Publl. Math. Debrecen
71/1-2 (2007), 141–163

On curvature decreasing property of a class of navigation problems

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Abstract. In Finsler geometry the flag curvature is an important quantity because it is a natural extension of the sectional curvature in Riemannian geometry. This note gives flag curvature a decreasing property via some navigation problems. As an application, we prove some rigidity results for Finsler metrics with special flag curvature properties.

1. Introduction

Finsler metrics arise naturally in many areas of mathematics as well as natural science. In Finsler geometry the flag curvature is an important quantity because it is a natural extension of the sectional curvature in Riemannian geometry. For a Finsler manifold (M, F), the flag curvature $K = K_F(y, \Pi)$ is a function of the tangent planes $\Pi \subset T_xM$ and direction $y \in \Pi\backslash\{0\}$. This quantity tells us how curved the space is at a point.

Recently, one of the important approaches in discussing Finsler metric is the navigation problem. For instance, BAO–ROBLES–SHEN have classified Randers metrics of constant flag curvature via the navigation problem in Riemannian manifolds [5]. Moreover ROBLES has determined the geodesics of these Randers metrics [16].

Randers metrics are among the simplest non-Riemannian Finsler metrics.

Mathematics Subject Classification: 53B40.
Key words and phrases: Finsler metric, flag curvature, navigation problem.

They are expressed in the form $F = \alpha + \beta$, where $\alpha := \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric on M and $\beta := b_i(x)y^i$ is a 1-form with $\|\beta\|_\alpha < 1$. Randers metrics arise naturally from physical applications [15].

The main technique of the navigation problem is described as follows. Given a Finsler metric F and a vector field V with $F(x, V_x) < 1$, define a new Finsler metric $\tilde{F}$ by

$$F\left(x, \frac{y}{\tilde{F}(x,y)} + V_x\right) = 1, \qquad \forall\, x \in M,\ y \in T_xM. \tag{1.1}$$

For non-collinear $u, v \in T_xM$, we denote the tangent plane $\mathrm{Span}\{u, v\}$ by $u \wedge v$. Using Chern connection and moving frames, this note gives the following

Theorem 1.1. *Let $F = F(x, y)$ be a Finsler metric on a manifold M and V a vector field on M with $F(x, V_x) < 1$. Let $\tilde{F} = \tilde{F}(x, y)$ denote the Finsler metric on M defined by* (1.1). *Suppose that V is homothetic with dilation c. Then the flag curvature of $\tilde{F}$ and F is related by*

$$K_{\tilde{F}}(y, y \wedge u) = K_F(\tilde{y}, \tilde{y} \wedge u) - c^2$$

where $\tilde{y} = y - F(x, y)V$.

Recall that a vector field V on M is called *homothetic with dilation c* if its flow ϕ_t satisfies

$$F(\phi_t(x), (\phi_t)_* y) = e^{2ct} F(x, y), \qquad \forall\, x \in M,\ y \in T_xM.$$

(For more details, see Section 5.) Some examples of non-trivial homothetic fields have been constructed in Sections 5 and 7.

Theorem 1.1 generalizes theorems previously only known in the case of Randers metrics [5, Theorem 3.1].

It is also worth mentioning the recent announcement of Foulon that if F is a Finsler metric and V is a Killing field, then F and $\tilde{F}$ have the same flag curvature, where $\tilde{F}$ denotes the navigation representation of (F, V). In this paper we give a *new geometric* proof of Foulon's claim.

For a non-Killing homothetic field Theorem 1.1 means that the navigation problem has the flag curvature decreasing property. After noting this interesting phenomenon, we investigate a series of rigidity theorems for Finsler metrics with special flag curvatures.

Mo–Shen's global rigidity theorem tells us that every Finsler metric on an n dimensional compact manifold of negative scalar flag curvature must be of Randers type for $n \geq 3$ [13]. Since navigation data preserve Randers type of Finsler metrics, on the basis of Theorem 1.1, we obtain the following

Theorem 1.2. *Let (M, F) be a compact Finsler manifold of dimension $n \geq 3$ and V a vector field with $F(x, V_x) < 1$. Suppose that V is a homothetic field with dilation c, and F is of scalar curvature $K(x, y)$, which satisfies*

$$\sup K(x, y) < c^2.$$

Then F is a Randers metric.

Theorem 1.2 greatly narrows down the possibility of compact Finsler manifolds satisfying given flag curvature conditions. We also obtain some other rigidity results in Section 7.

2. Preliminaries

In this section, we are going to give a brief description of the Chern connection, the flag curvature, and other quantities mentioned above. For more detail see [3].

Let M be a smooth manifold of dimension n, and let TM (resp. T^*M) be its tangent (resp. cotangent) bundle. By definition, a *Finsler metric* $F(x, y)$ on M is a nonnegative function on TM, which is positively y-homogeneous of degree one with positive definite fundamental tensor $g := g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$, where $g_{ij} := \frac{1}{2}\left[F^2\right]_{y^i y^j}(x, y)$. To characterize Riemannian metrics among Finsler metrics, we define the Cartan tensor A by

$$A := \frac{1}{4} F \frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k.$$

Throughout the paper, our index conventions are as follows:

$$1 \leq i, j, k, \dots \leq n, \quad 1 \leq \alpha, \beta, \gamma, \dots \leq n-1, \quad 1 \leq a, b, c, \dots \leq 2n-1.$$

Usually it is more convenient to do calculations on TM and to interpret them on SM, where SM is known to be the sphere bundle of M whose fibre at x consists of all directions in T_xM. Each geometrical quantity on TM, homogeneous of degree zero, is considered to sit on SM. In [7], a basis $\{\hat{e}_a\}_{a=1}^{2n-1}$ for $T(SM)$ is introduced to satisfy $G(\hat{e}_a, \hat{e}_b) = \delta_{ab}$ and $\hat{e}_n = \frac{y^i}{F} \frac{\delta}{\delta x^i}$, where

$$\hat{e}_i = u_i{}^j \frac{\delta}{\delta x^j}, \quad \hat{e}_{\bar{\alpha}} = u_\alpha{}^j F \frac{\partial}{\partial y^j}, \quad \bar{\alpha} := n + \alpha \tag{2.1}$$

and G is the Sasaki type Riemannian metric on SM. As its dual, a basis $\{\omega^a\}$ for T^*SM is given by

$$\omega^i = v_j{}^i \mathrm{d}x^j, \quad \omega^{\bar{\alpha}} = v_j{}^\alpha \frac{1}{F}\delta y^j. \tag{2.2}$$

Among them, $\omega^n = F_{y^i}\mathrm{d}x^i$ is the most important and it is called the *Hilbert form.*

In [7] CHERN claims that there is a unique set of connection 1-forms $\{\omega_i{}^j\}$ satisfying

$$\mathrm{d}\omega^i = \omega^j \wedge \omega_j{}^i \tag{2.3}$$

$$\delta_{ki}\omega_j{}^k + \delta_{kj}\omega_i{}^k = -2H_{ij\alpha}\omega^{\bar{\alpha}}, \quad H_{ijk} := A(e_i, e_j, e_k) \tag{2.4}$$

which, when collected together, are called the *Chern connection forms.*

It is an easy consequence that $\omega_n{}^\alpha = -\omega_\alpha{}^n = \omega^{\bar{\alpha}}$ and $\omega_n{}^n = 0$.

The curvature 2-forms $\Omega_i{}^j$ are defined by

$$\Omega_i{}^j = \mathrm{d}\omega_i{}^j - \omega_i{}^k \wedge \omega_k{}^j.$$

Let $R^\beta{}_\alpha := \Omega_n{}^\beta(e_\alpha, e_n)$. It naturally leads to a quantity, which is called the *Riemann tensor* [8]

$$R := R^\beta{}_\alpha \omega^\alpha \otimes e_\beta.$$

For a tangent plane $\Pi = \mathrm{Span}\{y, u\}$, $y, u \in T_xM$, we have the notion of *flag curvature*

$$K(y, y \wedge u) := \frac{g(R(u), u)}{g(e_n, e_n)g(u, u) - (g(e_n, u))^2}$$

where $\{e_i\}$ is a dual adapted orthonormal frame on the Finsler bundle. Take $h := (\omega^1)^2 + \cdots + (\omega^{n-1})^2$, then the flag curvature can be written as

$$K(y, y \wedge u) = \frac{h(R(u), u)}{h(u, u)}. \tag{2.5}$$

3. Description in terms of the Chern connection and its curvatures

In this section, we are going to use the Chern connection and its curvatures describing some concepts we will use later. We continue to use the notations of Section 2.

