{2}$ we even get

$$\left| \frac{x}{y} - \alpha^{(j)} \right| = \left| \frac{\gamma^{(j)}}{y} \right| \leq \frac{3.63241}{|y|^3}. \tag{1.7}$$

From this we are able to deduce for $|t| \geq 6$ and $|y| \geq 4$ that

$$|y| \geq 0.32706|t| \tag{1.8}$$

holds provided that $x \notin \{0, -y, ty\}$. In the sequel we will consider the quantity

$$\Lambda_j = \log \left| \frac{\gamma^{(j+1)}}{\gamma^{(j+2)}} \right| - \log \left| \frac{\alpha^{(j)} - \alpha^{(j+1)}}{\alpha^{(j)} - \alpha^{(j+2)}} \right|. \tag{1.9}$$

In Section 4 we derive an upper bound for $|\Lambda_j|$. In particular, we prove that for $|t| \geq 6$ and $|y| \geq 4$

$$|\Lambda_j| \leq \frac{8.79208}{|y|^3}. \tag{1.10}$$

If $\Re t = -\frac{1}{2}$ this can be improved to

$$|\Lambda_j| \leq \frac{8.03434}{|y|^3}. \tag{1.11}$$

Furthermore, if γ is as in (1.4) we get

$$|\Lambda_j| = \left| \log \left| \frac{\beta^{(j+1)}}{\beta^{(j+2)}} \right| + A_j \log |\alpha| + B_j \log |\alpha + 1| \right| \tag{1.12}$$

with $A_j, B_j \in \mathbb{Z}$ (cf. Lemma 5.1). For $\Re t = -\frac{1}{2}$, observing that $|\alpha| = |\alpha+1|$, this simplifies to

$$|\Lambda_j| = \left|\log\left|\frac{\beta^{(j+1)}}{\beta^{(j+2)}}\right| + C_j \log|\alpha|\right| \tag{1.13}$$

with $C_j \in \mathbb{Z}$. In Section 5 we combine lower estimates of (1.13) with the upper estimate (1.11) in order to find contradictions showing that for $\Re t = -\frac{1}{2}$ the relative Thue equation (1.1) has no non-trivial solutions (x, y) for $|t| \geq 6$ and $|y| \geq 4$ in all instances where $\Lambda_j \neq 0$. The cases $\Lambda_j = 0$ need a separate non-trivial treatment, and it turns out that these instances contribute some solutions. The case $|t| < 6$ and $|y| \geq 7$ is treated in Section 5 too. In Section 6 we consider the instance $|y| < 4$ for $|t| \geq 6$ resp. $|y| < 7$ for $|t| < 6$, in Section 7 we deal with $x \in \{0, -y, ty\}$ and in Section 8 we discuss the case where γ is associated to an integer in $\mathbb{Z}_k$. In Section 9 all solutions of the Thue equation (1.1) for $\Re t = -\frac{1}{2}$ will be derived.

2. Elements of small norm in $\mathbb{Z}_{k(\alpha)}$

In [5] it has been proved that for $|t| \geq 14$

$$|N_{k(\alpha)/k}(\gamma)| \leq |2t+1|$$

implies that γ is associated to an integer in $\mathbb{Z}_k$ or

$$\gamma = \mu' \beta \alpha^{b_1'} (\alpha^{(2)})^{b_2'} \tag{2.1}$$

where μ' is a unit of $\mathbb{Z}_k$, b_1', b_2' are rational integers and β can attain only finitely many values as listed in [5, Theorem 1.1]. Observe that in view of (1.2) identity (2.1) can also be written as

$$\gamma = \mu \beta \alpha^{b_1} (\alpha+1)^{b_2} \tag{2.2}$$

with μ being a unit of $\mathbb{Z}_k$ and b_1, b_2 being rational integers.

We use the abbreviation

$$b := \begin{cases} \dfrac{1+i\sqrt{D}}{2}, & \text{for } D \equiv 3 (\mathrm{mod} 4), \\ i\sqrt{D}, & \text{for } D \not\equiv 3 (\mathrm{mod} 4), \end{cases} \tag{2.3}$$

such that $\{1, b\}$ is an integer basis of k.

In the sequel we will use a refinement of the methods of [5] to find the following result.

Theorem 2.1. *For all t with $\Re t \le -\frac{1}{2}, \Im t > 0$ and $t \neq \frac{-1+3i\sqrt{3}}{2}$*

$$|N_{k(\alpha)/k}(\gamma)| \le |2t+1|$$

implies that γ is associated to an integer in $\mathbb{Z}_k$ or

$$\gamma = \mu\beta\alpha^{b_1}(\alpha+1)^{b_2} \tag{2.4}$$

where μ is a unit of $\mathbb{Z}_k$, $b_1, b_2 \in \mathbb{Z}$ and β is either an element of the triple $\{\alpha-1, -(2\alpha+1), \alpha+2\}$ (satisfying $N_{k(\alpha)/k}(\beta) = 2t+1$) or an element of the triples presented in the lists $\mathcal{L}(t, \alpha, \beta)$ below.

We distinguish two cases:

(1) $|t| \ge 6$:
Let
$M_1 := \{-1+10i\sqrt{2}, -5+9i\sqrt{2}, -4+9i\sqrt{2}, -3+9i\sqrt{2}, -2+9i\sqrt{2}, -1+9i\sqrt{2}, -6+8i\sqrt{2}, -5+8i\sqrt{2}, -4+8i\sqrt{2}, -3+8i\sqrt{2}, -2+8i\sqrt{2}, -1+8i\sqrt{2}, -7+7i\sqrt{2}, -6+7i\sqrt{2}, -5+7i\sqrt{2}, -4+7i\sqrt{2}, -3+7i\sqrt{2}, -2+7i\sqrt{2}, -1+7i\sqrt{2}, -7+6i\sqrt{2}, -6+6i\sqrt{2}, -5+6i\sqrt{2}, -4+6i\sqrt{2}, -3+6i\sqrt{2}, -2+6i\sqrt{2}, -1+6i\sqrt{2}, -7+5i\sqrt{2}, -6+5i\sqrt{2}, -5+5i\sqrt{2}, -4+5i\sqrt{2}, -3+5i\sqrt{2}, -2+5i\sqrt{2}, -1+5i\sqrt{2}, -7+4i\sqrt{2}, -6+4i\sqrt{2}, -5+4i\sqrt{2}, -4+4i\sqrt{2}, -3+4i\sqrt{2}, -2+4i\sqrt{2}, -6+3i\sqrt{2}, -5+3i\sqrt{2}\}$
and
$M_2 := \{-\frac{1}{2}+\frac{15i\sqrt{3}}{2}, -3+7i\sqrt{3}, -2+7i\sqrt{3}, -1+7i\sqrt{3}, -\frac{9}{2}+\frac{13i\sqrt{3}}{2}, -\frac{7}{2}+\frac{13i\sqrt{3}}{2}, -\frac{5}{2}+\frac{13i\sqrt{3}}{2}, -\frac{3}{2}+\frac{13i\sqrt{3}}{2}, -\frac{1}{2}+\frac{13i\sqrt{3}}{2}, -5+6i\sqrt{3}, -4+6i\sqrt{3}, -3+6i\sqrt{3}, -2+6i\sqrt{3}, -1+6i\sqrt{3}, -\frac{13}{2}+\frac{11i\sqrt{3}}{2}, -\frac{11}{2}+\frac{11i\sqrt{3}}{2}, -\frac{9}{2}+\frac{11i\sqrt{3}}{2}, -\frac{7}{2}+\frac{11i\sqrt{3}}{2}, -\frac{5}{2}+\frac{11i\sqrt{3}}{2}, -\frac{3}{2}+\frac{11i\sqrt{3}}{2}, -\frac{1}{2}+\frac{11i\sqrt{3}}{2}, -6+5i\sqrt{3}, -5+5i\sqrt{3}, -4+5i\sqrt{3}, -3+5i\sqrt{3}, -2+5i\sqrt{3}, -1+5i\sqrt{3}, -\frac{13}{2}+\frac{9i\sqrt{3}}{2}, -\frac{11}{2}+\frac{9i\sqrt{3}}{2}, -\frac{9}{2}+\frac{9i\sqrt{3}}{2}, -\frac{7}{2}+\frac{9i\sqrt{3}}{2}, -\frac{5}{2}+\frac{9i\sqrt{3}}{2}, -\frac{3}{2}+\frac{9i\sqrt{3}}{2}, -\frac{1}{2}+\frac{9i\sqrt{3}}{2}, -7+4i\sqrt{3}, -6+4i\sqrt{3}, -5+4i\sqrt{3}, -4+4i\sqrt{3}, -3+4i\sqrt{3}, -2+4i\sqrt{3}, -1+4i\sqrt{3}, -\frac{13}{2}+\frac{7i\sqrt{3}}{2}, -\frac{11}{2}+\frac{7i\sqrt{3}}{2}, -\frac{9}{2}+\frac{7i\sqrt{3}}{2}, -\frac{7}{2}+\frac{7i\sqrt{3}}{2}, -\frac{5}{2}+\frac{7i\sqrt{3}}{2}, -\frac{3}{2}+\frac{7i\sqrt{3}}{2}, -\frac{1}{2}+\frac{7i\sqrt{3}}{2}, -7+3i\sqrt{3}, -6+3i\sqrt{3}, -5+3i\sqrt{3}, -4+3i\sqrt{3}, -3+3i\sqrt{3}, -\frac{13}{2}+\frac{5i\sqrt{3}}{2}, -\frac{11}{2}+\frac{5i\sqrt{3}}{2}, -\frac{9}{2}+\frac{5i\sqrt{3}}{2}, -6+2i\sqrt{3}, -5+2i\sqrt{3}, -\frac{13}{2}+\frac{3i\sqrt{3}}{2}, -\frac{11}{2}+\frac{3i\sqrt{3}}{2}\}$. *In this case $\mathcal{L}(t, \alpha, \beta)$ is given in Table 1.*

(2) $|t| < 6$: *In this case, in addition to the instances in case (1) many other constellations can occur. Since the corresponding table $\mathcal{L}(t, \alpha, \beta)$ is very long we refrained from printing it here. We refer the reader to the homepage [6] where the complete list for all t can be found as* `pdf` *file.*

$\mathcal{L}(t,\alpha,\beta)$		
Discriminant	$N_{k(\alpha)/k}(\beta)$	β
$D=1$ and $\lvert t\rvert \ge 6$	$\frac{1+i}{2}(2t+1)-\frac{5}{2}(1-i)$ $\frac{1+i}{2}(2t+1)+\frac{5}{2}(1-i)$	$\{-(\alpha+b),(1-b)\alpha+1, b\alpha-1+b\}$ $\{-(\alpha+1-b),b\alpha+1, (1-b)\alpha-b\}$
$D=2$ and $t\in M_1$	$\frac{2+\sqrt{2}i}{2}(2t+1)+\frac{1}{2}(8-7\sqrt{2}i)$	$\{-(\alpha+1-b), b\alpha+1,(1-b)\alpha-b\}$
$D=3$ and $\lvert t\rvert \ge 6$	$\frac{i\sqrt{3}}{2}(2t+1)+\frac{7}{2}$ $\frac{2t+1}{2}-\frac{3i\sqrt{3}}{2}$ $\frac{2t+1}{2}+\frac{3i\sqrt{3}}{2}$	$\{\alpha-b,-((1+b)\alpha+1), b\alpha+1+b\}$ $\{-(\alpha+1-b),b\alpha+1, (1-b)\alpha-b\}$ $\{-(\alpha+b),(1-b)\alpha+1, b\alpha-1+b\}$
$D=3$ and $t\notin M_2$	$\frac{i\sqrt{3}}{2}(2t+1)-\frac{7}{2}$	$\{\alpha+1+b,b\alpha-1, -((1+b)\alpha+b)\}$
$D=3$ and $t=-\frac{1}{2}+\frac{7i\sqrt{3}}{2}$ $\lvert t\rvert=\sqrt{37}$	$5+6i\sqrt{3}$ $5-6i\sqrt{3}$ -12	$\{-(\alpha+1-2b),2b\alpha+1, -((2b-1)\alpha+2b)\}$ $\{\alpha+2-2b,(1-2b)\alpha-1, -((2-2b)\alpha+1-2b)\}$ $\{(\alpha+1-b)^2,(b\alpha+1)^2, ((1-b)\alpha-b)^2\}$
$D=7$ and $\lvert t\rvert \ge 6$	$2t+1-2i\sqrt{7}$	$\{-(\alpha+1-b),b\alpha+1, (1-b)\alpha-b\}$
$D=7$ and $t\in\{-1+3i\sqrt{7},-\frac{7}{2}+\frac{5i\sqrt{7}}{2}, -\frac{5}{2}+\frac{5i\sqrt{7}}{2},-\frac{3}{2}+\frac{5i\sqrt{7}}{2}, -\frac{1}{2}+\frac{5i\sqrt{7}}{2},-4+2i\sqrt{7}, -3+2i\sqrt{7},-\frac{9}{2}+\frac{3i\sqrt{7}}{2}\}$	$\frac{1+i\sqrt{7}}{2}(2t+1)+\frac{13-i\sqrt{7}}{2}$	$\{-(\alpha+2-b),(1-b)\alpha-1, (2-b)\alpha+1-b\}$
$D=7$ and $t=-\frac{1}{2}+\frac{5i\sqrt{7}}{2}$ $\lvert t\rvert=2\sqrt{11}$	$11+2i\sqrt{7}$	$\{-(\alpha-b),(1+b)\alpha+1, -(b\alpha+1+b)\}$
$D=11$ and $t=-\frac{1}{2}+\frac{5i\sqrt{11}}{2}$ $\lvert t\rvert=\sqrt{69}$	$5i\sqrt{11}$	$\{\alpha-1,-(2\alpha+1),\alpha+2\}$ or $\{-(\alpha+1-b),b\alpha+1, (1-b)\alpha-b\}$
$D=11$ and $t\in\{-3+2i\sqrt{11},-2+2i\sqrt{11}, -1+2i\sqrt{11},-\frac{7}{2}+\frac{3i\sqrt{11}}{2}\}$	$\frac{3}{2}(2t+1)-\frac{5i\sqrt{11}}{2}$	$\{-(\alpha+1-b),b\alpha+1, (1-b)\alpha-b\}$

Table 1.

Remark 2.2. In case $D=2$ we have for all $t\in\{-1+10i\sqrt{2},-5+9i\sqrt{2},-7+7i\sqrt{2},-7+4i\sqrt{2},-5+2i\sqrt{2}\}$ that $|N_{k(\alpha)/k}(\alpha-1)|=|N_{k(\alpha)/k}(\alpha+1-b)|=|2t+1|$, but none of the conjugates of $\alpha+1-b$ is associated to any of the conjugates of $\alpha-1$ since the quotient $\frac{N_{k(\alpha)/k}(\alpha+1-b)}{N_{k(\alpha)/k}(\alpha-1)}$ is not an element of $\mathbb{Z}_k$.