Lemma 3.1.

$$[\hat{e}_n, \hat{e}_\alpha] = \omega_\alpha{}^\beta(\hat{e}_n)\hat{e}_\beta + R^\beta{}_\alpha\, \hat{e}_{\bar{\beta}} \tag{3.1}$$

$$[\hat{e}_n, \hat{e}_{\bar{\alpha}}] = -\hat{e}_\alpha + \omega_\alpha{}^\beta(\hat{e}_n)\hat{e}_{\bar{\beta}}. \tag{3.2}$$

Proof. Direct computation yields

$$\omega^i[\hat{e}_n, \hat{e}_\alpha] = -\mathrm{d}\omega^i(\hat{e}_n, \hat{e}_\alpha) = -\omega^j \wedge \omega_j{}^i(\hat{e}_n, \hat{e}_\alpha)$$
$$= -\omega_n{}^i(\hat{e}_\alpha) + \omega_\alpha{}^i(\hat{e}_n) = \omega_\alpha{}^i(\hat{e}_n).$$

By Section 2,

$$\mathrm{d}\omega^{\bar{\alpha}} = \mathrm{d}\omega_n{}^\alpha = \Omega_n{}^\alpha + \omega_n{}^k \wedge \omega_k{}^\alpha = \Omega_n{}^\alpha + \omega^{\bar{\beta}} \wedge \omega_\beta{}^\alpha.$$

It follows that

$$\omega^{\bar{\beta}}[\hat{e}_n, \hat{e}_\alpha] = -\mathrm{d}\omega^{\bar{\beta}}(\hat{e}_n, \hat{e}_\alpha) = (-\Omega_n{}^\beta - \omega^{\bar{\gamma}} \wedge \omega_\gamma{}^\beta)(\hat{e}_n, \hat{e}_\alpha)$$
$$= -\Omega_n{}^\beta(\hat{e}_n, \hat{e}_\alpha) = R^\beta{}_\alpha.$$

Thus $[\hat{e}_n, \hat{e}_\alpha] = \omega_\alpha{}^\beta(\hat{e}_n)\hat{e}_\beta + R^\beta{}_\alpha\hat{e}_{\bar{\beta}}$. A similar calculation gives (3.2). □

There is a special vector field X on SM. We characterize it by the following

Proposition 3.2. *Let $\omega = \omega^n$ be the Hilbert form. Then there is a unique vector field X on SM satisfying*

$$\omega(X) = 1, \quad d\omega(X, \cdot) = 0. \tag{3.3}$$

Proof. Put $X = X^i\hat{e}_i + X^{\bar{\alpha}}\hat{e}_{\bar{\alpha}}$, then

$$X^n = \omega(X) = 1$$
$$X^\alpha = \omega^\alpha(X) = \omega^j \wedge \omega_j{}^n(X, \hat{e}_{\bar{\alpha}}) = \mathrm{d}\omega(X, \hat{e}_{\bar{\alpha}}) = 0$$
$$X^{\bar{\alpha}} = \omega^{\bar{\alpha}}(X) = \omega^j \wedge \omega_j{}^n(X, \hat{e}_\alpha) = \mathrm{d}\omega(X, \hat{e}_\alpha) = 0.$$

Thus we have $X = \hat{e}_n$. □

The vertical distribution $VSM := \mathrm{span}\{\hat{e}_{\bar{\alpha}}\}$ is naturally defined on SM, which is independent of the Finsler structure F. Actually, it can be realized as $\{v \in TSM \mid v(f) = 0, f \in C^\infty(M) \subset C^\infty(SM)\}$.

Proposition 3.3. *There is a unique $(1,1)$-tensor $\mathcal{V}$ satisfying*

$$\mathcal{V}(v) = \mathcal{V}(X) = 0, \quad \mathcal{V}[X, v] = -v, \quad \forall v \in VSM. \tag{3.4}$$

Proof. By the first equation of (3.4), we can write $\mathcal{V} = (A^i_\beta \hat{e}_i + A^{\bar{\gamma}}_\beta \hat{e}_{\bar{\gamma}}) \otimes \omega^\beta$. From (3.2), we have

$$-\hat{e}_{\bar{\alpha}} = \mathcal{V}[X, \hat{e}_{\bar{\alpha}}] = -(A^i_\beta \hat{e}_i + A^{\bar{\gamma}}_\beta \hat{e}_{\bar{\gamma}}) \delta_\alpha{}^\beta = -A^i_\alpha \hat{e}_i - A^{\bar{\gamma}}_\alpha \hat{e}_{\bar{\gamma}}.$$

A comparing of the coefficients yields

$$A^i_\alpha = 0, \quad A^{\bar{\gamma}}_\alpha = \delta^\gamma_\alpha.$$

Hence $\mathcal{V} = \hat{e}_{\bar{\alpha}} \otimes \omega^\alpha$, that is, it is uniquely determined. □

Remark. In natural coordinates one can also derive that

$$\mathcal{V}\left(\frac{\partial}{\partial x^i}\right) = F\frac{\partial}{\partial y^i}, \quad \mathcal{V}\left(F\frac{\partial}{\partial y^i}\right) = 0,$$

which provides another proof of Proposition 3.3.

Proposition 3.4. *Define $\mathcal{H}(v) := -[X, v] - \frac{1}{2}\mathcal{V}[X, [X, v]]$ for $v \in VSM$, then $\mathcal{H}$ is C^∞-linear and $\mathcal{H}(\hat{e}_{\bar{\alpha}}) = \hat{e}_\alpha$.*

Proof. For any $f \in C^\infty(SM)$, we have

$$\begin{aligned}
-\mathcal{H}(f \cdot v) &= [X, f \cdot v] + 1/2\mathcal{V}[X, [X, f \cdot v]] \\
&= X(f)v + f \cdot [X, v] + 1/2\mathcal{V}[X, X(f)v + f \cdot [X, v]] \\
&= X(f)v + f \cdot [X, v] - 1/2X(f)v + 1/2\mathcal{V}[X, f \cdot [X, v]] \\
&= 1/2X(f)v + f \cdot [X, v] + 1/2\mathcal{V}\big(X(f)[X, v] + f \cdot [X, [X, v]]\big) \\
&= 1/2X(f)v + f \cdot [X, v] - 1/2X(f)v + 1/2f \cdot \mathcal{V}[X, [X, v]] = -f \cdot \mathcal{H}(v),
\end{aligned}$$

which preserves the C^∞-linearity. By using (3.1) and (3.2), we have

$$\begin{aligned}
-\mathcal{H}(\hat{e}_{\bar{\alpha}}) &= [\hat{e}_n, \hat{e}_{\bar{\alpha}}] + \frac{1}{2}\mathcal{V}[\hat{e}_n, [\hat{e}_n, \hat{e}_{\bar{\alpha}}]] \\
&= -\hat{e}_\alpha + \omega_\alpha{}^\beta(\hat{e}_n)\hat{e}_{\bar{\beta}} + \frac{1}{2}\mathcal{V}[\hat{e}_n, -\hat{e}_\alpha + \omega_\alpha{}^\beta(\hat{e}_n)\hat{e}_{\bar{\beta}}] \\
&= -\hat{e}_\alpha + \omega_\alpha{}^\beta(\hat{e}_n)\hat{e}_{\bar{\beta}} - \frac{1}{2}\mathcal{V}[\hat{e}_n, \hat{e}_\alpha] + \frac{1}{2}\omega_\alpha{}^\beta(\hat{e}_n)\mathcal{V}[\hat{e}_n, \hat{e}_{\bar{\beta}}] \\
&= -\hat{e}_\alpha + \omega_\alpha{}^\beta(\hat{e}_n)\hat{e}_{\bar{\beta}} - \frac{1}{2}\mathcal{V}\big(\omega_\alpha{}^\beta(\hat{e}_n)\hat{e}_\beta + R^\beta{}_\alpha \hat{e}_{\bar{\beta}}\big) \\
&\quad + \frac{1}{2}\omega_\alpha{}^\beta(\hat{e}_n)\mathcal{V}\big(- \hat{e}_\beta + \omega_\beta{}^\gamma(\hat{e}_n)\hat{e}_{\bar{\gamma}}\big) \\
&= -\hat{e}_\alpha + \omega_\alpha{}^\beta(\hat{e}_n)\hat{e}_{\bar{\beta}} - \frac{1}{2}\mathcal{V}\big(\omega_\alpha{}^\beta(\hat{e}_n)\hat{e}_\beta\big) + \frac{1}{2}\omega_\alpha{}^\beta(\hat{e}_n)\mathcal{V}(-\hat{e}_\beta) = -\hat{e}_\alpha.
\end{aligned}$$

It follows that $\mathcal{H}(\hat{e}_{\bar{\alpha}}) = \hat{e}_\alpha$. □

Take $HSM := \mathcal{H}(VSM) = \operatorname{span}\{\hat{e}_\alpha\}$. Then $SM = HSM \oplus \operatorname{Span}\{X\} \oplus VSM$. Define

$$\mathcal{H}(u) = \mathcal{H}(X) = 0, \quad u \in HSM. \tag{3.5}$$

Then we have the following

Corollary 3.5. *$\mathcal{H}$ is a $(1,1)$-tensor on SM.*

From the proof of Proposition 3.4, we see that $\mathcal{H} = \hat{e}_\alpha \otimes \omega^{\bar{\alpha}}$.

Corollary 3.6. *The tensor $\mathcal{J} := \mathcal{H} - \mathcal{V}$ is an almost complex structure on $HSM \oplus VSM$.*

We denote the projection to VSM (resp. HSM) by $P_\mathcal{V} := \mathcal{V} \circ \mathcal{H}$ (resp. $P_\mathcal{H} := \mathcal{H} \circ \mathcal{V}$).

Proposition 3.7. *Define a map $\mathcal{R} : TSM \to TSM$ by*

$$\mathcal{R}(v) := P_\mathcal{V}[X, \mathcal{H}(v)], \quad v \in VSM \tag{3.6}$$

$$\mathcal{R}(u) := \mathcal{H}[X, u], \quad u \in HSM \tag{3.7}$$

$$\mathcal{R}(X) := 0. \tag{3.8}$$

Then $\mathcal{R}$ is actually the Riemann tensor with a slight difference. It commutes with the almost complex structure $\mathcal{J}$.

Proof. It is easy to check that $\mathcal{R}$ is C^∞-linear. By a direct computation, one obtains

$$\mathcal{R}(\hat{e}_\alpha) = \mathcal{H}[X, \hat{e}_\alpha] = \mathcal{H}\big(\omega_\alpha{}^\beta(\hat{e}_n)\hat{e}_\beta + R^\beta{}_\alpha \hat{e}_{\bar{\beta}}\big) = R^\beta{}_\alpha \hat{e}_\beta$$

$$\mathcal{R}(\hat{e}_{\bar{\alpha}}) = \mathcal{V}\mathcal{H}[X, \mathcal{H}(\hat{e}_{\bar{\alpha}})] = \mathcal{V}\mathcal{H}[X, \hat{e}_\alpha] = \mathcal{V}(R^\beta{}_\alpha \hat{e}_\beta) = R^\beta{}_\alpha \hat{e}_{\bar{\beta}}.$$

It follows that $\mathcal{R} = R^\beta{}_\alpha\big(\hat{e}_\beta \otimes \omega^\alpha + \hat{e}_{\bar{\beta}} \otimes \omega^{\bar{\alpha}}\big)$. The commutability is obvious. □

Finally we define the flag curvature alternatively on the projectively vertical subbundle $P(VSM)$.

Proposition 3.8. *Let*

$$h(v_1, v_2) := \begin{cases} d\omega([X, v_1], v_2) & \text{if } v_1, v_2 \in VSM \\ h(\mathcal{J}(v_1), \mathcal{J}(v_2)) & \text{if } v_1, v_2 \in HSM \\ 0 & \text{if } v_1 \in VSM,\ v_2 \in HSM. \end{cases}$$

Then h is an inner product on $HSM \oplus VSM$. In particular, $h|_{VSM \times VSM}$ is an inner product on VSM.

Proof. It is easy to see that h is C^∞-linear with respect to each index. Furthermore we have

$$\begin{aligned} h(\hat{e}_{\bar\alpha}, \hat{e}_{\bar\beta}) &= \mathrm{d}\omega([X, \hat{e}_{\bar\alpha}], \hat{e}_{\bar\beta}) = (\omega^\delta \wedge \omega_\delta{}^n)([\hat{e}_n, \hat{e}_{\bar\alpha}], \hat{e}_{\bar\beta}) \\ &= -\left(\sum \omega^\delta \wedge \omega^{\bar\delta}\right)(-\hat{e}_\alpha + \omega_\alpha{}^\gamma(X)\hat{e}_{\bar\gamma}, \hat{e}_{\bar\beta}) = \delta_{\alpha\beta}. \end{aligned}$$

Hence $h|_{VSM\times VSM} = \delta_{\alpha\beta}\omega^{\bar\alpha}\otimes\omega^{\bar\beta} = (\omega^{\bar 1})^2 + \cdots + (\omega^{\overline{n-1}})^2$ is an inner product on VSM. Extend it to $HSM \oplus VSM$. Then $h = (\omega^1)^2 + \cdots + (\omega^{n-1})^2 + (\omega^{\bar 1})^2 + \cdots + (\omega^{\overline{n-1}})^2$. □

Now the *flag curvature* is defined by

$$K([v]) := K(v) := \frac{h(\mathcal{R}(v), v)}{h(v,v)}, \quad v \in VSM, \tag{3.9}$$

where $[v]$ is the v-equivalent class of VSM with a positive number λ:

$$[v] := \{\lambda v \mid \lambda > 0\}, \quad v \in VSM.$$

The following lemma explains the relation between (3.9) and (2.5).

Lemma 3.9. *There is a globally defined $1-1$ map Φ between the flags $\{y\wedge u \mid u \in T_xM\}$ and the projectively vertical subbundle $P(VSM)$, such that*

$$K(y, y\wedge u) = K(\Phi(y\wedge u)),$$

where $P(VSM) := \{[v] \mid v \in VSM\} \simeq VSM/\mathbb{R}^+$. Furthermore, Φ is independent of the Finsler metric F.

Proof. Put $\Phi(y\wedge u) := [\mathcal{V}(u)]$, where $[v]$ is the equivalent class containing $v \in VSM$. It is clear that $\Phi(y\wedge u)$ is independent of the choice of u, that is, it is globally defined.

Without loss of generality, we assume that $u = u^\alpha e_\alpha \in T_xM$, where $\{e_i\}$ is the basis of T_xM dual to ω^i. Then $\Phi(y\wedge u) = [u^\alpha e_{\bar\alpha}]$. It follows that

$$K(y,u) = \frac{h(R(u),u)}{h(u,u)} = \frac{\delta_{\alpha\beta}R^\beta{}_\gamma u^\alpha u^\gamma}{\delta_{\alpha\gamma}u^\alpha u^\gamma} = \frac{h\big(\mathcal{R}(u^\alpha\hat{e}_{\bar\alpha}), u^\gamma\hat{e}_{\bar\gamma}\big)}{h(u^\alpha\hat{e}_{\bar\alpha}, u^\gamma\hat{e}_{\bar\gamma})} = K(\Phi(y\wedge u)).$$

The remark of Proposition 3.3 suggests that in natural coordinates $(x^i; y^i)$

$$\Phi\left(y\wedge\frac{\partial}{\partial x^i}\right) = \left[F\frac{\partial}{\partial y^i}\right] = \left[\frac{\partial}{\partial y^i}\right].$$

Thus Φ is independent of the Finsler metric F. □

Remark. It is straightforward to verify that the tensors $\mathcal{V}$, $\mathcal{H}$ and $\mathcal{R}$ can be localized. In fact, given two vector fields $u, v \in TSM$, if $u_{(x_0,[y_0])} = v_{(x_0,[y_0])}$, then

$$\mathcal{V}(u)|_{(x_0,[y_0])} = \mathcal{V}(v)|_{(x_0,[y_0])}.$$

The following two useful lemmas will be used later.

Lemma 3.10. *Using the above notations, we have the following identity for $v \in VSM$:*

$$P_{\mathcal{V}}[X, v] - \mathcal{V}[X, \mathcal{H}(v)] = 0. \tag{3.10}$$

Proof. Note that $[X, v] = -\mathcal{H}(v) - \frac{1}{2}\mathcal{V}[X, [X, v]]$. Thus its vertical part is $-\frac{1}{2}\mathcal{V}[X, [X, v]]$. Hence the left hand side of (3.10) is equal to

$$\begin{aligned}\mathcal{V}[X, -\frac{1}{2}[X, v] - \mathcal{H}(v)] &= \mathcal{V}\left[X, \frac{1}{2}[X, v] + \frac{1}{2}\mathcal{V}[X, [X, v]]\right] \\ &= \mathcal{V}\left[X, \frac{1}{2}[X, v]\right] - \frac{1}{2}\mathcal{V}[X, [X, v]] = 0.\end{aligned}$$

The desired identity is proved. □

Lemma 3.11. $\omega[X, v] = \omega[X, u] = 0$, $\omega[X, [X, v]] = \omega[X, [X, u]] = 0$, *where $v \in VSM$, $u \in HSM$.*

Proof. By (3.1) and (3.2), $[X, v]$ and $[X, u]$ have no X-component, thus $\omega[X, v] = \omega[X, u] = 0$. Using this fact again, we get the result. □

4. A dual construction on the co-sphere bundles

The geometry of the cotangent bundle is naturally dual to the geometry of the tangent bundle via the Legendre transformation [12]. In a similar manner, we claim that the geometry of the co-sphere bundle is dual to the geometry on the sphere bundle, which we have discussed in Section 3. However, we can derive more results because the cotangent bundle admits a canonical symplectic structure.