For the special case $\Re t=-\frac{1}{2}$ we achieve from Theorem 2.1 the following corollary:

Corollary 2.3. *For all t with $\Re t=-\frac{1}{2},\Im t>0$ and $t\neq\frac{-1+3i\sqrt{3}}{2}$*

$$|N_{k(\alpha)/k}(\gamma)|\leq|2t+1|$$

implies that γ is associated to an integer in $\mathbb{Z}_k$ or

$$\gamma=\mu\beta\alpha^{b_1}(\alpha+1)^{b_2}.$$

where μ is a unit of $\mathbb{Z}_k$, $b_1,b_2\in\mathbb{Z}$ and β is either an element of the triple $\{\alpha-1,-(2\alpha+1),\alpha+2\}$ with $N_{k(\alpha)/k}(\beta)=2t+1$ or an element of the triples given in the list $\mathcal{L}(t,\alpha,\beta)$ contained in Tables 2, 3 and 3a.

In the remaining part of this section we will prove Theorem 2.1.

At first we will give the proof for all t with $\Re t\leq-\frac{1}{2},\Im t>0$ and $|t|\geq 6$. After that we will treat the instances with small modulus.

Remark 2.4. To get the result for all t with $\Im t>0$ and $|t|\geq 6$ similar reflections as described in [5, Proposition 2.1] can be applied.

The quantity β from (2.1) can be written as

$$\beta=u+v\alpha+w\alpha^{(2)} \tag{2.5}$$

with $u,v,w\in\mathbb{Z}_k$.

In the sequel we will use estimates of the roots $\alpha^{(j)}$ of f_t in terms of t.

For two functions g and h and a positive number x_0 we will write $g(x)=L_{x_0}(h(|x|))$ if $|g(x)|\leq h(|x|)$ for all x with $|x|>x_0$. There is a root α of f_t such that we have the following estimates in terms of t (cf. [5, Lemma 3.1]).

$$\begin{aligned}\alpha&=t+\frac{2}{t}-\frac{1}{t^2}-\frac{3}{t^3}+L_6\left(\frac{4.6}{|t|^{7/2}}\right),\\ \alpha^{(2)}&=-1-\frac{1}{\alpha}=-1-\frac{1}{t}+\frac{2}{t^3}+L_6\left(\frac{1.5}{|t|^{7/2}}\right),\\ \alpha^{(3)}&=-\frac{1}{\alpha+1}=-\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+L_6\left(\frac{2}{|t|^{7/2}}\right).\end{aligned} \tag{2.6}$$

First of all we need bounds for $|v|$ and $|w|$ valid for $|t|\geq 6$.

$\mathcal{L}(t,\alpha,\beta)$		
Discriminant	$N_{k(\alpha)/k}(\beta)$	β
$D=3$ and $\lvert t\rvert \geq 14$	$\frac{i\sqrt{3}}{2}(2t+1)-\frac{7}{2}$	$\{\alpha+1+b, b\alpha-1, -((1+b)\alpha+b)\}$
$D=3$ and $t\notin\{-\frac{1}{2}+\frac{5i\sqrt{3}}{2}, -\frac{1}{2}+\frac{i\sqrt{3}}{2}\}$	$\frac{i\sqrt{3}}{2}(2t+1)+\frac{7}{2}$	$\{\alpha-b, -((1+b)\alpha+1), b\alpha+1+b\}$
$D=3$ and $t\neq-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	$\frac{2t+1}{2}-\frac{3i\sqrt{3}}{2}$ $\frac{2t+1}{2}+\frac{3i\sqrt{3}}{2}$	$\{-(\alpha+1-b), b\alpha+1, (1-b)\alpha-b\}$ $\{-(\alpha+b), (1-b)\alpha+1, b\alpha-1+b\}$
$D=3$ and $t=-\frac{1}{2}+\frac{7i\sqrt{3}}{2}$ $\lvert t\rvert=\sqrt{37}$	$5+6i\sqrt{3}$ $5-6i\sqrt{3}$ -12	$\{-(\alpha+1-2b), 2b\alpha+1, -((2b-1)\alpha+2b)\}$ $\{\alpha+2-2b, (1-2b)\alpha-1, -((2-2b)\alpha+1-2b)\}$ $\{(\alpha+1-b)^2, (b\alpha+1)^2, ((1-b)\alpha-b)^2\}$
$D=3$ and $t=-\frac{1}{2}+\frac{5i\sqrt{3}}{2}$ $\lvert t\rvert=\sqrt{19}$	-4	$\{\alpha-b, -((1+b)\alpha+1), b\alpha+1+b\}$ or $\{\alpha^2+(3-4b)\alpha+1-2b,$ $(2b-1)\alpha^2+(4b-1)\alpha+1,$ $-((2b-1)\alpha^2+\alpha+1-2b)\}$
	$2+3i\sqrt{3}$	$\{-(\alpha+1-2b), 2b\alpha+1, -((2b-1)\alpha+2b)\}$
	$2-3i\sqrt{3}$	$\{\alpha+2-2b, (1-2b)\alpha-1, -((2-2b)\alpha+1-2b)\}$
	$-4i\sqrt{3}$	$\{\alpha+2-3b, (1-3b)\alpha-1, -((2-3b)\alpha+1-3b)\}$
	$4-3i\sqrt{3}$	$\{\alpha+2-4b, (1-4b)\alpha-1, -((2-4b)\alpha+1-4b)\}$
	$4+3i\sqrt{3}$	$\{-(\alpha+3-4b), -((2-4b)\alpha-1),$ $(3-4b)\alpha+2-4b\}$
	-3	$\{(\alpha+1-b)^2, (b\alpha+1)^2, ((1-b)\alpha-b)^2\}$
	-5	$\{\alpha^2+(4-4b)\alpha-b,$ $(3b-3)\alpha^2+(4b-2)\alpha+1,$ $-(b\alpha^2-(2b-4)\alpha+3-3b)\}$
	8	$\{\alpha^2+(4-6b)\alpha-1-b,$ $(5b-4)\alpha^2+(6b-2)\alpha+1,$ $-((b+1)\alpha^2-(4b-6)\alpha+4-5b)\}$
$D=7$ and $t\neq-\frac{1}{2}+\frac{i\sqrt{7}}{2}$	$2t+1-2i\sqrt{7}$	$\{-(\alpha+1-b), b\alpha+1, (1-b)\alpha-b\}$
$D=7$ and $t=-\frac{1}{2}+\frac{5i\sqrt{7}}{2}$ $\lvert t\rvert=2\sqrt{11}$	$11-2i\sqrt{7}$ $11+2i\sqrt{7}$	$\{\alpha+2-b, (1-b)\alpha-1, -((2-b)\alpha+1-b)\}$ $\{-(\alpha-b), (1+b)\alpha+1, -(b\alpha+1+b)\}$

Table 2.

$\mathcal{L}(t,\alpha,\beta)$		
$D=7$ and	$4+i\sqrt{7}$	$\{-(\alpha-b),(1+b)\alpha+1,-(b\alpha+1+b)\}$
$t=-\frac{1}{2}+\frac{3i\sqrt{7}}{2}$	$4-i\sqrt{7}$	$\{\alpha+2-b,(1-b)\alpha-1,-((2-b)\alpha+1-b)\}$
$\lvert t\rvert=4$	$1-2i\sqrt{7}$	$\{\alpha+1-2b,-(2b\alpha+1),(2b-1)\alpha+2b\}$
	$1+2i\sqrt{7}$	$\{-(\alpha+2-2b),-((1-2b)\alpha-1),$ $(2-2b)\alpha+1-2b\}$
	$\frac{7}{2}+\frac{5i\sqrt{7}}{2}$	$\{-(2\alpha+1-b),(1+b)\alpha+2,(1-b)\alpha-1-b\}$
	$\frac{7}{2}-\frac{5i\sqrt{7}}{2}$	$\{2\alpha+2-b,-(b\alpha+2),-((2-b)\alpha-b)\}$
	-3	$\{\alpha^2+(2-2b)\alpha-b,(b-1)\alpha^2+2b\alpha+1,$ $-(b\alpha^2+2\alpha+1-b)\}$
	-5	$\{\alpha^2+(3-2b)\alpha-b,(b-2)\alpha^2+(2b-1)\alpha+1,$ $-(b\alpha^2+3\alpha+2-b)\}$
	-7	$\{\alpha^2+(4-2b)\alpha-b,(b-3)\alpha^2+(2b-2)\alpha+1,$ $-(b\alpha^2+4\alpha+3-b)\}$ or $\{(\alpha+1-b)^2,(b\alpha+1)^2,((1-b)\alpha-b)^2\}$
$D=11$ and $t=-\frac{1}{2}+\frac{5i\sqrt{11}}{2}$ $\lvert t\rvert=\sqrt{69}$	$5i\sqrt{11}$	$\{\alpha-1,-(2\alpha+1),\alpha+2\}$ or $\{-(\alpha+1-b),b\alpha+1,(1-b)\alpha-b\}$
$D=11$ and	$2i\sqrt{11}$	$\{-(\alpha+1-b),b\alpha+1,(1-b)\alpha-b\}$
$t=-\frac{1}{2}+\frac{3i\sqrt{11}}{2}$	$7+2i\sqrt{11}$	$\{-(\alpha-b),(1+b)\alpha+1,-(b\alpha+1+b)\}$
$\lvert t\rvert=5$	$7-2i\sqrt{11}$	$\{\alpha+2-b,(1-b)\alpha-1,-((2-b)\alpha+1-b)\}$
$D=15$ and $t=-\frac{1}{2}+\frac{3i\sqrt{15}}{2}$ $\lvert t\rvert=\sqrt{34}$	$3i\sqrt{15}$	$\{\alpha-1,-(2\alpha+1),\alpha+2\}$ or $\{-(\alpha+1-b),b\alpha+1,(1-b)\alpha-b\}$
$D=15$ and $t=-\frac{1}{2}+\frac{i\sqrt{15}}{2}$ $\lvert t\rvert=2$	3	$\{\alpha^2+\alpha+1,\alpha^2+\alpha+1,\alpha^2+\alpha+1\}$
$D=19$ and	-3	$\{2\alpha+3-b,(1-b)\alpha-2,-((3-b)\alpha+1-b)\}$
$t=-\frac{1}{2}+\frac{i\sqrt{19}}{2}$	2	$\{\alpha^2+\alpha+1,\alpha^2+\alpha+1,\alpha^2+\alpha+1\}$
$\lvert t\rvert=\sqrt{5}$	4	$\{(\alpha^2+\alpha+1)^2,(\alpha^2+\alpha+1)^2,(\alpha^2+\alpha+1)^2\}$
$D=23$ and $t=-\frac{1}{2}+\frac{i\sqrt{23}}{2}$ $\lvert t\rvert=\sqrt{6}$	$i\sqrt{23}$	$\{\alpha-1,-(2\alpha+1),\alpha+2\}$, $\{3\alpha+2-b,-((1+b)\alpha+3),-((2-b)\alpha-1-b)\}$ or $\{5\alpha+3-b,-((2+b)\alpha+5),-((3-b)\alpha-2-b)\}$
	$\frac{3}{2}+\frac{i\sqrt{23}}{2}$	$\{2\alpha+1-b,-((1+b)\alpha+2),-((1-b)\alpha-1-b)\}$
	$\frac{3}{2}-\frac{i\sqrt{23}}{2}$	$\{-(2\alpha+2-b),b\alpha+2,(2-b)\alpha-b\}$
$D=31$ and $t=-\frac{1}{2}+\frac{i\sqrt{31}}{2}$ $\lvert t\rvert=2\sqrt{2}$	$i\sqrt{31}$	$\{\alpha-1,-(2\alpha+1),\alpha+2\}$, $\{-(3\alpha+2-b),(1+b)\alpha+3,(2-b)\alpha-1-b\}$ or $\{-(5\alpha+4-3b),(1+3b)\alpha+5,(4-3b)\alpha-1-3b\}$
	$\frac{1}{2}-\frac{i\sqrt{31}}{2}$	$\{2\alpha+1-b,-((1+b)\alpha+2),-((1-b)\alpha-1-b)\}$
	$\frac{1}{2}+\frac{i\sqrt{31}}{2}$	$\{-(2\alpha+2-b),b\alpha+2,(2-b)\alpha-b\}$
	-3	$\{\alpha^2+\alpha+2,2\alpha^2+\alpha+1,2\alpha^2+3\alpha+2\}$, $\{\alpha^2+2\alpha+2,\alpha^2+1,2\alpha^2+2\alpha+1\}$ or $\{(b+1)\alpha^2+(b+2)\alpha+2,\alpha^2+b\alpha+1+b,$ $2\alpha^2-(b-2)\alpha+1\}$

Table 3.

		$\mathcal{L}(t, \alpha, \beta)$
$D = 35$ and $t = -\frac{1}{2} + \frac{i\sqrt{35}}{2}$ $\lvert t\rvert = 3$	$i\sqrt{35}$	$\{\alpha - 1, -(2\alpha + 1), \alpha + 2\}$, $\{-(2\alpha + 1 - b), (1 + b)\alpha + 2, (1 - b)\alpha - 1 - b\}$ or $\{-(2\alpha + 2 - b), b\alpha + 2, (2 - b)\alpha - b\}$
	-2	$\{\alpha^2 + \alpha + 1, \alpha^2 + \alpha + 1, \alpha^2 + \alpha + 1\}$
	4	$\{(\alpha^2 + \alpha + 1)^2, (\alpha^2 + \alpha + 1)^2, (\alpha^2 + \alpha + 1)^2\}$
	-5	$\{\alpha^2 + \alpha + 4, 4\alpha^2 + \alpha + 1, 4\alpha^2 + 7\alpha + 4\}$ $\{\alpha^2 + 2\alpha + 2, \alpha^2 + 1, 2\alpha^2 + 2\alpha + 1\}$ or
$D = 39, 43, 47, 51, 55$ and $t = -\frac{1}{2} + \frac{i\sqrt{D}}{2}$	$t^2 + t + 7$	$\{\alpha^2 + \alpha + 1, \alpha^2 + \alpha + 1, \alpha^2 + \alpha + 1\}$

Table 3a.

Lemma 2.5. *Let $\Re t \leq -\frac{1}{2}$ and v, w be defined as in* (2.5). *For $|t| \geq 6$ we can always choose the quantity β in a way such that*

$$|v|, |w| < 4.16955 \tag{2.7}$$

hold.

PROOF. These bounds can be computed along the same lines as in [5, Section 5] using the estimates (2.6). □

Now we want to compute a bound for $|u|$. From (2.6) we get

$$\alpha = t + L_6(0.383694), \quad \alpha^{(2)} = -1 + L_6(0.178761), \quad \alpha^{(3)} = L_6(0.202854). \tag{2.8}$$

Let

$$g_{v,w}(r) = (r + v\alpha + w\alpha^{(2)})(r + v\alpha^{(2)} + w\alpha^{(3)})(r + v\alpha^{(3)} + w\alpha)$$

be defined as in [5, Section 5]. The roots of $g_{v,w}(r)$ are

$$r_1 = -v\alpha - w\alpha^{(2)}, \quad r_2 = -v\alpha^{(2)} - w\alpha^{(3)}, \quad r_3 = -v\alpha^{(3)} - w\alpha. \tag{2.9}$$

Note that $g_{v,w}(u) = N_{k(\alpha)/k}(\beta)$. On the other hand $|g_{v,w}(r)|$ can be interpreted as the product of the distances of r from the points r_1, r_2, r_3.