By introducing (x^i, p_i) as natural coordinates on T^*M, i.e., $p = p_i \mathrm{d}x^i \in T^*_x M$, we get a symplectic form $\mathrm{d}p := \mathrm{d}p_i \wedge \mathrm{d}x^i$.

Definition 4.1 ([12]). For each function f on T^*M, there is a unique vector field X_f satisfying

$$\mathrm{d}p(X_f, \cdot) = -\mathrm{d}f,$$

which is called the *Hamiltonian vector field* for f.

Indeed, X_f can be locally expressed as

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial}{\partial p_i}.$$

Definition 4.2 ([12]). Given two functions $f, g : T^*M \to \mathbb{R}$, the *Poisson bracket* $\{f, g\}$ is defined by

$$\{f, g\} := \mathrm{d}p(X_f, X_g).$$

Using Definition 4.1, we have the following equivalent expressions:

$$\{f, g\} = -\mathrm{d}f(X_g) = \mathrm{d}g(X_f).$$

It is easy to verify that

Proposition 4.3 ([11]). $X_{\{f,g\}} = -[X_f, X_g]$.

We associate a Finsler metric F on M to a family of maps $L^F_x : T_xM \to T^*_xM$ defined by

$$(L^F_x(y))(u) := \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}F^2(x, y + tu)\bigg|_{t=0} = F\omega(u), \quad y, u \in T_xM.$$

We call the family $L^F := \{L^F_x \mid x \in M\}$ the Legendre transformation. In fact, $L^F : TM \to T^*M$ is a bundle map.

Note that L^F is a smooth diffeomorphism on $TM\backslash\{0\}$ and the function $H(x, p) := F(x, (L^F_x)^{-1}(p)), p \in T^*_xM$ is a Minkowski norm on T^*_xM [11].

Recall that a smooth manifold where each cotangent space T^*_xM is equipped with a Minkowski norm $H(x, p)$ smoothly depending on x, is called a *Cartan manifold* [12], and H is called a *Cartan metric*. The Hamiltonian vector field X_H for a given H has the local expression

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}.$$

By definition, H is p-homogeneous of degree one. It follows that $p(X_H) = p_i\mathrm{d}x^i(X_H) = H$.

In the rest of this section, unless otherwise stated, we always assume that $H : T^*M \to \mathbb{R}$ is the dual of $F : TM \to \mathbb{R}$ in the sense of the Legendre transformation $L^F : TM \to T^*M$, i.e., $H(x, p) = F(x, (L^F_x)^{-1}(p))$.

Lemma 4.4. *We have an alternative definition for H, that is*

$$H(x, p) = \max_{y \in T_xM\backslash\{0\}} \frac{p(y)}{F(x, y)}.$$

Proof. By homogeneity, we can restrict our considerations to the indicatrix

$$\{y \in T_xM \mid F(x,y) = 1\}.$$

Since the indicatrix is compact, there exists some point $y_0 \in T_xM\backslash\{0\}$, such that

$$\frac{p(y_0)}{F(x,y_0)} = \max_{y\in T_xM} \frac{p(y)}{F(x,y)}.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{p(y_0+tu)}{F(x,y_0+tu)}\bigg|_{t=0} = 0, \quad \forall u \in T_xM.$$

Expanding this equation yields

$$\frac{p(u)}{F(x,y_0)} - \frac{p(y_0)}{F^2(x,y_0)}\,\frac{\mathrm{d}}{\mathrm{d}t}F(y_0+tu)\bigg|_{t=0} = 0, \quad \forall u \in T_xM.$$

Together with the definition of $L_x^F(y_0)$, we have

$$p(u) = \frac{p(y_0)}{F^2(x,y_0)}(L_x^F(y_0))(u), \quad \forall u \in T_xM,$$

hence

$$p = \frac{p(y_0)}{F^2(x,y_0)}L_x^F(y_0).$$

Taking $H(x,\cdot\,)$ on both sides, we have

$$H(x,p) = \frac{p(y_0)}{F^2(x,y_0)}H(x,L_x^F(y_0)) = \frac{p(y_0)}{F(x,y_0)} = \max_{y\in T_xM}\frac{p(y)}{F(x,y)}. \qquad \square$$

Remark. From the above proof, it is clear that the maximizer of $p(y)/F(x,y)$ can be taken as $y = (L_x^F)^{-1}(p)$. H is called the *co-Finsler metric* of F [17].

Note that $p = L^F(y) = F(x,y)\omega \in T_x^*M$. If we view ω as a 1-form on TM, then $\omega^\flat$, its dual quantity on T^*M, satisfies

$$\omega^\flat := \big((L^F)^{-1}\big)^*\omega = \big((L^F)^{-1}\big)^*\frac{L_x^F(y)}{F(x,y)} = \frac{p}{H}.$$

Lemma 4.5. *The Hamiltonian vector field X_H satisfies*

$$\omega^\flat(X_H) = 1, \quad d\omega^\flat(X_H,\cdot) = 0,$$

i.e., $X_H = (L^F)_*X$. We also denote X_H by $X^\flat$.

Proof. From previous analysis $\omega^\flat(X_H) = p(X_H)/H = 1$. By Definition 4.1 we have $\mathrm{d}H(X_H) = -\mathrm{d}p(X_H, X_H) = 0$. It follows that

$$\begin{aligned}\mathrm{d}\omega^\flat(X_H,\cdot) &= \mathrm{d}(p/H)(X_H,\cdot) = \mathrm{d}\frac{1}{H}\wedge p(X_H,\cdot) + \frac{1}{H}\mathrm{d}p(X_H,\cdot)\\ &= -\frac{1}{H^2}\mathrm{d}H\wedge p(X_H,\cdot) - \frac{1}{H}\mathrm{d}H\\ &= -\frac{1}{H^2}\mathrm{d}H(X_H)\cdot p + \frac{1}{H^2}p(X_H)\mathrm{d}H - \frac{1}{H}\mathrm{d}H\\ &= \frac{1}{H}\mathrm{d}H - \frac{1}{H}\mathrm{d}H = 0.\end{aligned}$$

Comparing this with Proposition 3.2 we conclude that X_H is dual to X. □

A quick glance shows that both $\omega^\flat$ and $X^\flat$ are sitting on S^*M, the co-sphere bundle of M. Hence they can be used to define other quantities on S^*M just as we have done in Section 2. Those quantities naturally become the duals of their corresponding quantities on SM without any change of formulation, because they are all defined by exterior differentiation and Lie brackets, which have good commutability with pullbacks and tangent maps. To reduce the amount of symbols, we denote each dual quantity on S^*M by simply adding a '$^\flat$' to the corresponding symbol on SM, as we have done with ω and X. The dual of the vertical distribution $\mathcal{V}$ is now denoted by $\mathcal{V}^\flat$. It is characterized by

Proposition 4.6. *Given a Cartan metric H on M, there is a unique $(1,1)$-tensor $\mathcal{V}^\flat$ satisfying*

$$\mathcal{V}^\flat(v) = \mathcal{V}^\flat(X^\flat) = 0, \quad \mathcal{V}^\flat[X^\flat, v] = -v, \quad \forall v \in VS^*M, \tag{4.1}$$

where $VS^*M := \{v \in TSM \mid v(f) = 0, \forall f \in C^\infty(M) \subset C^\infty(S^*M)\}$ is the dual of VSM, which is independent of the metric H.

We omit the proof because of duality. For the same reason, we can prove the remaining results of Section 3 step by step and we obtain

Proposition 4.7. *Define $\mathcal{H}^\flat(v) := -[X^\flat, v] - \frac{1}{2}\mathcal{V}[X^\flat, [X^\flat, v]]$ for $v \in VS^*M$, then $\mathcal{H}^\flat$ is C^∞-linear. Additionally, set $HS^*M := \mathcal{H}^\flat(VS^*M)$ and define*

$$\mathcal{H}^\flat(u) := \mathcal{H}^\flat(X^\flat) := 0, \quad u \in HS^*M,$$

then $\mathcal{H}$ is a $(1,1)$-tensor on S^*M.

Proposition 4.8. *The tensor $\mathcal{J}^\flat := \mathcal{H}^\flat - \mathcal{V}^\flat$ is an almost complex structure on $HS^*M \oplus VS^*M$.*

We denote the projection to VS^*M (resp. HS^*M) by $P^\flat_{\mathcal{V}} := \mathcal{V}^\flat \circ \mathcal{H}^\flat$ (resp. $P^\flat_{\mathcal{H}} := \mathcal{H}^\flat \circ \mathcal{V}^\flat$). Define

$$\begin{aligned} \mathcal{R}^\flat(v) &:= P^\flat_{\mathcal{V}}[X^\flat, \mathcal{H}^\flat(v)], && v \in VS^*M \\ \mathcal{R}^\flat(u) &:= \mathcal{H}^\flat[X^\flat, u], && u \in HS^*M \\ \mathcal{R}^\flat(X^\flat) &:= 0, \end{aligned}$$

then the *flag curvature* on $P(VS^*M)$ is given by

$$K^\flat(v) := \frac{h^\flat(\mathcal{R}^\flat(v), v)}{h^\flat(v,v)}, \quad v \in TS^*M, \tag{4.2}$$

where

$$h^\flat(v_1, v_2) := \begin{cases} \mathrm{d}\omega^\flat([X, v_1], v_2), & \text{if } v_1, v_2 \in VS^*M; \\ h^\flat(\mathcal{J}^\flat(v_1), \mathcal{J}^\flat(v_2)), & \text{if } v_1, v_2 \in HS^*M; \\ 0, & \text{if } v_1 \in VS^*M,\ v_2 \in HS^*M. \end{cases}$$

The following observations will be used in the proof of our main results. They are the duals of Lemma 3.10 and Lemma 3.11.

Lemma 4.9.

$$P^\flat_{\mathcal{V}}[X^\flat, v] - \mathcal{V}^\flat[X^\flat, \mathcal{H}^\flat(v)] = 0, \quad v \in VS^*M. \tag{4.3}$$

Lemma 4.10. $\omega^\flat[X^\flat, v] = \omega^\flat[X^\flat, u] = 0$, $\omega^\flat[X^\flat, [X^\flat, v]] = \omega^\flat[X^\flat, [X^\flat, u]] = 0$, *where* $v \in VS^*M, u \in HS^*M$.

5. Homothetic maps and homothetic fields

In this section we discuss homothetic maps and homothetic fields for a Finsler (resp. Cartan) manifold, the latter being of special importance for the navigation problem.

Note that a local diffeomorphism $\varphi : M \to M$ can be (locally) lifted to the maps $\breve{\varphi} : TM_0 \to TM_0$ and $\tilde{\varphi} : T^*M_0 \to T^*M_0$, where

$$\begin{aligned} \breve{\varphi}(x, y) &:= (\varphi(x), \varphi_*(y)), \quad y \in T_xM; \\ \tilde{\varphi}(x, p) &:= (\varphi(x), (\varphi^*)^{-1}(p)), \quad p \in T^*_xM. \end{aligned}$$

With this notation, an isometry φ of a Finsler manifold (M, F) (resp. Cartan manifold (M, H)) actually means

$$F = \breve{\varphi}^* F \quad (\text{resp. } H = \tilde{\varphi}^* H),$$

i.e. $F(x, y) = F(\varphi(x), \varphi_*(y))$ (resp. $H(x, p) = H(\varphi(x), (\varphi^*)^{-1}(p))$).

Definition 5.1. For a Finsler manifold (M,F) (resp. Cartan manifold (M,H)), the map $\varphi : M \to M$ is called homothetic, if there exists a constant $\lambda > 0$, s.t.

$$\breve{\varphi}^* F = \lambda F \quad (\text{resp. } \tilde{\varphi}^* H = \lambda^{-1} H). \tag{5.1}$$

An immediate consequence of Definition 5.1 is the following

Lemma 5.2. *For a homothetic map φ on a Finsler manifold (M,F) (resp. Cartan manifold (M,H)), we have*

1. $\breve{\varphi}^*\omega = \lambda\omega$ *(resp.* $\tilde{\varphi}^*\omega^\flat = \lambda\omega^\flat$*);*
2. $\breve{\varphi}_* X = \lambda \cdot X$ *(resp.* $\tilde{\varphi}_* X^\flat = \lambda \cdot X^\flat$*);*
3. $\breve{\varphi}_* \circ \mathcal{V} = \lambda^{-1} \cdot \mathcal{V} \circ \breve{\varphi}_*$ *(resp.* $\tilde{\varphi}_* \circ \mathcal{V}^\flat = \lambda^{-1} \cdot \mathcal{V}^\flat \circ \tilde{\varphi}_*$*);*
4. $\breve{\varphi}_* \circ \mathcal{H} = \lambda \cdot \mathcal{H} \circ \breve{\varphi}_*$. *(resp.* $\tilde{\varphi}_* \circ \mathcal{H}^\flat = \lambda \cdot \mathcal{H}^\flat \circ \tilde{\varphi}_*$*).*

Proof. We only prove the Finslerian case. Rewriting (5.1) as

$$F(\varphi(x), \varphi_*(y)) = \lambda F(x,y), \quad \forall y \in T_x M$$

we have

$$F(\varphi(x), \varphi_*(y+tu)) = \lambda F(x, y+tu), \quad \forall y, u \in T_x M, \ t \in \mathbb{R}.$$

Differentiating the above equation with respect to t at $t = 0$ yields $\omega(\varphi_*(u)) = \lambda\omega(u)$ for $u \in T_x M$. It follows that $\breve{\varphi}^*\omega = \lambda\omega$.

One can verify that $\frac{1}{\lambda}\breve{\varphi}_* X$ satisfies (3.3). Thus $\breve{\varphi}_* X = \lambda X$.

For a vertical vector field v and $f \in C^\infty(M)$ we have

$$(\breve{\varphi}_* v)(f) = v(\breve{\varphi}^* f) = v(f \circ \varphi) = 0.$$

From this we obtain that $\breve{\varphi}_* v$ is vertical. Thus, $\breve{\varphi}_* \circ \mathcal{V}(v) = \lambda^{-1} \cdot \mathcal{V} \circ \breve{\varphi}_*(v) = 0$.

Form the second equation in Lemma 5.2 we have $\breve{\varphi}_* \circ \mathcal{V}(X) = \lambda^{-1} \cdot \mathcal{V} \circ \breve{\varphi}_*(X)$. It follows that $\lambda^{-1} \cdot \mathcal{V} \circ \breve{\varphi}_*[X,v] = \lambda^{-1} \cdot \mathcal{V}[\breve{\varphi}_*(X), \breve{\varphi}_*(v)] = \lambda^{-1} \cdot \mathcal{V}[\lambda X, \breve{\varphi}_*(v)] = -\breve{\varphi}_*(v) = \breve{\varphi}_* \circ \mathcal{V}[X,v]$. Thus $\breve{\varphi}_* \circ \mathcal{V} = \lambda^{-1} \cdot \mathcal{V} \circ \breve{\varphi}_*$.

From the above results and the definition of $\mathcal{H}$ we obtain $\breve{\varphi}_* \circ \mathcal{H} = \lambda \cdot \mathcal{H} \circ \breve{\varphi}_*$. □

Recall that a (local) *flow* on a manifold M is a map $\phi : (-\epsilon, \epsilon) \times M \to M$, also denoted by $\phi_t := \phi(t, \cdot)$, satisfying

- $\phi_0 = \mathrm{id} : M \to M$;
- $\phi_s \circ \phi_t = \phi_{s+t}$ for any $s, t \in (-\epsilon, \epsilon)$ with $s + t \in (-\epsilon, \epsilon)$.

Hence, the lift of a flow ϕ_t on M is again a flow $\breve{\phi}_t$ (resp. $\tilde{\phi}_t$) on TM (resp. T^*M),

$$\breve{\phi}_t(x,y) := (\phi_t(x), \phi_{t*}(y)), \quad (\text{resp. } \tilde{\phi}_t(x,p) := (\phi_t(x), (\phi_t^*)^{-1}(p))) \tag{5.2}$$

By the relationship of vector fields and flows, (5.2) induces a natural way to lift a vector field u on M to a vector field X_u (resp. X_u^*) on TM (resp. T^*M). In natural coordinates, we have

$$\begin{aligned} X_u &= u^i \frac{\partial}{\partial x^i} + y^j \frac{\partial u^i}{\partial x^j}\frac{\partial}{\partial y^i} \in \Gamma(T(TM_0)) \\ X_u^* &= u^i \frac{\partial}{\partial x^i} - p_j \frac{\partial u^j}{\partial x^i}\frac{\partial}{\partial p_i} \in \Gamma(T(T^*M_0)), \end{aligned} \tag{5.3}$$

where $u = u^i \frac{\partial}{\partial x^i}$.