We will use

Lemma 2.6 (cf. [5, Lemma 5.2]). *Let $R \in \mathbb{R}$ and let $z, z_1, z_2, z_3 \in \mathbb{C}$ be disjoint. For $i \in \{1, 2, 3\}$ set $d_i = |z - z_i|$ and $d := \max_{i,j} |z_i - z_j|$. If $d_i \geq R$ for each $i \in \{1, 2, 3\}$ then*

$$d_1 d_2 d_3 \geq R^2(d - R).$$

Applying Lemma 2.6 for $z = r$ and $z_i = r_i$, $i \in \{1, 2, 3\}$ and choosing R such that $R^2(d - R) > |2t + 1|$ implies that $|g_{v,w}(r)| = |r - r_1||r - r_2||r - r_3| > |2t + 1|$ if $|r - r_i| \geq R$ for all i. Therefore we only have to analyse values of u which fulfil the inequality $|u - r_i| < R$ for at least one $i \in \{1, 2, 3\}$.

In contrast to [5, Section 5], we have to split up the cases $|w| = \max(|v|,|w|) > 0$ and $|v| = \max(|v|, |w|) > 0$ depending on $M := \max(|v|, |w|)$ to get a lower bound for d.

- $|w| \geq |v|$:
  (1) $M = 1$: We have to distinguish two cases.
    (a) $|v| = 0$ and $|w| = 1$: We get

$$d \geq |r_3 - r_2| \geq |w|(|t| - L_6(0.586548)) = |t| - L_6(0.586548).$$

Setting $R = \sqrt{3.35}$ yields the inequality $R^2(d - R) > |2t| \geq |2t + 1|$ for $|t| \geq 6$ and $\Re t \leq -\frac{1}{2}$.

    (b) $|v| = 1$ and $|w| = 1$: We have to distinguish the cases $w = v$, $w = -v$ (for all D), $w = iv$, $w = -iv$ (for $D = 1$) as well as $w = bv$, $w = -bv$, $w = (1 - b)v$, $w = -(1 - b)v$ (for $D = 3$).

$\underline{w = v}$: Using

$$d \geq |r_1 - r_2| \geq |v|(|t| - L_6(0.586548)) = |t| - L_6(0.586548)$$

and setting $R = \sqrt{3.35}$ yields the inequality $R^2(d-R) > |2t| \geq |2t+1|$ for $|t| \geq 6$ and $\Re t \leq -\frac{1}{2}$.

$\underline{w = -v}$: The inequality $R^2(d - R) > |2t| \geq |2t + 1|$ for $|t| \geq 6$ and $\Re t \leq -\frac{1}{2}$ is fulfilled by setting

$$d \geq |r_1 - r_3| \geq |v|(2|t| - L_6(2.149)) = 2|t| - L_6(2.149)$$

and $R = \sqrt{1.39}$.

$\underline{w = iv \text{ for } D = 1}$: Using

$$d \geq |r_1 - r_3| \geq |v|(\sqrt{2}|t| - L_6(2.149)) = \sqrt{2}|t| - L_6(2.149)$$

and setting $R = \sqrt{2.53}$ yields the inequality $R^2(d-R) > |2t| \geq |2t+1|$ for $|t| \geq 6$ and $\Re t \leq -\frac{1}{2}$.

$\underline{w = -iv \text{ for } D = 1}$: This case is analogous to the case $w = iv$.

$\underline{w = bv \text{ for } D = 3}$: In this case it does not suffice to use the shortened expansions of α in (2.8) to get an R which fulfils $R^2(d - R) > |2t| \geq |2t + 1|$ for $|t| \geq 6$ and $\Re t \leq -\frac{1}{2}$, but it is necessary to use the asymptotic expansions (2.6). We get

$$\begin{aligned} d \geq |r_2 - r_3| &= |b\alpha - \alpha^{(2)} + (1 - b)\alpha^{(3)}| \\ &= \left| bt + \frac{2b}{t} - \frac{b}{t^2} - \frac{3b}{t^3} + L_6\left(\frac{4.6}{|t|^{7/2}}\right) + 1 + \frac{1}{t} - \frac{2}{t^3} \right. \end{aligned}$$

$$+ L_6\left(\frac{1.5}{|t|^{7/2}}\right) - \frac{1-b}{t} + \frac{1-b}{t^2} + \frac{1-b}{t^3} + L_6\left(\frac{2}{|t|^{7/2}}\right)\Bigg|$$
$$= |t| - L_6(1.58464).$$

Setting $R = \sqrt{6.3}$ yields the inequality $R^2(d-R) > |2t| \geq |2t+1|$ for $|t| \geq 6$ and $\Re t \leq -\frac{1}{2}$.
$w = -bv$ for $D = 3$: This case is analogous to the case $w = bv$.
$w = \pm(1-b)v$ for $D = 3$: These cases are treated in the same way as the case $w = bv$. Here, setting $R = \sqrt{6.3}$ yields the inequality $R^2(d-R) > |2t| \geq |2t+1|$ for $|t| \geq 6$ and $\Re t \leq -\frac{1}{2}$ too.

(2) $M \neq 1$: We have

$$d \geq |r_3 - r_2| \geq |w|(|t| - L_6(1.968163)).$$

Since $M \neq 1$, $M = \sqrt{2}$ is the worst case, in this case setting $R = \sqrt{\frac{4.29}{M}}$ yields the inequality $R^2(d-R) > |2t+1|$ for $|t| \geq 6$ and $\Re t \leq -\frac{1}{2}$.

- $|v| > |w|$:

(1) $M = 1$, i.e., $|w| = 0$ and $|v| = 1$: We get

$$d \geq |r_1 - r_2| = |v|(|t| - L_6(1.56246)) = |t| - L_6(1.56246).$$

Setting $R = \sqrt{6.1}$ yields the inequality $R^2(d-R) > |2t| \geq |2t+1|$ for $|t| \geq 6$ and $\Re t \leq -\frac{1}{2}$.

(2) $M \neq 1$. Since $M \neq 1$, $M = \sqrt{2}$ is the worst case. Here we have to distinguish two cases.

(a) $|w| = 0$ and $|v| = \sqrt{2}$: Here we get

$$d \geq |r_1 - r_2| = |v|(|t| - L_6(1.56246)) = \sqrt{2}(|t| - L_6(1.56246)).$$

Setting $R = \sqrt{\frac{3.64}{M}}$ yields the inequality $R^2(d-R) > |2t| \geq |2t+1|$ for $|t| \geq 6$ and $\Re t \leq -\frac{1}{2}$.

(b) $|w| = 1$ and $|v| = \sqrt{2}$: We have

$$\begin{aligned} d \geq |r_1 - r_2| &\geq |v||t| - (|w| + |v|L_6(0.383694) + |w|L_6(0.178761) \\ &\quad + |v| + |v|L_6(0.178761) + |w|L_6(0.202854)) \\ &= \sqrt{2}|t| - L_6(3.59126). \end{aligned}$$

Setting $R = \sqrt{\frac{3.13}{M}}$ yields the inequality $R^2(d-R) > |2t| \geq |2t+1|$ for $|t| \geq 6$ and $\Re t \leq -\frac{1}{2}$.

Setting $R := \sqrt{6.3}$ is appropriate for all cases. Thus we get the following lemma.

Lemma 2.7. *Let* u*,* v*,* w *be defined as in* (2.5) *and* r_1*,* r_2*,* r_3 *be defined as in* (2.9)*. If the minimal distance of* u *to one of the points* r_1*,* r_2*,* r_3 *is greater than* $R = \sqrt{6.3}$*, then* $|N_{k(\alpha)/k}(u + v\alpha + w\alpha^{(2)})| > |2t + 1|$.

Thus for the proof of Theorem 2.1 it suffices to consider the instances where u is within one of the balls of radius R about r_1, r_2 or r_3.

In order to determine all the candidates u we approximate r_1, r_2 and r_3 by points of the lattice $\mathbb{Z}_k$. In particular, set

$$p_1 := -vt + w, \quad p_2 := -v, \quad p_3 := -wt \tag{2.10}$$

(cf. [5, (14)]). From (2.8) and (2.9) it follows that

$$|r_i - p_i| \leq ML_6(0.38369) \qquad (1 \leq i \leq 3).$$

Setting

$$\hat{R} := R + 0.38369M < 4.11,$$

it suffices for the proof of Theorem 2.1 to consider all numbers u with distance less than $\hat{R}$ from at least one of the points p_i.

We now insert these bounds for u, v, w into a modification of the `Mathematica`® program described in [5, Section 6]. The modification concerns the list of discriminants D which have to be analysed ($D = 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 23, 31, 35, 39, 43, 47, 51, 55, 59, 67, 71$) as well as $t = c_1 + c_2 b$ which has to fulfil $\Re t \leq -\frac{1}{2}$, $\Im t > 0$ and $|t| \geq 6$. We check for each number $\beta = u + v\alpha + w\alpha^{(2)}$ whether the inequality $|N_{k(\alpha)/k}(\beta)| \leq |2t + 1|$ is fulfilled or cannot be decided to be true or false. Each number whose norm either fulfils the norm inequality or cannot be linked to $|2t+1|$ is written in a list. After that the program tries to find associated elements in this list by generating for each element three normal forms and comparing all of these triples. If one triple can be computed by multiplying another triple by a unit of $\mathbb{Z}_k$, we can drop one of these triples. Therefore we obtain for all t with $\Re t \leq -\frac{1}{2}$, $\Im t > 0$ and $|t| \geq 6$ a list of all elements of small norm.

Now we consider all values t with $|t| < 6$. As we only have a few values t which fulfil $|t| < 6$, we can treat each of these values t separately. In these instances we can compute the exact roots $\alpha^{(1)}$, $\alpha^{(2)}$, $\alpha^{(3)}$ of f_t and insert them in the estimate (cf. [5, Equation (4)])

$$\begin{aligned} &|(t^2 + t + 7)v|, |(t^2 + t + 7)w| \\ &\leq |N_{k(\alpha)/k}(\gamma)|^{1/3}|\alpha|^{2/3}(|\alpha - \alpha^{(2)}| + |\alpha^{(2)} - \alpha^{(3)}| + |\alpha^{(3)} - \alpha|) \end{aligned}$$

to calculate bounds for $|v|, |w|$. In contrast to the case $|t| \geq 6$ where we consider three different balls of radius R about r_1, r_2 and r_3 (cf. Lemma 2.7) to get a bound for $|u|$, r_1, r_2, r_3 now lie within one ball with radius $R = \sqrt[3]{|2t+1|}$. These bounds are used in a second `Mathematica`® program, which checks for each t and $\beta = u + v\alpha + w\alpha^{(2)}$ with u, v, w within the computed bounds whether the inequality $|N_{k(\alpha)/k}(\beta)| \leq |2t+1|$ is fulfilled and if two numbers β which satisfy this norm inequality are associated. Furthermore, the program tests if there are elements which are not associated but whose norms have the some absolute value. For this purpose the program proceeds in the following way. All elements which are associated in $\mathbb{Z}_k$ are listed. Afterwards all quotients r_1, r_2 resp. r_3 of one element with the conjugates of another element of this list are determined and the quantities $s_j = |r_j - \mu\alpha^{j_1}(\alpha+1)^{j_2}|$ ($j \in \{1, 2, 3\}$) where μ is a unit of $\mathbb{Z}_k$ (cf. (2.2)) are computed. If $s_j = 0$, these two elements are associated and we can drop one of them. Thus we obtain a list of all elements of small norm for all t with $\Re t \leq -\frac{1}{2}$, $\Im t > 0$ and $|t| < 6$.

Combining all lists, we get the list given in Theorem 2.1.

Remark 2.8. Using the above-mentioned `Mathematica`® program we only obtain one element of each triple given in Theorem 2.1. To get all three values of a triple, we have to conjugate α and use (1.2).
For example: `Mathematica`® returns $\alpha - 1$. Then the conjugates are

$$\alpha^{(2)} - 1 = -\frac{\alpha+1}{\alpha} - 1 = -\frac{2\alpha+1}{\alpha}, \qquad \alpha^{(3)} - 1 = -\frac{1}{\alpha+1} - 1 = -\frac{\alpha+2}{\alpha+1}.$$

Since the denominators are absorbed by the powers of α resp. $\alpha + 1$ in $\gamma = \mu\beta\alpha^{b_1}(\alpha+1)^{b_2}$ and the sign is absorbed by μ, we arrive at the triple $\{\alpha - 1, 2\alpha + 1, \alpha + 2\}$ in this case.

3. An upper bound for $|x/y - \alpha^{(j)}|$

Let j be defined as in (1.5) and denote by $D_{f_t} = (t^2+t+7)^2$ the discriminant of f_t. Then (cf. [9, Lemma 1])

$$D_{f_t} = \left(\frac{\gamma^{(j+1)}}{y}\right)^2 \left(\frac{\gamma^{(j+2)}}{y}\right)^4 \left(1 - \frac{\gamma^{(j+1)}}{\gamma^{(j+2)}}\right)^2 \left(1 - \frac{\gamma^{(j)}}{\gamma^{(j+2)}}\right)^2 \left(1 - \frac{\gamma^{(j)}}{\gamma^{(j+1)}}\right)^2 \quad (3.1)$$

with $\gamma^{(j)} = x - \alpha^{(j)}y$ from (1.5). We need the following auxiliary result.

Lemma 3.1.

$$\max_{\substack{z_1,z_2\in\mathbb{C}\\ |z_1|\le 1,\,|z_2|\le 1,\,|z_2|\le|z_1|}} \left|(1-z_1)\,(1-z_2)\left(1-\frac{z_2}{z_1}\right)\right| = 3\sqrt{3}.$$

Proof. Let $g(u,w) := (1-u)(1-w)(1-uw)$ and

$$M := \max_{\substack{|u|\le 1\\ |w|\le 1}} |g(u,w)|.$$

Since

$$\left|g\left(\frac{-1+i\sqrt{3}}{2},\frac{-1+i\sqrt{3}}{2}\right)\right| = 3\sqrt{3},$$

we have $M \ge 3\sqrt{3}$. In order to prove the reverse inequality let

$$A := \{z : |z| \le 1\} \cap \{z : |1-z| \le 1\} \quad \text{and} \quad B := \{z : |z| \le 1\}\backslash A.$$

We distinguish two cases:

Case 1: $\{u,w,uw\}\cap A \ne \emptyset$. Thus there exists a $z \in \{u,w,uw\}\cap A$ for which we have $|1-z| \le 1$, and, hence, $|g(u,w)| \le 1\cdot 2\cdot 2 = 4 < 3\sqrt{3}$.