Remark. X_u is not dual to X_u^* in general.

Definition 5.3. A vector field V on a Finsler manifold (M,F) (resp. Cartan manifold (M,H)) is said to be a *homothetic field of F (resp. H) with dilation c*, if the corresponding flow ϕ_t is homothetic, i.e.,

$$\breve{\phi}_t^* F = e^{2ct}F \quad (\text{resp. } \tilde{\phi}_t^* H = e^{-2ct}H). \tag{5.4}$$

In particular V is called a *Killing field* if $c = 0$.

Differentiating (5.4) with respect to t at $t = 0$ yields

Lemma 5.4. *V is a homothetic field of F (resp. H) if and only if $X_V(F) = 2cF$ (resp. $X_V(H) = -2cH$).*

Example. Consider the domain

$$M := \left\{x = (x^1, \dots, x^n) \in \mathbb{R}^n \mid x^1 > 0\right\}.$$

Note that $|x| \neq 0$ for $\forall x \in M$. Let

$$F(x,y) = \frac{x^1}{|x|}|y|, \quad y \in T_xM$$

be a reversible Finsler metric. Put $V(x) = \sum_i x^i \dfrac{\partial}{\partial x^i}$. It is easy to see that

$$\frac{\partial |x|}{\partial x^i} = \frac{x^i}{|x|}$$

for $i = 1, \dots, n$. Hence

$$\frac{\partial F}{\partial x^j} = \begin{cases} \dfrac{(x^2)^2 + \cdots + (x^n)^2}{|x|^3}|y| & \text{if } j = 1, \\ -\dfrac{x^1 x^j}{|x|^3}|y| & \text{if } j \geq 2. \end{cases}$$

It follows that

$$\begin{aligned} X_V(F) &= V^i \frac{\partial F}{\partial x^i} + y^j \sum_i \frac{\partial V^j}{\partial x^i}\frac{\partial F}{\partial y^i} = x^i \frac{\partial F}{\partial x^i} + y^j \sum_i \delta_i^j \frac{\partial F}{\partial y^i} \\ &= x^1 \frac{(x^2)^2 + \cdots + (x^n)^2}{|x|^3}|y| + \Sigma_{i\geq 2} x^i \left(-\frac{x^1 x^i}{|x|^3}|y|\right) + y^i \frac{\partial F}{\partial y^i} = F. \end{aligned}$$

Hence V is a homothetic field of F with dilation $\frac{1}{2}$.

An important property of homothetic fields is the invariance of horizontal and vertical distributions under Lie derivatives.

Lemma 5.5. *Given a homothetic field V on a Finsler manifold (M, F) (resp. Cartan manifold (M, H)), we have*

1. *$[X_V, u]$ is vertical for $u \in VSM$ (resp. $[X_V^*, u]$ is vertical for $u \in VS^*M$);*
2. *$[X_V, w]$ is horizontal for $w \in HSM$ (resp. $[X_V^*, w]$ is horizontal for $w \in HS^*M$).*

Proof. As before, we only prove the Finslerian case. Let ϕ_t (resp. $\breve{\phi}_t$) be the flow generated by V (resp. X_V).

As indicated in the proof of Lemma 5.2, $\breve{\phi}_{t*}u$ is vertical. Hence $[X_V, u] = \lim_{t\to 0} \frac{u - \breve{\phi}_{t*}u}{t}$ is vertical.

Considering $w \in HSM$, we write $w = \mathcal{H}(u)$, where $u \in VSM$. Then from Lemma 5.2 we get that

$$\breve{\phi}_{t*}w = \breve{\phi}_{t*}(\mathcal{H}(u)) = e^{2ct}\mathcal{H}(\breve{\phi}_{t*}u).$$

Thus $\breve{\phi}_{t*}w$ is horizontal. Hence $[X_V, w] = \lim_{t\to 0} \frac{w - \breve{\phi}_{t*}w}{t}$ is horizontal, too. □

We provide here the following lemma with Cartan version.

Lemma 5.6. *For a homothetic field V on a Cartan manifold (M, H) with dilation c, we have $[X^\flat, X_V^*] = -2cX^\flat$.*

Proof. Actually, X_V^* is the Hamiltonian vector field of $p(V)$. In view of Proposition 4.3, we have $[X^\flat, X_V^*] = [X_H, X_{p(V)}] = -X_{\{H, p(V)\}} = X_{X_V^*(H)} = X_{(-2cH)} = -2cX_H$. □

6. Navigation problem

Recall that a navigation problem makes use of a Finsler metric F and a vector field V with $F(x, V_x) < 1$, and produces a new Finsler metric $\tilde{F}$ by solving the equation

$$F\big(x, y + \tilde{F}(x,y)V\big) = \tilde{F}(x,y). \tag{6.1}$$

With the help of the Legendre transformation, we obtain Cartan metrics $H(x,p)$ and $\tilde{H}(x,p)$ on M, i.e., $H(x,p) := F(x, (L_x^F)^{-1}(p))$, and $\tilde{H}(x,p) := \tilde{F}(x, (L_x^{\tilde{F}})^{-1}(p))$.

Lemma 6.1 ([17])**.** *The above H and $\tilde{H}$ are related by*

$$H(x,p) = \tilde{H}(x,p) + p(V). \tag{6.2}$$

Proof. By Lemma 4.4,

$$\begin{aligned} H(p) &= \max_{y\in T_xM} \frac{p(y)}{F(x,y)} = \max_{y\in T_xM} \frac{p(y+\tilde{F}v)}{F(x, y+\tilde{F}v)} \\ &= \max_{y\in T_xM} \frac{p(y+\tilde{F}v)}{\tilde{F}(x,y)} = \max_{y\in T_xM} \frac{p(y)}{\tilde{F}(x,y)} + p(V) \\ &= \tilde{H}(x,p) + p(V). \end{aligned}$$ □

Remark. In the above proof we have used the fact that $y \to y + \tilde{F}v$ is a $1-1$ map on T_xM. By the remark of Lemma 4.4 we see that if $(L_x^F)^{-1}(p) = y$, then $(L_x^{\tilde{F}})^{-1}(p) = \lambda(y + \tilde{F}v)$, where $\lambda > 0$.

The simple relation (6.2) suggests that the curvatures of the Cartan metrics H and $\tilde{H}$ may be related by simple rules. Let $\omega^\flat, X^\flat, \mathcal{V}^\flat, \mathcal{H}^\flat, \mathcal{R}^\flat, h^\flat, K^\flat$ be quantities associated with H, and $\tilde{\omega}^\flat, \tilde{X}^\flat, \tilde{\mathcal{V}}^\flat, \tilde{\mathcal{H}}^\flat, \tilde{\mathcal{R}}^\flat, \tilde{h}^\flat, \tilde{K}^\flat$ be quantities associated with $\tilde{H}$, as their meaning was explained in Section 4.

By definition of the Hamiltonian vector field, we get

Lemma 6.2. $X^\flat = \tilde{X}^\flat + X_V^*$.

Proof.

$$\begin{aligned} X^\flat &= X_H = X_{\tilde{H}} + X_{p(V)} \\ &= X_{\tilde{H}} + X_V^* = \tilde{X}^\flat + X_V^*. \end{aligned}$$ □

Lemma 6.3. $\mathcal{V}^\flat = \tilde{\mathcal{V}}^\flat - \tilde{\mathcal{V}}^\flat(X_V^*) \otimes \omega^\flat$.

Proof. We will check that the right hand side satisfies (4.1).

It is clear that $(\tilde{\mathcal{V}}^\flat - \tilde{\mathcal{V}}^\flat(X_V^*)\otimes\omega^\flat)(v) = 0$, for $v \in VS^*M$.

By Lemma 6.2, $\tilde{\mathcal{V}}^\flat(X^\flat) - \tilde{\mathcal{V}}^\flat(X_V^*)\otimes\omega^\flat(X^\flat) = \tilde{\mathcal{V}}^\flat(\tilde{X}^\flat) = 0$.

By Lemma 4.10 and Lemma 5.5

$$\begin{aligned}\tilde{\mathcal{V}}^\flat[X^\flat, v] - \tilde{\mathcal{V}}^\flat(X_V^*)\otimes\omega^\flat[X^\flat, v] &= \tilde{\mathcal{V}}^\flat[X^\flat, v] \\ &= \tilde{\mathcal{V}}^\flat[\tilde{X}^\flat + X_V^*, v] = \tilde{\mathcal{V}}^\flat[\tilde{X}^\flat, v] = -v.\end{aligned}$$

Hence $\tilde{\mathcal{V}}^\flat - \tilde{\mathcal{V}}^\flat(X_V^*)\otimes\omega^\flat$ satisfies all the equations in (4.1), and it must coincide with $\mathcal{V}^\flat$. □

Lemma 6.3 tells us that $\mathcal{V}^\flat = \tilde{\mathcal{V}}^\flat$ on $HS^*M \oplus VS^*M$.

Lemma 6.4. *Let V be a homothetic field of H with dilation c. Then $\mathcal{H}^\flat(v) = \tilde{\mathcal{H}}^\flat(v) + c\cdot v$ for $v \in VS^*M$ and $\mathcal{H}^\flat(u) = \tilde{\mathcal{H}}^\flat(u) - c\cdot u + c^2\tilde{\mathcal{V}}^\flat(u)$ for $u \in HS^*M$.*

Proof. From Proposition 4.7 and Lemma 4.10, we obtain

$$-\mathcal{H}^\flat(v) = [X^\flat, v] + \frac{1}{2}\mathcal{V}^\flat[X^\flat,[X^\flat, v]] = [X^\flat, v] + \frac{1}{2}\tilde{\mathcal{V}}^\flat[X^\flat,[X^\flat, v]]$$

and

$$\begin{aligned}\tilde{\mathcal{V}}^\flat[X^\flat,[X^\flat,v]] &= \tilde{\mathcal{V}}^\flat[\tilde{X}^\flat,[X^\flat,v]] + \tilde{\mathcal{V}}^\flat[X_V^*,[X^\flat,v]] \\ &= \tilde{\mathcal{V}}^\flat[\tilde{X}^\flat,[\tilde{X}^\flat,v]] + \tilde{\mathcal{V}}^\flat[\tilde{X}^\flat,[X_V^*,v]] + \tilde{\mathcal{V}}^\flat[X_V^*,[X^\flat,v]] \\ &= \tilde{\mathcal{V}}^\flat[\tilde{X}^\flat,[\tilde{X}^\flat,v]] - [X_V^*,v] + \tilde{\mathcal{V}}^\flat[X_V^*,[X^\flat,v]] \\ &= \tilde{\mathcal{V}}^\flat[\tilde{X}^\flat,[\tilde{X}^\flat,v]] - [X_V^*,v] - \tilde{\mathcal{V}}^\flat[X^\flat,[v,X_V^*]] - \tilde{\mathcal{V}}^\flat[v,[X_V^*,X^\flat]] \\ &= \tilde{\mathcal{V}}^\flat[\tilde{X}^\flat,[\tilde{X}^\flat,v]] - 2[X_V^*,v] - \tilde{\mathcal{V}}^\flat[v,[X_V^*,X^\flat]],\end{aligned}$$

where $[X_V^*, X^\flat] = 2cX^\flat$ (by Lemma 5.6). It follows that $\tilde{\mathcal{V}}[v,[X_V^*,X^\flat]] = 2c\cdot v$.

Combining these results, we have

$$\mathcal{H}^\flat(v) = \tilde{\mathcal{H}}^\flat(v) + c\cdot v, \quad v \in VS^*M. \tag{6.3}$$

For $u \in HS^*M$ we write $u = \mathcal{H}^\flat(v)$, where $v \in VS^*M$. We have

$$\begin{aligned}\tilde{\mathcal{H}}^\flat(u) - c\cdot u + c^2\cdot\tilde{\mathcal{V}}^\flat(u) &= \tilde{\mathcal{H}}^\flat(\mathcal{H}^\flat(v)) - c\cdot\mathcal{H}^\flat(v) + c^2\cdot v \\ &= \tilde{\mathcal{H}}^\flat(\tilde{\mathcal{H}}^\flat(v) + cv) - c\cdot(\tilde{\mathcal{H}}^\flat(v) + cv) + c^2\cdot v \\ &= 0 = \mathcal{H}^\flat(u).\end{aligned}$$

It follows that $\mathcal{H}^\flat(u) = \tilde{\mathcal{H}}^\flat(u) - c\cdot u + c^2\cdot\tilde{\mathcal{V}}^\flat(u)$. □

Lemma 6.5. *With the same assumption,* $P^\flat_{\mathcal{V}} = P^\flat_{\tilde{\mathcal{V}}} - c\cdot\tilde{\mathcal{V}}^\flat$ *on* $HS^*M \oplus VS^*M$.

Proof. For $v \in VS^*M$,

$$P^\flat_{\mathcal{V}}(v) = \mathcal{V}^\flat \circ \mathcal{H}^\flat(v) = \tilde{\mathcal{V}}^\flat \circ (\tilde{\mathcal{H}}^\flat(v) + c\cdot v) = P^\flat_{\tilde{\mathcal{V}}}(v) = P^\flat_{\tilde{\mathcal{V}}}(v) - c\cdot\tilde{\mathcal{V}}^\flat(v).$$

For $u \in HS^*M$ we write $u = \mathcal{H}^\flat(v)$, where $v \in VS^*M$. Then

$$\begin{aligned} P^\flat_{\tilde{\mathcal{V}}}(u) - c\tilde{\mathcal{V}}^\flat(u) &= \tilde{\mathcal{V}}^\flat\tilde{\mathcal{H}}^\flat(\mathcal{H}^\flat(v)) - c\tilde{\mathcal{V}}^\flat(\mathcal{H}^\flat(v)) \\ &= \tilde{\mathcal{V}}^\flat\tilde{\mathcal{H}}^\flat(\tilde{\mathcal{H}}^\flat(v) + cv) - c\tilde{\mathcal{V}}^\flat(\tilde{\mathcal{H}}^\flat(v) + cv) = 0 = P^\flat_{\mathcal{V}}(u). \end{aligned}$$ □

Lemma 6.6. *For a homothetic field* V *of* H *with dilation* c *we have* $\mathcal{R}^\flat(v) = \tilde{\mathcal{R}}^\flat(v) + c^2 v$ *where* $v \in VS^*M$.

Proof. By Lemma 5.5, $[X^*_V, \mathcal{H}(v)]$ is horizontal. It follows that

$$\mathcal{R}^\flat(v) = P^\flat_{\mathcal{V}}[X^\flat, \mathcal{H}^\flat(v)] = P^\flat_{\mathcal{V}}[\tilde{X}^\flat + X^*_V, \mathcal{H}^\flat(v)] = P^\flat_{\mathcal{V}}[\tilde{X}^\flat, \mathcal{H}^\flat(v)].$$

Note that $[X^\flat, \mathcal{H}^\flat(v)]$ has no $X^\flat$-component. Now we have

$$\begin{aligned} \mathcal{R}^\flat(v) &= P^\flat_{\tilde{\mathcal{V}}}[\tilde{X}^\flat, \mathcal{H}^\flat(v)] - c\cdot\tilde{\mathcal{V}}^\flat[\tilde{X}^\flat, \mathcal{H}^\flat(v)] \\ &= P^\flat_{\tilde{\mathcal{V}}}[\tilde{X}^\flat, \tilde{\mathcal{H}}^\flat(v) + cv] - c\cdot\tilde{\mathcal{V}}^\flat[\tilde{X}^\flat, \tilde{\mathcal{H}}^\flat(v) + cv] \\ &= \tilde{\mathcal{R}}^\flat(v) + P^\flat_{\tilde{\mathcal{V}}}[\tilde{X}^\flat, cv] - c\cdot\tilde{\mathcal{V}}^\flat[\tilde{X}^\flat, \tilde{\mathcal{H}}^\flat(v)] + c^2 v \\ &= \tilde{\mathcal{R}}^\flat(v) + c\cdot\big(P^\flat_{\tilde{\mathcal{V}}}[\tilde{X}^\flat, v] - \tilde{\mathcal{V}}^\flat[\tilde{X}^\flat, \tilde{\mathcal{H}}^\flat(v)]\big) + c^2 v = \tilde{\mathcal{R}}^\flat(v) + c^2 v. \end{aligned}$$

At the last step we have used (4.3). □

Lemma 6.7. $h^\flat(v_1, v_2) = (\tilde{H}/H)\tilde{h}^\flat(v_1, v_2), v_1, v_2 \in VS^*M$.