Case 2: $\{u,w,uw\}\cap A = \emptyset$. In this case let $u := \rho e^{i\varphi}$ and $w := \sigma e^{i\psi}$ be the polar representations of u and w. Since $u,w,uw \notin A$, the distances of these points to 1 do not become smaller with increasing ρ or σ. Thus we have to take $\rho=\sigma=1$ to get the maximum. Since $|1-u| = 2|\sin\frac{\varphi}{2}|$, $|1-w| = 2|\sin\frac{\psi}{2}|$ and $|1-uw| = 2|\sin\frac{\varphi+\psi}{2}|$ we may write

$$|g(u,w)| \le 8 \max_{\varphi,\psi,\varphi+\psi\in\left(\frac{\pi}{3},\frac{5\pi}{3}\right)} \left|\sin\frac{\varphi}{2}\sin\frac{\psi}{2}\sin\frac{\varphi+\psi}{2}\right|$$

in this case. $\varphi,\psi,\varphi+\psi \in \{\frac{\pi}{3},\frac{5\pi}{3}\}$ yield $|g(u,w)| \le 4$, since in this case $|1-u| = 1$, $|1-w| = 1$ or $|1-uw|=1$. Let $\alpha := \frac{\varphi}{2}$, $\beta := \frac{\psi}{2}$ and $h(\alpha,\beta) := \sin\alpha\sin\beta\sin(\alpha+\beta)$.

The relative maxima of $h(\alpha,\beta)$ fulfil $\frac{\partial h}{\partial\alpha} = 0$, i.e., $\cos\alpha\sin(\alpha+\beta) + \sin\alpha\cos(\alpha+\beta) = 0$ or $-\tan(\alpha+\beta) = \tan\alpha$ ($\cos\alpha \ne 0$); if $\cos\alpha=0$, $\cos(\alpha+\beta)=0$, too, and we get a contradiction. Furthermore $\frac{\partial h}{\partial\beta} = 0$, i.e., $\cos\beta\sin(\alpha+\beta) + \sin\beta\cos(\alpha+\beta) = 0$ or $-\tan(\alpha+\beta) = \tan\beta$ ($\cos\beta \ne 0$); if $\cos\beta = 0$, $\cos(\alpha+\beta)=0$, too, and we get a contradiction. Thus $\tan\alpha = \tan\beta = -\tan(\alpha+\beta)$. Since $\alpha,\beta,\alpha+\beta \in \left(\frac{\pi}{6},\frac{5\pi}{6}\right)$, $\alpha=\beta=\pi-(\alpha+\beta)$ yielding $\alpha=\beta=\frac{\pi}{3}$ and $\varphi=\psi=\frac{2\pi}{3}$, which corresponds to $u = w = \frac{-1+i\sqrt{3}}{2}$. Therefore $M = 3\sqrt{3}$. $\square$

Let (x, y) be a solution of the Thue equation (1.1) and $\gamma^{(j)} = x - \alpha^{(j)}y$. We can now apply Lemma 3.1 to (3.1) by setting $z_1 = \frac{\gamma^{(j)}}{\gamma^{(j+1)}}$ and $z_2 = \frac{\gamma^{(j)}}{\gamma^{(j+2)}}$. This yields $|t^2+t+7|^2 \leq \left|\frac{\gamma^{(j+1)}}{y}\right|^2 \left|\frac{\gamma^{(j+2)}}{y}\right|^4 (3\sqrt{3}\,)^2$. Multiplying this inequality with $\left|\frac{\gamma^{(j+1)}}{y}\right|^2 \left|\frac{\gamma^{(j)}}{y}\right|^4$ and using the fact that $|N(\gamma)| = |\gamma^{(j)}||\gamma^{(j+1)}||\gamma^{(j+2)}| = |\ell|$ we get

$$\left|\frac{\gamma^{(j)}}{y}\right| \leq \left(\frac{(3\sqrt{3})^2|\ell|^4}{|t^2+t+7|^2}\right)^{1/6} \frac{1}{|y|^2} \tag{3.2}$$

because $|\gamma^{(j)}| \leq |\gamma^{(j+1)}|$.

Observing that $|\ell| \leq |2t+1|$ and applying the triangular inequality (3.2) implies the following estimate valid for $|t| \geq 6$.

$$\left|\frac{\gamma^{(j)}}{y}\right| = \min_{l \in \{1,2,3\}} \left|\frac{x}{y} - \alpha^{(l)}\right| \leq \frac{2.9958}{|y|^2}. \tag{3.3}$$

For $\Re t = -\frac{1}{2}$, using $|t| = |t+1|$ and $t(t+1) = -|t|^2$, we even gain the sharper estimate

$$\left|\frac{\gamma^{(j)}}{y}\right| \leq \frac{2.9482}{|y|^2}. \tag{3.4}$$

In the following we proceed by bootstrapping similar to [9, p. 263f]. Let $|y| \geq 4$. Then (3.3) implies that

$$\min_{l \in \{1,2,3\}} \left|\frac{x}{y} - \alpha^{(l)}\right| = \left|\frac{x}{y} - \alpha^{(j)}\right| \leq 0.18723.$$

Thus for $l \in \{j+1, j+2\}$ using (2.6) we have

$$\begin{aligned} |x - \alpha^{(l)}y| &\geq \left(|\alpha^{(l)} - \alpha^{(j)}| - \left|\frac{x}{y} - \alpha^{(j)}\right|\right)|y| \geq \Big(|\alpha^{(l)} - \alpha^{(j)}| - 0.18723\Big)|y| \\ &\geq \Big(|\alpha^{(3)} - \alpha^{(2)}| - 0.18723\Big)|y| \geq (0.96097 - 0.18723)|y| = 0.773747|y|. \end{aligned}$$

Furthermore, again using (2.6) we see that there exists $l \in \{j+1, j+2\}$ such that

$$\begin{aligned} |x - \alpha^{(l)}y| &\geq \left(|\alpha^{(l)} - \alpha^{(j)}| - \left|\frac{x}{y} - \alpha^{(j)}\right|\right)|y| \geq \Big(|\alpha^{(l)} - \alpha^{(j)}| - 0.18723\Big)|y| \\ &\geq \Big(\min(|\alpha^{(2)} - \alpha|, |\alpha^{(3)} - \alpha|) - 0.18723\Big)|y| \\ &\geq (|t| - 1.56246 - 0.18723)|y| = (|t| - 1.74969)|y|. \end{aligned}$$

Inserting these estimates into the relation

$$\left|\frac{x}{y}-\alpha^{(j)}\right|=\frac{|N(\gamma)|}{|x-\alpha^{(j+1)}y||x-\alpha^{(j+2)}y||y|}$$

yields

$$\left|\frac{x}{y}-\alpha^{(j)}\right|\leq\frac{|2t+1|}{0.773747\cdot||t|-1.74969||y|^3}\leq\frac{3.95297}{|y|^3}.$$

Summing up we proved the following result.

Lemma 3.2. *Let $|t|\geq 6$, $|y|\geq 4$ and let $(x,y)\in\mathbb{Z}_k^2$ be a solution of* (1.1). *Then we have*

$$\left|\frac{x}{y}-\alpha^{(j)}\right|=\min_{l\in\{1,2,3\}}\left|\frac{x}{y}-\alpha^{(l)}\right|\leq\frac{3.95297}{|y|^3}.$$

Corollary 3.3. *Let $(x,y)\in\mathbb{Z}_k^2$ be a solution of* (1.1). *For $\Re t=-\frac{1}{2}$ we have the better estimate*

$$\left|\frac{x}{y}-\alpha^{(j)}\right|\leq\frac{3.63241}{|y|^3}.$$

In the sequel we use the notation

$$\alpha^{(j)}=\lfloor\alpha^{(j)}\rfloor+\{\alpha^{(j)}\},\tag{3.5}$$

where $\lfloor\alpha^{(1)}\rfloor=t$, $\lfloor\alpha^{(2)}\rfloor=-1$ and $\lfloor\alpha^{(3)}\rfloor=0$ are the "integer parts" of $\alpha^{(j)}$ ($j\in\{1,2,3\}$). Then by Lemma 3.2 for $|t|\geq 6$, $|y|\geq 4$ and $x\notin\{0,-y,ty\}$

$$\frac{3.95297}{|y|^3}\geq\left|\frac{x-\lfloor\alpha^{(j)}\rfloor y}{y}-\{\alpha^{(j)}\}\right|\geq\frac{1}{|y|}-|\{\alpha^{(j)}\}|.$$

The latter inequality follows from the fact that $|z_1-z_2|\geq 1$ for $z_1,z_2\in\mathbb{Z}_k$, $z_1\neq z_2$. Therefore,

$$\frac{1}{|y|}\left(1-\frac{3.95297}{|y|^2}\right)\leq|\{\alpha^{(j)}\}|.$$

Using the expansions (2.6) for $\alpha^{(j)}$ we have $|\{\alpha^{(j)}\}|\leq|\{\alpha\}|\leq\frac{2.302166}{|t|}$ and therefore

$$|y|\geq\frac{0.75294}{|\{\alpha^{(j)}\}|}\geq\frac{0.75294}{|\{\alpha^{(1)}\}|}\geq 0.32706|t|.$$

Thus we proved the following lemma.

Lemma 3.4. *Let $|t|\geq 6$ and $|y|\geq 4$ and let $(x,y)\in\mathbb{Z}_k^2$ be a solution of* (1.1) *with $x\notin\{0,-y,ty\}$. Then*

$$|y|\geq 0.32706|t|.$$

Corollary 3.5. *Let $(x,y)\in\mathbb{Z}_k^2$ be a solution of* (1.1) *with $x\notin\{0,-y,ty\}$. For $\Re t=-\frac{1}{2}$ we get the sharper estimate*

$$|y|\geq 0.33576|t|.$$

4. An upper bound for Λ_j

In order to get an upper estimate for Λ_j defined in (1.9) we start from the equality (compare [2, Equation (14)])

$$|\gamma^{(l)}| = |x - \alpha^{(l)}y| = |y||\alpha^{(j)} - \alpha^{(l)}| \left|1 + \frac{x/y - \alpha^{(j)}}{\alpha^{(j)} - \alpha^{(l)}}\right|$$

valid for $l \in \{j+1, j+2\}$. Using Lemma 3.2 we find that for j

$$\left|\frac{x/y - \alpha^{(j)}}{\alpha^{(j)} - \alpha^{(l)}}\right| \leq \frac{3.95297}{|y|^3} \frac{1}{|\alpha^{(2)} - \alpha^{(3)}|} \leq \frac{4.11349}{|y|^3}.$$

Therefore,

$$\log|\gamma^{(l)}| = \log|y| + \log|\alpha^{(j)} - \alpha^{(l)}| + R_l \tag{4.1}$$

where $|R_l| \leq \frac{4.39604}{|y|^3}$ for $|y| \geq 4$ (here we use $|\log|1+z|| \leq |\log(1+z)| \leq |\log(1-|z|)| = \log\frac{1}{1-|z|} \leq \frac{|z|}{1-|z|}$ for $|z| \leq 1$).

Taking the difference of the instances $l = j+1$ and $l = j+2$ of (4.1) we arrive at the following estimate.

Lemma 4.1. *For $|t| \geq 6$, $|y| \geq 4$, $(x, y) \in \mathbb{Z}_k^2$ a solution of (1.1), $x \notin \{0, -y, ty\}$ and j defined in Lemma 3.2 we have*

$$|\Lambda_j| \leq \frac{8.79208}{|y|^3}.$$

For $\Re t = -\frac{1}{2}$, we gain the following result by arguing along the same lines.

Lemma 4.2. *With the conditions of Lemma 4.1 and $\Re t = -\frac{1}{2}$ we get*

$$|\Lambda_j| \leq \frac{8.03434}{|y|^3}.$$

5. Almost solution-free regions

Lemma 5.1. *Let Λ_j be defined as in (1.9) and let (x, y) be a solution of (1.1). If $\gamma = x - \alpha y$ is not associated to an integer in $\mathbb{Z}_k$ then we have*

$$|\Lambda_j| = |\log|\delta_j| + A_j \log|\alpha| + B_j \log|\alpha + 1||$$

with $A_j, B_j \in \mathbb{Z}$ and with δ_j being quotient of two cyclically succeeding elements in one of the triples written in Theorem 2.1.

PROOF. Let Λ_j be defined as in (1.9) and γ as in (2.4). Inserting this representation of γ into (1.9) we get

$$\Lambda_j = \log\left|\frac{\beta^{(j+1)}}{\beta^{(j+2)}}\right| + \log\left|\frac{\alpha^{(j+1)b_1}(\alpha^{(j+1)}+1)^{b_2}}{\alpha^{(j+2)b_1}(\alpha^{(j+2)}+1)^{b_2}}\right| - \log\left|\frac{\alpha^{(j)}-\alpha^{(j+1)}}{\alpha^{(j)}-\alpha^{(j+2)}}\right|.$$

Since $\alpha^{(j)}, \alpha^{(j+1)}, \alpha^{(j+2)}$ can be written in terms of α and $\alpha+1$, we get the result. □

Corollary 5.2. *Let the assumptions of Lemma 5.1 be in force. If* $\Re t = -\frac{1}{2}$ *then*

$$|\Lambda_j| = |\log|\delta_j| + C_j \log|\alpha||$$

with $C_j \in \mathbb{Z}$ *and where* δ_j *is the quotient of two cyclically succeeding elements in one of the triples in the list of Corollary 2.3.*

PROOF. Note that for $\Re t = -\frac{1}{2}$ we have $|\alpha| = |\alpha+1|$ (compare [2, Section 7]). □

In the remaining part of this paper we will always assume that $\Re t = -\frac{1}{2}$. In this special case we will solve the Thue equation (1.1) completely. At first we will show that under some special restrictions on x and y, these values do not yield a solution of the Thue equation. After that we will consider all pairs (x, y) which do not fulfil these restrictions with respect to their contribution to the set of solutions of the Thue equation (1.1).

In the following we combine the results of Section 2 and Section 4 in order to determine all solutions $(x, y) \in \mathbb{Z}_k^2$ of the relative Thue equation (1.1) with the following properties:

$$\begin{cases} \bullet\ \ |y| \geq 4 \text{ (for } |t| \geq 6) \text{ resp. } |y| \geq 7 \text{ (for } |t| < 6), \\ \bullet\ \ x \notin \{0, -y, ty\} \text{ and} \\ \bullet\ \ \gamma = x - \alpha y \text{ is not associated to an integer.} \end{cases} \tag{5.1}$$

For this purpose all instances of β in the list occurring in Corollary 2.3 have to be considered. Inserting all instances of β into the representation of $|\Lambda_j|$ of Corollary 5.2 and using the upper bound of $|\Lambda_j|$ of Lemma 4.2 as well as the above properties we first prove that we always get a contradiction iff $\Lambda_j \neq 0$. Therefore under the present assumptions only instances of β yielding $\Lambda_j = 0$ may lead to solutions of the Thue equation (1.1) and will have to be considered later on.

(i) $\{\alpha-1, 2\alpha+1, \alpha+2\}$ (this triple occurs for all D and arbitrary t)

(i.1) $\beta^{(j+1)} = \alpha - 1 : \delta_j = \frac{\beta^{(j+1)}}{\beta^{(j+2)}} = \frac{\alpha-1}{\alpha^{(2)}-1} = -\alpha\frac{\alpha-1}{2\alpha+1}$,
let $\tilde{\delta_j} = \frac{\alpha-1}{2\alpha+1} = \frac{1}{2}\left(1 - \frac{3}{2\alpha+1}\right)$.
Then

$$\log|\tilde{\delta_j}| = -\log 2 + \log\left|1 - \frac{3}{2\alpha+1}\right|.$$

Using (2.6) this yields

$$\log|\tilde{\delta_j}| = -\log 2 + L_6\left(\frac{2.489}{|t|}\right).$$

Applying Lemma 4.2 we get for $|y| \geq 4$

$$|\Lambda_j| = \left|(C_j+1)\log|\alpha| - \log 2 + L_6\left(\frac{2.489}{|t|}\right)\right| \leq \frac{8.03434}{|y|^3} \leq 0.12554.$$

For $|t| \geq 6$ the left hand side gets minimal for $C_j + 1 = 0$ in which case it is bounded from below by 0.2784. This yields a contradiction to Lemma 4.2.

(i.2) $\beta^{(j+1)} = 2\alpha + 1 : \delta_j = \frac{\beta^{(j+1)}}{\beta^{(j+2)}} = \frac{2\alpha+1}{2\alpha^{(2)}+1} = -\alpha\frac{2\alpha+1}{\alpha+2}$, let $\tilde{\delta_j} = \frac{2\alpha+1}{\alpha+2}$.
Since $|\alpha+2| = |\alpha-1|$, the modulus $|\tilde{\delta_j}|^{-1}$ is the same as the modulus $\tilde{\delta_j}$ in case (i.1). Thus it can be treated in the same way as case (i.1).

(i.3) $\beta^{(j+1)} = \alpha + 2 : \delta_j = \frac{\beta^{(j+1)}}{\beta^{(j+2)}} = \frac{\alpha+2}{\alpha^{(2)}+2} = -\alpha\frac{\alpha+2}{\alpha-1}$, let $\tilde{\delta_j} = \frac{\alpha+2}{\alpha-1}$.
Since $|\alpha+2| = |\alpha-1|$, $|\tilde{\delta_j}| = 1$ and $\log|\tilde{\delta_j}| = 0$.
Using Lemma 4.2 we get for $|y| \geq 4$

$$|\Lambda_j| = |(C_j+1)\log|\alpha|| \leq 0.12554.$$

Again we get an obvious contradiction for $C_j + 1 \neq 0$ and $|t| \geq 6$. For $C_j + 1 = 0$ we gain $\Lambda_j = 0$. So this instance has to be inserted into the list for further consideration.

(ii) $\{\alpha + 1 - b, b\alpha + 1, (1-b)\alpha - b\}$ ($D = 3, 7$ and t with $|t| \geq 6$ as well as $D = 11$ and $t = -\frac{1}{2} + \frac{5i\sqrt{11}}{2}$)

(ii.1) $\beta^{(j+1)} = \alpha + 1 - b : \delta_j = \frac{\beta^{(j+1)}}{\beta^{(j+2)}} = \frac{\alpha+1-b}{\alpha^{(2)}+1-b} = -\alpha\frac{\alpha+1-b}{b\alpha+1}$, let $\tilde{\delta_j} = \frac{\alpha+1-b}{b\alpha+1}$.
$\underline{D=3}$: Since $|\alpha+1-b| = |b\alpha+1|$, $|\tilde{\delta_j}| = 1$ and $\log|\tilde{\delta_j}| = 0$.
Using Lemma 4.2 we get for $|y| \geq 4$

$$|\Lambda_j| = |(C_j+1)\log|\alpha|| \leq 0.12554.$$

For $|t| \geq 6$ and $C_j + 1 \neq 0$ we get a contradiction. For $C_j + 1 = 0$ we gain $\Lambda_j = 0$ and this instance has to be noted for further consideration.

$\underline{D=7}$: In this case we may rewrite $\tilde{\delta_j}$ as follows:
$\tilde{\delta_j} = \frac{\alpha+1-b}{b\alpha+1} = \frac{1}{b}\frac{b\alpha+b-b^2}{b\alpha+1} = \frac{1}{b}\left(1+\frac{1}{b\alpha+1}\right)$ using $b^2 = b-2$. Then

$$\log|\tilde{\delta_j}| = -\log|b| + \log\left|1+\frac{1}{b\alpha+1}\right|.$$

Using (2.6) this yields

$$\log|\tilde{\delta_j}| = -\log\sqrt{2} + L_6\left(\frac{1.00965}{|t|}\right).$$

Applying Lemma 4.2 we get for $|y| \geq 4$

$$|\Lambda_j| = \left|(C_j+1)\log|\alpha| - \log\sqrt{2} + L_6\left(\frac{1.00965}{|t|}\right)\right| \leq 0.12554.$$

For $|t| \geq 6$ the left hand side gets minimal for $C_j+1=0$ in which case it is bounded from below by 0.1783. This yields a contradiction to Lemma 4.2.
$\underline{D=11 \text{ and } t = -\frac{1}{2}+\frac{5i\sqrt{11}}{2}}$: In this case we rewrite $\tilde{\delta_j}$ as follows:
$\tilde{\delta_j} = \frac{\alpha+1-b}{b\alpha+1} = \frac{1}{b}\frac{b\alpha+b-b^2}{b\alpha+1} = \frac{1}{b}\left(1+\frac{2}{b\alpha+1}\right)$ using $b^2 = b-3$. Then

$$\log|\tilde{\delta_j}| = -\log|b| + \log\left|1+\frac{2}{b\alpha+1}\right|.$$

Since we know t, we can compute the exact value of α. This yields

$$\log|\tilde{\delta_j}| = -\log\sqrt{3} + \log 0.85158.$$

Using Lemma 4.2 we get for $|y| \geq 4$

$$|\Lambda_j| = \left|(C_j+1)\log 8.05941 - \log\sqrt{3} + \log 0.85158\right| \leq 0.12554.$$

The left hand side gets minimal for $C_j+1=0$ in which case it is bounded from below by 0.70997. This yields a contradiction to Lemma 4.2 again.

(ii.2) $\beta^{(j+1)} = b\alpha+1 : \delta_j = \frac{\beta^{(j+1)}}{\beta^{(j+2)}} = \frac{b\alpha+1}{b\alpha^{(2)}+1} = \alpha\frac{b\alpha+1}{(1-b)\alpha-b}$, let $\tilde{\delta_j} = \frac{b\alpha+1}{(1-b)\alpha-b}$.
Since $|b\alpha+1| = |(1-b)\alpha-b|$, $|\tilde{\delta_j}| = 1$ and $\log|\tilde{\delta_j}| = 0$.
Using Lemma 4.2 we get for $|y| \geq 4$

$$|\Lambda_j| = |(C_j+1)\log|\alpha|| \leq 0.12554.$$

For $|t| \geq 6$ and $C_j+1 \neq 0$ we get a contradiction. For $C_j+1=0$ we gain $\Lambda_j = 0$ and the instance enters into the list below.

(ii.3) $\beta^{(j+1)} = (1-b)\alpha - b$: $\frac{\beta^{(j+1)}}{\beta^{(j+2)}} = \frac{(1-b)\alpha-b}{(1-b)\alpha^{(2)}-b} = -\alpha\frac{(1-b)\alpha-b}{\alpha+1-b}$, $\tilde{\delta_j} = \frac{(1-b)\alpha-b}{\alpha+1-b}$. Since $|b\alpha+1| = |(1-b)\alpha-b|$, the modulus $|\tilde{\delta_j}|^{-1}$ is the same as the modulus $\tilde{\delta_j}$ in case (ii.1). Thus it can be treated in the same way as case (ii.1).

Each triple occurring in Corollary 2.3 can be treated in the same way as cases (i) or (ii). Triples which only appear for one special t can be treated analogously to case (ii.1), $D = 11$ and $t = -\frac{1}{2} + \frac{5i\sqrt{11}}{2}$, since in this case we can compute the root α of f_t exactly.

We now deal with the cases that yield $\Lambda_j = 0$. Essentially these cases can be subdivided into two subclasses. The cases including infinite families of parameters t and the cases including only finitely many parameters. As one naturally expects, the first subclass is more difficult to settle. For this reason we start with summing up all cases belonging to this subclass in Table 4.

	Discriminant	t	β
Class 1	all D	all t	$\alpha+2$
Class 2	$D=3$	$t=-\frac{1}{2}+\frac{ci\sqrt{3}}{2}$	$\alpha+1-b$
Class 3		$t=-\frac{1}{2}+\frac{ci\sqrt{3}}{2}$	$\alpha+b$
Class 4		$t=-\frac{1}{2}+\frac{ci\sqrt{3}}{2}$	$\alpha+1+b$
Class 5		$t=-\frac{1}{2}+\frac{ci\sqrt{3}}{2}$	$\alpha+2-b$
Class 6	$D=7$	$t=-\frac{1}{2}+\frac{ci\sqrt{7}}{2}$	$b\alpha+1$

Table 4.

Now we show how these infinite families can be treated. First note that by [1, Lemma 4.5] it is sufficient to look for solutions (x, y) satisfying

$$\gamma^{(3)} = |x - \alpha^{(3)}y| = \min_{i\in\{1,2,3\}} \gamma_i, \tag{5.2}$$

i.e., $j = 3$ provided that $|t| \geq 6$ (and $|y| \geq 3$; "small" t will be treated later on). Thus

$$\begin{aligned}\Lambda_j = \Lambda_3 &= \log\left|\frac{\beta}{\beta^{(2)}}(\alpha^{(3)})^{1-b_1-2b_2}(\alpha^{(3)}+1)^{2b_1+b_2-1}\right| \\ &= \log\left|\frac{\beta}{\beta^{(2)}}\alpha^{2b_1+b_2-1}(\alpha+1)^{-b_1+b_2}\right|.\end{aligned}$$

In the above table all instances have $\beta = u + v\alpha$ with $u, v \in \mathbb{Z}_k$. Thus

$$\Lambda_3 = \log\left|\frac{u+v\alpha}{v+(v-u)\alpha}\right| + (b_1+2b_2)\log|\alpha|$$

where the first logarithm equals 0 and $b_1 + 2b_2 = 0$, too. Therefore $b_1 = -2b_2$, and with $m = -b_2$ we obtain

$$\gamma = \gamma_m = \mu\beta \left(\frac{\alpha^2}{\alpha+1}\right)^m \qquad (m \in \mathbb{Z}). \tag{5.3}$$

It is easily checked that (5.2) implies that $m \geq 0$ if $|t| \geq 6$.

Observe that solutions of the diophantine equation correspond to values m where $\gamma_m = x - \alpha y$, $x, y \in \mathbb{Z}_k$, i.e., those instances where $x_3(m) = 0$ in the representation

$$\gamma_m = x_1(m) + x_2(m)\alpha + x_3(m)\alpha^2 \qquad (x_\ell(m) \in \mathbb{Z}_k)$$

of γ_m. Regarding $\frac{\alpha^2}{\alpha+1} = -\alpha^2 + (t+1)\alpha + 1$, we easily find

$$\mathbf{x}(m) = (x_\ell(m))_{1\leq \ell \leq 3} = A^m \mathbf{x}(0) \tag{5.4}$$

with

$$A = \begin{pmatrix} 1 & -1 & 2 \\ t+1 & -t-1 & 2t+3 \\ -1 & 2 & t-3 \end{pmatrix} \quad \text{and} \quad \mathbf{x}(0) = \mu \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}. \tag{5.5}$$

Denoting

$$p_m := \begin{cases} \dfrac{x_3(m)}{\mu(2t+1)}, & \text{for } \beta = \alpha + 2, \\ \dfrac{x_3(m)}{\mu(2u-v)}, & \text{otherwise} \end{cases} \tag{5.6}$$

a short calculation shows that the following recurrence holds (in $\mathbb{Z}$):

$$p_m + 3p_{m-1} + Lp_{m-2} + p_{m-3} = 0 \qquad (m \geq 3). \tag{5.7}$$

Here $L := -(t^2 + t + 4) \in \mathbb{Z}$, $p_0 = 0$, $p_1 \in \{0, 1\}$ and $p_2 = p_2(c) \in \mathbb{Z}$ (recall that c is defined by $t = \frac{-1+ic\sqrt{D}}{2}$) depending on u and v.

The solution of (5.7) is a sequence of polynomials

$$p_m = p_m(L, c)$$

with coefficients in $\mathbb{Z}$. Considering (5.7) we find

$$p_m(L, c) \equiv p_m(3, c) \bmod (L-3). \tag{5.8}$$

Computing

$$p_m(3, c) = (-1)^m \left(\frac{p_2(c) + 2p_1}{2} m^2 - \frac{p_2(c) + 4p_1}{2} m\right)$$

we get the following result.

Lemma 5.3. *Let $p_m = p_m(L, c)$ be defined by (5.7). Then γ_m corresponds to a solution of the diophantine equation if and only if $p_m(L, c) = 0$. Furthermore, in this instance $L - 3 = |t|^2 - 7$ must fulfil*

$$|t|^2 - 7 \,\Big|\, \frac{p_2(c) + 2p_1}{2} m^2 - \frac{p_2(c) + 4p_1}{2} m.$$

The divisibility relation contained in Lemma 5.3 yields the following lower estimates for those m corresponding to a solution of the diophantine equation.