Proof. By using the definitions of $h^\flat$ and of $\tilde{H}$, we obtain

$$h^\flat(v_1, v_2) = -\omega^\flat[[X^\flat, v_1], v_2] = -(p/H)[[X^\flat, v_1], v_2],$$

and

$$\tilde{h}(v_1, v_2) = -\tilde{\omega}^\flat[[\tilde{X}^\flat, v_1], v_2] = -\tilde{\omega}^\flat[[X^\flat, v_1], v_2] = -(p/\tilde{H})[[X^\flat, v_1], v_2].$$

Hence $h^\flat(v_1, v_2) = (\tilde{H}/H)\tilde{h}^\flat(v_1, v_2)$. □

From Lemma 6.6 and Lemma 6.7 we have the following

Lemma 6.8. $K^\flat(v) = \tilde{K}^\flat(v) + c^2$ *for* $v \in VS^*M$.

7. Main results and examples

Now we are going to prove our main theorem on a navigation problem in terms of a homothetic field.

Proof of Theorem 1.1. Applying the Legendre transformations in Section 6, we get two Cartan metrics $H(x,p)$, and $\tilde{H}(x,p)$, where V is a homothetic field of H with dilation c, $H(x,p) := F(x,(L^F_x)^{-1}p)$ and $\tilde{H}(x,p) := \tilde{F}(x,(L^{\tilde{F}}_x)^{-1}p)$. By Lemma 6.8 we have

$$\big[K^\flat(v)\big]_{(x,[p])} = \big[\tilde{K}^\flat(v)\big]_{(x,[p])} + c^2, \quad v \in VS^*M.$$

Pulling back to the sphere bundle, we have

$$\big[K((L^F)^{-1}_* v)\big]_{(x,[y])} = \big[\tilde{K}((L^{\tilde{F}})^{-1}_* v)\big]_{(x,[y+\tilde{F}v])} + c^2, \quad v \in VSM.$$

Note that

$$(L^F)^{-1}_* \frac{\partial}{\partial p_i} = \frac{\partial^2 H^2/2}{\partial p_i \partial p_j}\frac{\partial}{\partial y^j} = H\frac{\partial^2 H}{\partial p_i \partial p_j}\frac{\partial}{\partial y^j} = H\frac{\partial^2 \tilde{H}}{\partial p_i \partial p_j}\frac{\partial}{\partial y^j}, \tag{7.1}$$

where we have used the fact that $\dfrac{\partial H}{\partial p_j}\dfrac{\partial}{\partial y^j} = \dfrac{y^j}{F}\dfrac{\partial}{\partial y^j} = 0$ on SM. Similarly,

$$(L^{\tilde{F}})^{-1}_* \frac{\partial}{\partial p_i} = \tilde{H}\frac{\partial^2 \tilde{H}}{\partial p_i \partial p_j}\frac{\partial}{\partial y^j}. \tag{7.2}$$

Combining (7.1) with (7.2), we have $(L^{\tilde{F}})^{-1}_* v = \tilde{H}/H \cdot u$, where $(L^F)^{-1}_* v := u$. It follows that

$$\big[K(u)\big]_{(x,[y])} = \big[\tilde{K}(u)\big]_{(x,[y+\tilde{F}v])} + c^2.$$

By Lemma 3.9 we get the desired result. □

Proof of Theorem 1.2. Given the hypotheses of Theorem 1.2 the conclusion of Theorem 1.1 implies that $\sup \tilde{K}(x,y)$ is negative; thus by the rigidity result of Mo and Shen [13], $\tilde{F}$ must be of Randers type; consequently, by reconstructing F from $\tilde{F}$, we see that F is a Randers metric as well. □

Corollary 7.1. *Let $F = F(x,y)$ be a Finsler metric on a manifold M and V a vector field on M with $F(x,V_x) < 1$. Let $\tilde{F} = \tilde{F}(x,y)$ denote the Finsler metric on M defined by (1.1). Suppose that V is homothetic with dilation c. If F is of scalar curvature, then $\tilde{F}$ is also of scalar curvature. Moreover, if F has constant curvature, then so does $\tilde{F}$.*

Recall that a Finsler metric F is said to be of *scalar curvature* if its flag curvature $K_F(y,\Pi)=K_F(y)$ is a scalar function on the slit tangent bundle $TM\setminus\{0\}$. Furthermore, F is said to be of *constant (flag) curvature* if $K_F(y,\Pi)=\text{constant}$. Using SHEN's rigidity result in [19], we get

Corollary 7.2. *Let (M,F) be a closed Finsler manifold and V a vector field on M with $F(x,V_x)<1$. Suppose that V is homothetic with dilation c and the flag curvature K of F satisfies*

$$\sup K < c^2.$$

If F has constant S-curvature then it must be Riemannian.

For details about (constant) S-curvature, see [18], [19]. Similarly, we have AKBAR-ZADEH and NUMATA' type rigidity theorems (see [1], [8], [14]). Finally we give some examples.

Example 1. Given a Minkowski norm $\varphi:\mathbf{E}\to\mathbb{R}$ on a vector space $\mathbf{E}$, one can construct a convex domain $\Omega:=\{v\in\mathbf{E}\mid\varphi(v)<1\}$, $T_x\Omega\simeq\mathbf{E}$. Thus $(\Omega,F(x,y):=\varphi(y))$ is a Minkowski manifold and has constant curvature $K=0$. For each $x\in\Omega$, identify $T_x\Omega$ with $\mathbf{E}$. Then $V_x:=x$ is a vector field on Ω satisfying $F(x,V_x)=\varphi(x)<1$. Moreover, we have

$$X_V(F)=V^i\frac{\partial F}{\partial x^i}+y^j\frac{\partial V^j}{\partial x^i}\frac{\partial F}{\partial y^i}=y^j\delta_i^j\frac{\partial\varphi(y)}{\partial y^j}=F.$$

Hence V is a homothetic field of F with dilation $c=1/2$. The produced Finsler metric $\tilde{F}$, known as the *Funk metric* on Ω, also has constant curvature $\tilde{K}=K-c^2=-1/4$.

Example 2. As in [3], we substitute (x^1,x^2) by (x,y) and (y^1,y^2) by (p,q) for a two dimensional manifold. Let $M=\{(x,y)\in\mathbb{R}^2\mid y>1\}$ be a half-plane. Let

$$F(x,y;p,q)=\frac{\sqrt{p^2+2q^2}}{y}+\frac{q}{y}$$

be a Randers metric on M. One can verify that $V=(1,0)$ is a Killing field for F and $F(x,V)<1$. The corresponding navigation problem creates again a Randers metric

$$\tilde{F}(x,y;p,q)=\frac{\sqrt{2pqy+2q^2y^2-q^2+y^2p^2}}{y^2-1}+\frac{p+yq}{y^2-1}.$$

A direct calculation shows that their Gauss curvatures are given respectively by

$$\begin{aligned}K(x,y;p,q)&=-1/16(48q^8+192p^2q^6+165p^4q^4+50p^6q^2+5p^8)/(p^2+q^2)^4\\&\quad+1/4\sqrt{p^2+2q^2}(8q^6+30p^2q^4+19q^2p^4+3p^6)q/(p^2+q^2)^4\end{aligned}$$

and by

$$\begin{aligned}\tilde{K}(x,y;p,q) &= -(48y^4q^8 - 3q^8 + 36yq^7p + 18p^2q^6 + 192y^4p^2q^6 - 126y^2p^2q^6 \\ &\quad + 204y^3p^3q^5 - 56yp^3q^5 + 36y^2p^4q^4 + 165y^4p^4q^4 - 3p^4q^4 + 4q^3p^5y \\ &\quad + 120y^3p^5q^3 + 50y^4p^6q^2 + 18q^2y^2p^6 + 12p^7y^3q + 5y^4p^8)/[16y^4(p^2+q^2)^4] \\ &\quad + q(8y^3q^6 + 30y^3p^2q^4 + 19y^3p^4q^2 + 3y^3p^6 - 4y^2pq^5 + 23y^2p^3q^3 + 9y^2p^5q \\ &\quad + 2yq^6 - 13p^2yq^4 + 3yp^4q^2 + 3q^5p - 3p^3q^3) \\ &\qquad \cdot \sqrt{2pqy + 2q^2y^2 - q^2 + y^2p^2}/[4y^4(p^2+q^2)^4].\end{aligned}$$

By a direct computation, one can verify

$$K(x,y;p+\tilde{F},q) = \tilde{K}(x,y;p,q).$$

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(Received January 20, 2006; revised June 13, 2006)