- *Class 1:* $\beta = \alpha + 2$ and D is arbitrary. Then $p_1 = 0$, $p_2 = 1$ and $|t|^2 - 7 \mid \frac{m(m-1)}{2}$ implies

$$m = 0 \text{ or } m = 1 \text{ or } (m \geq 2 \text{ and } m \geq 1.4121020211|t|) \quad \text{for } |t| \geq 48. \tag{5.9}$$

- *Class 2:* $\beta = \alpha + 1 - b$ and $D = 3$. Then $p_1 = 1$, $p_2 = \frac{c-7}{2}$ with $c \geq 5$. After a short calculation we gain

$$m = 0 \text{ or } (m = 2 \text{ and } c = 7) \text{ or } (m \geq 3 \text{ and } m \geq 1.8612097182|t|^{1/2}) \quad \text{for } |t| \geq 48. \tag{5.10}$$

- *Class 3:* $\beta = \alpha + b$ and $D = 3$. Then $p_1 = 1$, $p_2 = -\frac{c+7}{2}$, $c \neq 3$ and we gain

$$m = 0 \textit{ or } (m \geq 1 \text{ and } m \geq 2^{1/2}3^{1/4}|t|^{1/2} - \sqrt{3} \geq 1.6112097|t|^{1/2}) \quad \text{for } |t| \geq 48. \tag{5.11}$$

- *Class 4:* $\beta = \alpha + 1 + b$ and $D = 3$. Then $p_1 = 1$, $p_2 = -\frac{3c+7}{2}$, $c \geq 5$ and we gain

$$m = 0 \text{ or } (m \geq 4 \text{ and } m \geq 2^{1/2}3^{-1/4}|t|^{1/2} \geq 1.07456993182|t|^{1/2}) \quad \text{for } |t| \geq 48. \tag{5.12}$$

- *Class 5:* $\beta = \alpha + 2 - b$ and $D = 3$. Then $p_1 = 1$, $p_2 = \frac{3c-7}{2}$, $c \neq 3$ and we gain

$$m = 0 \text{ or } (m \geq 4 \text{ and } m \geq 2^{1/2}3^{-1/4}|t|^{1/2} \geq 1.07456993182|t|^{1/2}) \quad \text{for } |t| \geq 48. \tag{5.13}$$

- *Class 6:* $\beta = b\alpha + 1$ and $D = 7$. Then $p_1 = 1$, $p_2 = -\frac{c-7}{2}$ and we gain

$$m = 0 \text{ or } (m = 2 \text{ and } c = 7) \text{ or } (m \geq 3 \text{ and } m \geq 2^{1/2}7^{1/4}|t|^{1/2} \geq 2.30032663379|t|^{1/2}) \quad \text{for } |t| \geq 48. \tag{5.14}$$

Observe that all instances $m = 0$, i.e., $\gamma = \mu(u + v\alpha)$ correspond to values y with $|y| < 4$, in contradiction to our assumption $|y| \geq 4$ in this section.

Class 1 with $m = 1$ yields $\gamma = \mu(\alpha+2)\frac{\alpha^2}{\alpha+1} = 1+(t+1)\alpha$ which corresponds to the solution $(1, -1-t)$ (this is solution [D.2] in Theorem 9.2).

Class 2 with $m = 2$ and $c = 7$, i.e., $t = \frac{-1+7i\sqrt{3}}{2}$ yields $\gamma = \mu(\alpha+1-b)\big(\frac{\alpha^2}{\alpha+1}\big)^2$ which corresponds to the solution $(5(1-b), -(19+14b))$ listed in Theorem 9.2 as solution [3.5].

Moreover, Class 6 with $m = 2$ and $c = 7$, i.e., $t = \frac{-1+7i\sqrt{7}}{2}$ yields $\gamma = \mu(b\alpha + 1)\big(\frac{\alpha^2}{\alpha+1}\big)^2$ which corresponds to the solution $(-12, -35 + 82b)$ listed in Theorem 9.2 as solution [7.2].

In what follows we combine the above estimates for m in the remaining cases with the estimate contained in [1, Theorem 1] in order to reduce the possible values of "large" t such that γ_m may correspond to a solution of the diophantine equation (1.1) to a finite set.

Let $m \geq 2$ and

$$\gamma = \gamma_m = \mu(u + v\alpha)\left(\frac{\alpha^2}{\alpha+1}\right)^m = x - \alpha y. \tag{5.15}$$

Then

$$|x - \alpha^{(3)}y| = \frac{|u - v + u\alpha|}{|\alpha|^{2m+1}} \leq \frac{2}{|\alpha|^{2m}} \tag{5.16}$$

holds for $|t| \geq 6$ for Class 1 to Class 6 ($\beta = u + v\alpha$). Furthermore, denoting $\rho = \frac{x/y - \alpha^{(3)}}{\alpha - \alpha^{(3)}}$, we have

$$\gamma = x - \alpha y = y(\alpha - \alpha^{(3)})(1 + \rho).$$

Using $|\alpha - \alpha^{(3)}| = \frac{|\alpha|^2 - 1}{|\alpha|}$ we obtain

$$|y| = \frac{|u + v\alpha||\alpha|}{(|\alpha|^2 - 1)|1 + \rho|}|\alpha|^m \leq \frac{|u||\alpha| + |v|}{|\alpha| - 1}\,\frac{1}{|1 + \rho|}|\alpha|^m. \tag{5.17}$$

Now

$$\frac{1}{|\alpha - \alpha^{(3)}|} = \frac{|\alpha|}{|\alpha|^2 - 1} \leq \frac{1}{|\alpha| - 1} \leq \frac{1}{0.99912221198|t|} \tag{5.18}$$

where the last inequality is valid for $|t| \geq 48$ because of the representation of α in terms of t in (2.6). Combining (5.18) with Corollary 3.3 we have

$$\left|\frac{1}{1 + \rho}\right| \leq 1.00121012929$$

for $|y| \geq 4$, $|t| \geq 48$. Thus the following upper estimates for $|y|$ follow for $|t| \geq 48$.

- *Classes 1 to 5:*
$$|y| \le 1.065174501|\alpha|^m. \tag{5.19}$$
- *Class 6:*
$$|y| \le 1.467399495|\alpha|^m. \tag{5.20}$$

Now we are ready to make use of [1, Theorem 1] which states that

$$\left|\alpha^{(3)} - \frac{x}{y}\right| > \frac{1}{746|t||y|^{\kappa+1}} \tag{5.21}$$

for $x, y \in \mathbb{Z}_k$ with
$$|y| \ge 0.0773|t|, \tag{5.22}$$
where
$$\kappa \le 1 + \frac{2.13}{\log|t|} + \frac{6.8}{\log^2|t|} \tag{5.23}$$

(observe that, by (5.17), (5.22) holds for $m \ge 1$ and $|t| \ge 6$). Combining (5.16) and (5.21) yields
$$\frac{1}{746|t||y|^\kappa} < \left|x - \alpha^{(3)}y\right| \le \frac{2}{|\alpha|^{2m}}$$
so that $|\alpha|^{2m} < 1492|t||y|^\kappa$. Using (5.19), (5.20) and (2.6) this implies

$$|\alpha|^{(2-\kappa)(m-1)} < \begin{cases} 1493.31082(1.06517501)^\kappa & \text{in Classes 1 to 5,} \\ 1493.31082(1.46739949)^\kappa & \text{in Class 6.} \end{cases} \tag{5.24}$$

Now we may insert the lower estimates for m in terms of $|t|$ from (5.9) to (5.14) to find, together with (5.23), the following estimate for $|t|$ (since otherwise (5.24) leads to a contradiction).

- *Class 1:* $|t| \le 54$, i.e., $c^2D < 11663$.
- *Class 2:* $|t| \le 81$, i.e., $c \le 93$.
- *Class 3:* $|t| \le 86$, i.e., $c \le 99$.
- *Class 4:* $|t| \le 108$, i.e., $c \le 123$.
- *Class 5:* $|t| \le 108$, i.e., $c \le 123$.
- *Class 6:* $|t| \le 76$, i.e., $c \le 57$.

Thus it remains to consider the case $\Lambda_j = 0$ only for these finitely many cases as well as for the (finitely many) cases coming from instances $\Lambda_j = 0$ having $|t| < 6$.

For all triples where $|t| < 6$, the upper bound for $|\Lambda_j|$ of Lemma 4.1 is not valid anymore, but Λ_j can be computed according to Section 3 and Section 4 for each t with $|t| < 6$ individually: We insert the value of a special t into (3.2) to get an upper bound for $\left|\frac{\gamma^{(j)}}{y}\right|$ which can be improved using bootstrapping. Using this result and (4.1) we finally gain an upper bound for $|\Lambda_j|$. In this case we achieve a contradiction for all instances of β, if we assume $|y| \geq 7$, again except some cases where $\Lambda_j = 0$.

Finally we end up with the following complete list of instances of $\beta = \beta^{(j+1)}$ which yield $\Lambda_j = 0$ in the considerations described above. (see Tables 5 to 14; the meaning of the column "corresponding equation" will be explained later.)

for all D			
t	β	$N_{k(\alpha)/k}(\beta)$	Corresponding Equation
$t = -\frac{1}{2} + \frac{ci\sqrt{D}}{2}$ with $c^2 D < 11663$	$\alpha+2$	$2t+1$	$D(p^3 + xp^2c) + 9x^2p + x^3c = \pm 8c$

Table 5.

In order to treat the finitely many remaining cases contained in this table we employ the computer algebra system KASH3 [4]. To do this it is necessary to transform the relative diophantine equations under discussion into real ones. The transformed equations are contained in the above table. We demonstrate the transformation process for the following two instances.

Case $\beta = \alpha + 2$: Then

$$x(\alpha+1) + y = (\alpha+1)(x - \alpha^{(3)}y) = (\alpha+1)\gamma_m^{(3)} = \frac{2\alpha+1}{|\alpha|^{2m}}(-1)^m \in i\mathbb{R}.$$

A short calculation shows $x \in \mathbb{Z}$ and $\Re y = -\frac{x}{2}$, so that

$$y = -\frac{x}{2} + i\sqrt{D}\frac{p}{2} = -\frac{x}{2} + \left(t + \frac{1}{2}\right)\frac{p}{c}, \qquad p \in \mathbb{Z}.$$

Therefore the relative Thue equation (1.1) reads

$$F(x,y) = \frac{2t+1}{8c}(D(p^3 + xp^2c) + 9xp^2 + x^3c) = \pm(2t+1).$$

(Note that the unit μ on the right hand side of the equation has to be ± 1 since the left hand side of the equation is real.) Therefore, the real equation

$$D(p^3 + xp^2c) + 9xp^2 + x^3c = \pm 8c$$

remains to be solved.

$D=3$			
t	β	$N_{k(\alpha)/k}(\beta)$	Corresponding Equation
$t=-\frac{1}{2}+\frac{ci\sqrt{3}}{2}$ with $c\le 93$	$\alpha+1-b$	$-\frac{2t+1}{2}+\frac{3i\sqrt{3}}{2}$	$3(p^3+\tilde{x}p^2c)+9\tilde{x}^2p+\tilde{x}^3c$ $=\pm 4(c-3)$ with $\tilde{x}=bx$
$t=-\frac{1}{2}+\frac{ci\sqrt{3}}{2}$ with $c\le 99$	$\alpha+b$	$-\frac{2t+1}{2}-\frac{3i\sqrt{3}}{2}$	$3(p^3+\tilde{x}p^2c)+9\tilde{x}^2p+\tilde{x}^3c$ $=\pm 4(c+3)$ with $\tilde{x}=(1-b)x$
$t=-\frac{1}{2}+\frac{ci\sqrt{3}}{2}$ with $c\le 123$	$\alpha+1+b$	$(2t+1)\frac{i\sqrt{3}}{2}-\frac{7}{2}$	$3(p^3+\tilde{x}p^2c)+9\tilde{x}^2p+\tilde{x}^3c=$ $\pm 12(3c+7)$ with $\tilde{x}=(2-b)x$
$t=-\frac{1}{2}+\frac{ci\sqrt{3}}{2}$ with $c\le 123$	$\alpha+2-b$	$(2t+1)\frac{i\sqrt{3}}{2}+\frac{7}{2}$	$3(p^3+\tilde{x}p^2c)+9\tilde{x}^2p+\tilde{x}^3c=$ $\pm 12(3c-7)$ with $\tilde{x}=(1+b)x$
$-\frac{1}{2}+\frac{5i\sqrt{3}}{2}$	$(1-3b)\alpha-1$	$-4i\sqrt{3}$	$3(p^3+5xp^2)+9x^2p+5x^3=\pm 32$
	$(\alpha+1-b)^2$	-3	$45\tilde{x}^3-27\tilde{p}\tilde{x}^2+15\tilde{p}^2\tilde{x}-\tilde{p}^3=$ ± 24 with $\tilde{p}=\frac{p}{1-b},\tilde{x}=-\frac{xi\sqrt{3}}{3}$
	$\alpha^2+(3-4b)\alpha+1-2b$	-4	$45\tilde{x}^3-27\tilde{p}\tilde{x}^2+15\tilde{p}^2\tilde{x}-\tilde{p}^3=$ ± 32 with $\tilde{p}=\frac{p}{b},\tilde{x}=-\frac{xi\sqrt{3}}{3}$
	$(2b-1)\alpha^2+(4b-1)\alpha+1$	-4	$45\tilde{x}^3-27p\tilde{x}^2+15p^2\tilde{x}-p^3=$ ± 32 with $\tilde{x}=-\frac{xi\sqrt{3}}{3}$
	$b\alpha^2+(4-2b)\alpha+3-3b$	5	$45\tilde{x}^3-27\tilde{p}\tilde{x}^2+15\tilde{p}^2\tilde{x}-\tilde{p}^3=$ ± 40 with $\tilde{p}=\frac{p}{b},\tilde{x}=-\frac{xi\sqrt{3}}{3}$
	$\alpha^2+(4-6b)\alpha-1-b$	8	$45\tilde{x}^3-27\tilde{p}\tilde{x}^2+15\tilde{p}^2\tilde{x}-\tilde{p}^3=$ ± 64 with $\tilde{p}=\frac{p}{1-b},\tilde{x}=-\frac{xi\sqrt{3}}{3}$
$-\frac{1}{2}+\frac{7i\sqrt{3}}{2}$	$(\alpha+1-b)^2$	-12	$63\tilde{x}^3-27\tilde{p}\tilde{x}^2+21\tilde{p}^2\tilde{x}-\tilde{p}^3=$ ± 96 with $\tilde{p}=\frac{p}{1-b},\tilde{x}=-\frac{xi\sqrt{3}}{3}$

Table 6.

Case $D=3$, $\beta=\alpha+1-b$: Then

$$\frac{1}{1-b}(x(\alpha+1)+y)=\frac{\alpha+1-b}{|\alpha|^{2m}}(-1)^m\in i\mathbb{R}.$$

It follows that $\tilde{x}:=bx\in\mathbb{Z}$ and $\Re(by)=-\frac{bx}{2}$ so that

$$y=-\frac{x}{2}+\left(t+\frac{1}{2}\right)\frac{p}{bc},\quad p\in\mathbb{Z}.$$

$D=7$			
t	β	$N_{k(\alpha)/k}(\beta)$	Corresponding Equation
$t=-\frac{1}{2}+\frac{ci\sqrt{7}}{2}$ with $c\le 57$	$b\alpha+1$	$2t+1-2i\sqrt{7}$	$7(p^3+xp^2c)+9x^2p+x^3c=\pm 8(c-2)$
$-\frac{1}{2}+\frac{3i\sqrt{7}}{2}$	$2\alpha+1-b$	$-\frac{7}{2}-\frac{5i\sqrt{7}}{2}$	$7(p^3+3\tilde{x}p^2)+9\tilde{x}^2p+3\tilde{x}^3=\pm 2$ with $\tilde{x}=\frac{b}{2}x$
	$(2-b)\alpha-b$	$-\frac{7}{2}+\frac{5i\sqrt{7}}{2}$	$7(p^3+3\tilde{x}p^2)+9\tilde{x}^2p+3\tilde{x}^3=\pm 2$ with $\tilde{x}=\frac{1-b}{2}x$
	$\alpha^2+(2-2b)\alpha-b$	-3	$147\hat{x}^3-63p\hat{x}^2+21p^2\hat{x}-p^3=\pm 24$
	$\alpha^2+(3-2b)\alpha-b$	-5	$147\hat{x}^3-63p\hat{x}^2+21p^2\hat{x}-p^3=\pm 40$
	$\alpha^2+(4-2b)\alpha-b$	-7	$147\hat{x}^3-63p\hat{x}^2+21p^2\hat{x}-p^3=\pm 56$
	$(b\alpha+1)^2$	-7	$147\hat{x}^3-63p\hat{x}^2+21p^2\hat{x}-p^3=\pm 56$ with $\hat{x}=-\frac{xi\sqrt{7}}{7}$

Table 7.

$D=11$			
t	β	$N_{k(\alpha)/k}(\beta)$	Corresponding Equation
$-\frac{1}{2}+\frac{3i\sqrt{11}}{2}$	$b\alpha+1$	$2i\sqrt{11}$	$11(p^3+3xp^2)+9x^2p+3x^3=\pm 16$
$-\frac{1}{2}+\frac{5i\sqrt{11}}{2}$	$b\alpha+1$	$5i\sqrt{11}$	$11(p^3+5xp^2)+9x^2p+5x^3=\pm 40$

Table 8.

$D=15$			
t	β	$N_{k(\alpha)/k}(\beta)$	Corresponding Equation
$-\frac{1}{2}+\frac{i\sqrt{15}}{2}$	$\alpha^2+\alpha+1$	3	$225\tilde{x}^3-135p\tilde{x}^2+15p^2\tilde{x}-p^3=\pm 24$ with $\tilde{x}=-\frac{xi\sqrt{15}}{15}$
$-\frac{1}{2}+\frac{3i\sqrt{15}}{2}$	$b\alpha+1$	$3i\sqrt{15}$	$15(p^3+5xp^2)+9x^2p+3x^3=\pm 24$

Table 9.

This yields the real equation

$$3(p^3+\tilde{x}p^2c)+9\tilde{x}p^2+\tilde{x}^3c=\pm 4(c-3).$$

Proceeding in a similar manner all remaining cases of t yield real Thue equations, and we end up with the list given in the above table.

$D = 19$			
t	β	$N_{k(\alpha)/k}(\beta)$	Corresponding Equation
$-\frac{1}{2}+\frac{i\sqrt{19}}{2}$	$\alpha^2+\alpha+1$	2	$361^2\tilde{x}^3 - 171p\tilde{x}^2 + 19p^2\tilde{x} - p^3 = \pm 16$
	$(\alpha^2+\alpha+1)^2$	4	$361^2\tilde{x}^3 - 171p\tilde{x}^2 + 19p^2\tilde{x} - p^3 = \pm 32$
			with $\tilde{x} = -\frac{xi\sqrt{19}}{19}$

Table 10.

$D = 23$			
t	β	$N_{k(\alpha)/k}(\beta)$	Corresponding Equation
$-\frac{1}{2}+\frac{i\sqrt{23}}{2}$	$(1+b)\alpha+3$	$-i\sqrt{23}$	$23(p^3+xp^2)+9x^2p+x^3 = \pm 8$
	$(2+b)\alpha+5$	$-i\sqrt{23}$	$23(p^3+xp^2)+9x^2p+x^3 = \pm 8$

Table 11.

$D = 31$			
t	β	$N_{k(\alpha)/k}(\beta)$	Corresponding Equation
$-\frac{1}{2}+\frac{i\sqrt{31}}{2}$	$(1+b)\alpha+3$	$i\sqrt{31}$	$31(p^3+xp^2)+9x^2p+x^3 = \pm 8$
	$(1+3b)\alpha+5$	$i\sqrt{31}$	$31(p^3+xp^2)+9x^2p+x^3 = \pm 8$
	$2\alpha^2+\alpha+1$	-3	$961\tilde{x}^3 - 279p\tilde{x}^2 + 31p^2\tilde{x} - p^3 = \pm 24$
	$\alpha^2+2\alpha+2$	-3	$961\tilde{x}^3 - 279p\tilde{x}^2 + 31p^2\tilde{x} - p^3 = \pm 24$
	$\alpha^2+b\alpha+1+b$	-3	$961\tilde{x}^3 - 279p\tilde{x}^2 + 31p^2\tilde{x} - p^3 = \pm 24$
			with $\tilde{x} = -\frac{xi\sqrt{31}}{31}$

Table 12.

$D = 35$			
t	β	$N_{k(\alpha)/k}(\beta)$	Corresponding Equation
$-\frac{1}{2}+\frac{i\sqrt{35}}{2}$	$4\alpha^2+\alpha+1$	-5	$1225\tilde{x}^3 - 315p\tilde{x}^2 + 35p^2\tilde{x} - p^3 = \pm 40$
	$\alpha^2+2\alpha+2$	-5	$1225\tilde{x}^3 - 315p\tilde{x}^2 + 35p^2\tilde{x} - p^3 = \pm 40$
	$\alpha^2+\alpha+1$	-2	$1225\tilde{x}^3 - 315p\tilde{x}^2 + 35p^2\tilde{x} - p^3 = \pm 16$
	$(\alpha^2+\alpha+1)^2$	4	$1225\tilde{x}^3 - 315p\tilde{x}^2 + 35p^2\tilde{x} - p^3 = \pm 32$
			with $\tilde{x} = -\frac{xi\sqrt{35}}{35}$

Table 13.

$D = 39, 43, 47, 51, 55$			
t	β	$N_{k(\alpha)/k}(\beta)$	Corresponding Equation
$-\frac{1}{2}+\frac{i\sqrt{D}}{2}$	$\alpha^2+\alpha+1$	t^2+t+7	$D^2\tilde{x}^3 - 9Dp\tilde{x}^2 + Dp^2\tilde{x} - p^3 = \pm 8(t^2+t+7)$
			with $\tilde{x} = -\frac{xi\sqrt{D}}{D}$

Table 14.

Applying KASH3 we gain the solutions [3.5], [7.2], [19.1], [23.1], [43.1] and [51.1] contained in the tables in Theorem 9.2.

6. The case of small y

Let $|t| \geq 6$ and $|y| < 4$. Using the notation (3.5) we get from (3.4)

$$\left|x - \lfloor\alpha^{(j)}\rceil y\right| \leq \frac{2.9482}{|y|} + |\{\alpha^{(j)}\}||y|.$$

Applying the expansion (2.6) we find for the "fractional" parts of $\alpha^{(j)}$ for $|t| \geq 6$

$$|\{\alpha\}| \leq 0.383694, \quad |\{\alpha^{(2)}\}| \leq 0.178761, \quad |\{\alpha^{(3)}\}| \leq 0.202854.$$

Therefore we obtain the following estimates for $\left|x - \lfloor\alpha^{(j)}\rceil y\right|$:

$$\begin{aligned}
|x - \lfloor\alpha\rceil y| &\leq \frac{2.9482}{|y|} + 0.383694|y|,\\
\left|x - \lfloor\alpha^{(2)}\rceil y\right| &\leq \frac{2.9482}{|y|} + 0.178761|y|,\\
\left|x - \lfloor\alpha^{(3)}\rceil y\right| &\leq \frac{2.9482}{|y|} + 0.202854|y|.
\end{aligned}$$

Using Corollary 3.5 we get for $|t| \geq 6$ that $|y| \geq 2.01456 \geq 2$ and so the above estimates imply for $2 \leq |y| < 4$:

$$\left|x - \lfloor\alpha\rceil y\right| \leq 3.00888, \quad \left|x - \lfloor\alpha^{(2)}\rceil y\right| \leq 2.18914, \quad \left|x - \lfloor\alpha^{(3)}\rceil y\right| \leq 2.28552.$$

Now we compute all x and y which fulfil the above inequalities and insert them into the Thue equation (1.1). The right hand side ℓ of the Thue equation runs through the column $N_{k(\alpha)/k}(\gamma)$ in Corollary 2.3, since these are the only cases where the inequality $|F_t(x,y)| = |\ell| = |N_{k(\alpha)/k}(x - \alpha y)| \leq |2t+1|$ holds. We solve equation (1.1) for all possible values x, y and ℓ in terms of t and check if $t \in \mathbb{Z}_k$ and $\Re t = -\frac{1}{2}$ holds using `Mathematica`®. Proceeding like that we get the following pairs of solutions (x,y) of $F_t(x,y) = \ell$ with $|y| < 4$ and $|t| \geq 6$ written in Theorem 9.2:

$D \equiv 3 \pmod 4$: $[D.1]$.

$D = 3$: In addition to $[D.1]$ we get the solutions: [3.1], [3.2], [3.3], [3.4], [3.6], [3.7].

$\underline{D=7:}$ In addition to $[D.1]$ we get the solutions: [7.1], [7.3], [7.4].

$\underline{D=11:}$ In addition to $[D.1]$ we get the solution: [11.1].

Let $|t|<6$ and $|y|<7$. Using inequality (3.2) we have

$$|x|-|\alpha^{(j)}||y| \le |x-\alpha^{(j)}y| \le \left(\frac{(3\sqrt{3})^2|\ell|^4}{|t^2+t+7|^2}\right)^{1/6}\frac{1}{|y|}$$

and therefore

$$|x| \le \left(\frac{(3\sqrt{3})^2|\ell|^4}{|t^2+t+7|^2}\right)^{1/6}\frac{1}{|y|} + \max_{j\in\{1,2,3\}}|\alpha^{(j)}||y|.$$

Since we can compute the roots $\alpha^{(j)}$ of $f_t(x)$ for every t with $|t|<6$ explicitly we can insert the exact values of t and $|\alpha^{(j)}|$ into the above inequality and derive an upper bound for $|x|$. Now we can solve the Thue equation (1.1) for all x,y which fulfil the inequalities and all ℓ from Corollary 2.3 (column $N_{k(\alpha)/k}(\gamma)$) in terms of t and check whether $t\in\mathbb{Z}_k$ and $\Re t=-\frac{1}{2}$ holds using `Mathematica`®.

In this manner we get the following pairs of solutions (x,y) of $F_t(x,y)=\ell$ with $|y|<7$ and $|t|<6$ presented in the subsequent Theorem 9.2:

$\underline{D\equiv 3 \pmod 4:}$ $[D.1]$, $[D.2]$.

$\underline{D=3:}$ In addition to $[D.1]$, $[D.2]$ we get the solutions: [3.2], [3.3], [3.4], [3.8], [3.9], [3.10], [3.11], [3.12], [3.13], [3.14] for $t=-\frac{1}{2}+\frac{5i\sqrt{3}}{2}$.

$\underline{D=7:}$ In addition to $[D.1]$, $[D.2]$ we get the solutions: [7.1], [7.5], [7.6], [7.7], [7.8], [7.9], [7.10], [7.11], [7.12] for $t=-\frac{1}{2}+\frac{3i\sqrt{7}}{2}$.

$\underline{D=11:}$ In addition to $[D.1]$, $[D.2]$ we get the solutions: [11.2], [11.3] for $t=-\frac{1}{2}+\frac{3i\sqrt{11}}{2}$, [11.4], [11.5] for $t=-\frac{1}{2}+\frac{i\sqrt{11}}{2}$.

$\underline{D=15:}$ In addition to $[D.1]$, $[D.2]$ we get the solutions: [15.1] for $t=-\frac{1}{2}+\frac{3i\sqrt{15}}{2}$, [15.2] for $t=-\frac{1}{2}+\frac{i\sqrt{15}}{2}$.

$\underline{D=19:}$ In addition to $[D.1]$, $[D.2]$ we get the solutions: [19.2], [19.3] for $t=-\frac{1}{2}+\frac{i\sqrt{19}}{2}$.

$\underline{D=23:}$ In addition to $[D.1]$, $[D.2]$ we get the solutions: [23.2], [23.3], [23.4], [23.5], [23.6], [23.7] for $t=-\frac{1}{2}+\frac{i\sqrt{23}}{2}$.

$\underline{D=31:}$ In addition to $[D.1]$, $[D.2]$ we get the solutions: [31.1], [31.2], [31.3], [31.4] for $t=-\frac{1}{2}+\frac{i\sqrt{31}}{2}$.

$\underline{D=35:}$ In addition to $[D.1]$, $[D.2]$ we get the solutions: [35.1], [35.2] for $t=-\frac{1}{2}+\frac{i\sqrt{35}}{2}$.

7. The exceptional values $x \in \{0, -y, ty\}$

In Section 3, the values $x = 0$, $x = -y$ and $x = ty$ are excluded for $|t| \geq 6$ to avoid that $|x - \lfloor \alpha^{(j)} \rceil y| = 0$. Now we will study these special instances.

In case $x = 0$, the Thue equation (1.1) reads $F_t(0, y) = -y^3 = \ell$. In order to solve this equation we have to compute the cubic roots of $-\ell$. Since these roots have to be elements of $\mathbb{Z}_k$, not all values ℓ of the table of Corollary 2.3 (column $N_{k(\alpha)/k}(\gamma)$) can be taken. The list of all possible ℓ and t can be calculated using a `Mathematica`® program. This program computes for a list of t with $\Re t = -\frac{1}{2}$ and $\Im t > 0$ the cubic roots y of all $\pm\ell$ listed in the table of Corollary 2.3 and checks, if these roots y are elements of $\mathbb{Z}_k$ for a special t with $|t| \geq 6$. We get the following solutions (x, y) presented in Theorem 9.2:

$\underline{D \equiv 3 \pmod 4}$: $[D.3]$.

$\underline{D = 3}$: In addition to $[D.3]$ we get the solutions: [3.16], [3.17], [3.18], [3.19].

$\underline{D = 7}$: In addition to $[D.3]$ we get the solution: [7.13].

In case $x = -y$, the Thue equation (1.1) reads

$$\begin{aligned} F_t(-y, y) &= (-y)^3 - (t-1)(-y)^2 y - (t+2)(-y)y^2 - y^3 \\ &- y^3 - (t-1)y^3 + (t+2)y^3 - y^3 = y^3 = \ell \end{aligned}$$

Since all values of $\pm\ell$ are listed in the table of Corollary 2.3, the case $x = -y$ is equivalent to the case $x = 0$ and we get the following list of solutions (x, y) written in Theorem 9.2:

$\underline{D \equiv 3 \pmod 4}$: $[D.3]$.

$\underline{D = 3}$: In addition to $[D.3]$ we get the solutions: [3.16], [3.17], [3.18], [3.19].

$\underline{D = 7}$: In addition to $[D.3]$ we get the solution: [7.13].

In case $x = ty$, the Thue equation (1.1) reads

$$F_t(ty, y) = (ty)^3 - (t-1)(ty)^2 y - (t+2)tyy^2 - y^3 = -2ty^3 - y^3 = \ell.$$

In order to solve this equation, we have to compute the cubic roots of $-\frac{\ell}{2t+1}$. These cubic roots have to be elements of $\mathbb{Z}_k$ and so we only get a solution if $\ell = \pm(2t+1)$. In this case we solve $y^3 = \pm 1$, i.e., y is the corresponding root of unity.

For $x = ty$ we get solution $[D.2]$ of Theorem 9.2.

8. γ associated to an integer

Finally we have to analyse the case where γ is associated to an integer. Let $\gamma^{(1)}, \gamma^{(2)}$ and $\gamma^{(3)}$ be defined as in (1.5). Since γ is associated to an integer, $\gamma^{(1)}, \gamma^{(2)}$ and $\gamma^{(3)}$ are associated to the same integer and we can rewrite $N_{k(\alpha)/k}(x - \alpha y) = F_t(x, y) = \ell$ as $\gamma^{(1)} \cdot \gamma^{(2)} \cdot \gamma^{(3)} = \ell = \mu r^3$ with $|\mu| = 1$ and $r \in \mathbb{Z}_k$. Dividing this equation by r^3 we get $\frac{\gamma^{(1)}}{r} \cdot \frac{\gamma^{(2)}}{r} \cdot \frac{\gamma^{(3)}}{r} = \mu$. Now we are able to use [1] where all solutions for $\Re t = -\frac{1}{2}$ to

$$\hat{x}^3 - (t-1)\hat{x}^2\hat{y} - (t+2)\hat{x}\hat{y}^2 - \hat{y}^3 = \mu$$

are derived. Setting $x = \hat{x}r$ and $y = \hat{y}r$, we get all solutions for $\Re t = -\frac{1}{2}$ to (1.1). Since we have to compute the cubic roots of ℓ and these roots have to be elements of $\mathbb{Z}_k$, there are only solutions for special ℓ depending on D and t (cf. the special cases $x = 0, x = -y$). The list of solutions can be obtained by using a `Mathematica`® program. This program computes for a list of t with $\Re t = -\frac{1}{2}$ and $\Im t > 0$ the cubic roots r of ℓ, multiplies each $\hat{x}$ and $\hat{y}$ with r and checks, if these values are elements of $\mathbb{Z}_k$ for a special t. Thereby we get the following solutions (x, y) written in Theorem 9.2:

$\underline{D \equiv 3 \pmod 4}$: $[D.3]$.

$\underline{D = 3}$: In addition to $[D.3]$ we get the solutions: [3.15] for $t = -\frac{1}{2} + \frac{5i\sqrt{3}}{2}$, [3.16], [3.17], [3.18], [3.19].

$\underline{D = 7}$: In addition to $[D.3]$ we get the solution: [7.13].

9. Solutions of the Thue Equation for all t with $\Re t = -\frac{1}{2}$

In this section we state the main theorem.

for all D				
	x	y	ℓ	restrictions on t
$[D.1]$	-1	-1	$2t+1$	for all D and t
$[D.2]$	1	$-1-t$	$-2t-1$	for all D and t
$[D.3]$	0	$-(2k+1)(1-2b)$	$2t+1$	for all D and $t = -\frac{1}{2} + \frac{ci\sqrt{D}}{2}$, $c = D(2k+1)^3, k \geq 0$

Table 15.

$D = 3$				
	x	y	ℓ	restrictions on t
[3.1]	$1+b$	-1	$\frac{i\sqrt{3}}{2}(2t+1)-\frac{7}{2}$	for all $t \neq -\frac{1}{2}+\frac{ki\sqrt{3}}{2}$, $k \in \{1,5,7,9,11,13,15\}$
[3.2]	$-b$	-1	$\frac{i\sqrt{3}}{2}(2t+1)+\frac{7}{2}$	for all $t \neq -\frac{1}{2}+\frac{i\sqrt{3}}{2}$
[3.3]	$-(1-b)$	1	$\frac{2t+1}{2}-\frac{3i\sqrt{3}}{2}$	for all $t \neq -\frac{1}{2}+\frac{i\sqrt{3}}{2}$
[3.4]	$-b$	1	$\frac{2t+1}{2}+\frac{3i\sqrt{3}}{2}$	for all $t \neq -\frac{1}{2}+\frac{i\sqrt{3}}{2}$
[3.5]	$5-5b$	$-(19+14b)$	$2i\sqrt{3}$	$t=-\frac{1}{2}+\frac{7i\sqrt{3}}{2}$
[3.6]	b	$2-b$	$5-6i\sqrt{3}$	$t=-\frac{1}{2}+\frac{7i\sqrt{3}}{2}$
[3.7]	$-(1-b)$	2	$5+6i\sqrt{3}$	$t=-\frac{1}{2}+\frac{7i\sqrt{3}}{2}$
[3.8]	$2+3b$	$1-b$	$7i\sqrt{3}$	$t=-\frac{1}{2}+\frac{5i\sqrt{3}}{2}$
[3.9]	$-(7-6b)$	$1+b$	-4	$t=-\frac{1}{2}+\frac{5i\sqrt{3}}{2}$
[3.10]	$-(1+2b)$	$-(1-b)$	$4i\sqrt{3}$	$t=-\frac{1}{2}+\frac{5i\sqrt{3}}{2}$
[3.11]	$-(1+b)$	$-(1-b)$	$2+3i\sqrt{3}$	$t=-\frac{1}{2}+\frac{5i\sqrt{3}}{2}$
[3.12]	$-(1+b)$	$2b$	$2-3i\sqrt{3}$	$t=-\frac{1}{2}+\frac{5i\sqrt{3}}{2}$
[3.13]	$-(1+3b)$	$-(1-b)$	$4+3i\sqrt{3}$	$t=-\frac{1}{2}+\frac{5i\sqrt{3}}{2}$
[3.14]	$2+2b$	$1-b$	$4-3i\sqrt{3}$	$t=-\frac{1}{2}+\frac{5i\sqrt{3}}{2}$
[3.15]	0	-2	8	$t=-\frac{1}{2}+\frac{5i\sqrt{3}}{2}$
[3.16]	0	$-k(1-2b)$	$\frac{2t+1}{2}-\frac{3i\sqrt{3}}{2}$	$t=-\frac{1}{2}+\frac{ci\sqrt{3}}{2}$, $c=6k^3+3, k \geq 1$
[3.17]	0	$-k(1-2b)$	$\frac{2t+1}{2}+\frac{3i\sqrt{3}}{2}$	$t=-\frac{1}{2}+\frac{ci\sqrt{3}}{2}$, $c=6k^3-3, k \geq 2$
[3.18]	0	$3k+2$	$\frac{i\sqrt{3}}{2}(2t+1)-\frac{7}{2}$	$t=-\frac{1}{2}+\frac{ci\sqrt{3}}{2}$, $c=\frac{2(3k+2)^3-7}{3}, k \geq 1$
[3.19]	0	$3k+1$	$\frac{i\sqrt{3}}{2}(2t+1)+\frac{7}{2}$	$t=-\frac{1}{2}+\frac{ci\sqrt{3}}{2}$, $c=\frac{2(3k+1)^3+7}{3}, k \geq 1$

Table 16.

Lemma 9.1. *Let (x,y) be a solution of $F_t(x,y)=\ell$. Then $(-(x+y),x)$ and $(y,-(x+y))$ are solutions of $F_t(x,y)=\ell$ too and $(-x,-y)$ is a solution of $F_t(x,y)=-\ell$. In case $D=3$ $(-bx,-by)$ and $(-(1-b)x,-(1-b)y)$ are solutions of $F_t(x,y)=\ell$ too.*

Proof. Immediate. The last assertion follows from $b^2=b-1$ for $b=\frac{1}{2}+\frac{i\sqrt{3}}{2}$. $\square$

$D=7$				
	x	y	ℓ	restrictions on t
[7.1]	1	$-b$	$2t+1-2i\sqrt{7}$	for all $t \neq -\frac{1}{2}+\frac{i\sqrt{7}}{2}$
[7.2]	-12	$-(35-82b)$	$5i\sqrt{7}$	$t=-\frac{1}{2}+\frac{7i\sqrt{7}}{2}$
[7.3]	1	$-(1+b)$	$11+2i\sqrt{7}$	$t=-\frac{1}{2}+\frac{5i\sqrt{7}}{2}$
[7.4]	-1	$-(1-b)$	$11-2i\sqrt{7}$	$t=-\frac{1}{2}+\frac{5i\sqrt{7}}{2}$
[7.5]	1	$-(1+b)$	$4+i\sqrt{7}$	$t=-\frac{1}{2}+\frac{3i\sqrt{7}}{2}$
[7.6]	-1	$-(1-b)$	$4-i\sqrt{7}$	$t=-\frac{1}{2}+\frac{3i\sqrt{7}}{2}$
[7.7]	1	$1-2b$	$1+2i\sqrt{7}$	$t=-\frac{1}{2}+\frac{3i\sqrt{7}}{2}$
[7.8]	-1	$2b$	$1-2i\sqrt{7}$	$t=-\frac{1}{2}+\frac{3i\sqrt{7}}{2}$
[7.9]	2	$-(1+b)$	$\frac{7}{2}+\frac{5i\sqrt{7}}{2}$	$t=-\frac{1}{2}+\frac{3i\sqrt{7}}{2}$
[7.10]	-2	b	$\frac{7}{2}-\frac{5i\sqrt{7}}{2}$	$t=-\frac{1}{2}+\frac{3i\sqrt{7}}{2}$
[7.11] [7.12]	$5-b$ $3+2b$	$-b$ $1-b$	-7	$t=-\frac{1}{2}+\frac{3i\sqrt{7}}{2}$
[7.13]	0	$-(2k+1)(1-2b)$	$2t+1-2i\sqrt{7}$	$t=-\frac{1}{2}+\frac{ci\sqrt{7}}{2}$, $c=7(2k+1)^3+2, k\geq 0$

Table 17.

$D=11$				
	x	y	ℓ	restrictions on t
[11.1]	1	$-b$	$5i\sqrt{11}$	$t=-\frac{1}{2}+\frac{5i\sqrt{11}}{2}$
[11.2]	1	$-b$	$2i\sqrt{11}$	$t=-\frac{1}{2}+\frac{3i\sqrt{11}}{2}$
[11.3]	1	$-(1+b)$	$7+2i\sqrt{11}$	$t=-\frac{1}{2}+\frac{3i\sqrt{11}}{2}$
[11.4] [11.5]	2 2	$2-b$ $-(3+b)$	$i\sqrt{11}$	$t=-\frac{1}{2}+\frac{i\sqrt{11}}{2}$

Table 18.

$D=15$				
	x	y	ℓ	restrictions on t
[15.1]	1	$-b$	$3i\sqrt{15}$	$t=-\frac{1}{2}+\frac{3i\sqrt{15}}{2}$
[15.2]	$3+b$	$4-b$	$i\sqrt{15}$	$t=-\frac{1}{2}+\frac{i\sqrt{15}}{2}$

Table 19.

Combining the last four sections we finally get the following theorem concerning the solutions of the Thue equation (1.1).

$D=19$				
	x	y	ℓ	restrictions on t
[19.1]	-85	$36+13b$	$i\sqrt{19}$	$t=-\frac{1}{2}+\frac{i\sqrt{19}}{2}$
[19.2]	2	$1-b$	3	$t=-\frac{1}{2}+\frac{i\sqrt{19}}{2}$
[19.3]	-2	$2+b$		

Table 20.

$D=23$				
	x	y	ℓ	restrictions on t
[23.1]	17	$-(7+3b)$	$i\sqrt{23}$	$t=-\frac{1}{2}+\frac{i\sqrt{23}}{2}$
[23.2]	-3	$1+b$		
[23.3]	-5	$2+b$		
[23.4]	2	$1-b$	$\frac{3}{2}+\frac{i\sqrt{23}}{2}$	$t=-\frac{1}{2}+\frac{i\sqrt{23}}{2}$
[23.5]	-2	$1+b$		
[23.6]	2	$-b$	$\frac{3}{2}-\frac{i\sqrt{23}}{2}$	$t=-\frac{1}{2}+\frac{i\sqrt{23}}{2}$
[23.7]	-2	$2+b$		

Table 21.

$D=31$				
	x	y	ℓ	restrictions on t
[31.1]	3	$-(1+b)$	$i\sqrt{31}$	$t=-\frac{1}{2}+\frac{i\sqrt{31}}{2}$
[31.2]	5	$-(1+3b)$		
[31.3]	2	$-b$	$\frac{1}{2}+\frac{i\sqrt{31}}{2}$	$t=-\frac{1}{2}+\frac{i\sqrt{31}}{2}$
[31.4]	-2	$1+b$	$\frac{1}{2}-\frac{i\sqrt{31}}{2}$	$t=-\frac{1}{2}+\frac{i\sqrt{31}}{2}$

Table 22.

Theorem 9.2. *Let $k:=\mathbb{Q}(\sqrt{-D})$, $\mathbb{Z}_k$ be the corresponding ring of integers and $t,\ell\in\mathbb{Z}_k$ with $t\notin\mathbb{Z}$ and $1<|\ell|\le|2t+1|$. For squarefree $D\equiv 3 \pmod 4$ let $b=\frac{1}{2}+\frac{i\sqrt{D}}{2}$. For $\Re t=-\frac{1}{2}, \Im t>0$ and $t\neq\frac{-1+3i\sqrt{3}}{2}$ the only solutions $(x,y)\in\mathbb{Z}_k^2$ (apart from the obvious additional solutions according to Lemma 9.1) of the family of relative Thue equations*

$$F_t(x,y):=x^3-(t-1)x^2y-(t+2)xy^2-y^3=\ell$$

are listed in Tables 15 to 25 depending on the discriminant D.

$D=35$				
	x	y	ℓ	restrictions on t
[35.1]	2	$-(1+b)$	$i\sqrt{35}$	$t=-\frac{1}{2}+\frac{i\sqrt{35}}{2}$
[35.2]	2	$-b$		

Table 23.

$D=43$				
	x	y	ℓ	restrictions on t
[43.1]	$2b-1$	$17-b$	-4	$t=-\frac{1}{2}+\frac{i\sqrt{43}}{2}$

Table 24.

$D=51$				
	x	y	ℓ	restrictions on t
[51.1]	32	$-(3+26b)$	$i\sqrt{51}$	$t=-\frac{1}{2}+\frac{i\sqrt{51}}{2}$

Table 25.

10. Concluding remarks

In the general case where we drop the restriction $\Re t=-\frac{1}{2}$ we have to deal with the more complex linear form in logarithms

$$|\Lambda_j|=|\log|\delta_j|+A_j\log|\alpha|+B_j\log|\alpha+1||\,.$$

In this case from our results we are able to make a statement about those values ℓ that may result in solutions of the Thue equation (1.1). For all t with $\Im t>0$ the values of column $N_{k(\alpha)/k}(\gamma)$ in Theorem 2.1 are the only possible values ℓ of the Thue equation (1.1), such that pairs of solutions $x,y\in\mathbb{Z}_k$ exist, since these are the only cases where the inequality $|F_t(x,y)|=|\ell|=|N_{k(\alpha)/k}(x-\alpha y)|\leq|2t+1|$ holds.

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