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An order result for the exponential divisor function

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Abstract. The integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i} > 1$ if $b_i \mid a_i$ for every $i \in \{1, 2, \dots, s\}$. Let $\tau^{(e)}(n)$ denote the number of exponential divisors of n, where $\tau^{(e)}(1) = 1$ by convention. The aim of the present paper is to establish an asymptotic formula with remainder term for the r-th power of the function $\tau^{(e)}$, where $r \geq 1$ is an integer. This improves an earlier result of M. V. Subbarao [5].

1. Introduction

Let $n > 1$ be an integer of canonical form $n = \prod_{i=1}^{s} p_i^{a_i}$. The integer d is called an *exponential divisor* of n if $d = \prod_{i=1}^{s} p_i^{b_i}$, where $b_i \mid a_i$ for every $i \in \{1, 2, \dots, s\}$, notation: $d \mid_e n$. By convention $1 \mid_e 1$.

Let $\tau^{(e)}(n)$ denote the number of exponential divisors of n. The function $\tau^{(e)}$ is called the *exponential divisor function*. J. Wu [7] showed, improving an earlier result of M. V. Subbarao [5], that

$$\sum_{n \leq x} \tau^{(e)}(n) = Ax + Bx^{1/2} + O(x^{2/9} \log x), \tag{1}$$

where

$$A := \prod_{p} \left(1 + \sum_{a=2}^{\infty} \frac{\tau(a) - \tau(a-1)}{p^a}\right),$$

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$$B := \prod_p \left(1 + \sum_{a=5}^{\infty} \frac{\tau(a) - \tau(a-1) - \tau(a-2) + \tau(a-3)}{p^{a/2}}\right),$$

τ denoting the usual divisor function. The O-term can further be improved.

Other properties of the function $\tau^{(e)}$, compared with those of the divisor function τ were investigated in papers [1], [2], [4], [5].

M. V. Subbarao [5] remarked that for every positive integer r,

$$\sum_{n \le x} (\tau^{(e)}(n))^r \sim A_r x, \tag{2}$$

where

$$A_r := \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{(\tau(a))^r - (\tau(a-1))^r}{p^a}\right). \tag{3}$$

It is the aim of the present paper to establish the following more precise asymptotic formula for the r-th power of the function $\tau^{(e)}$, where $r \ge 1$ is an integer:

$$\sum_{n \le x} (\tau^{(e)}(n))^r = A_r x + x^{1/2} P_{2^r-2}(\log x) + O(x^{u_r+\varepsilon}), \tag{4}$$

for every $\varepsilon > 0$, where A_r is given by (3), P_{2^r-2} is a polynomial of degree $2^r - 2$ and $u_r := \frac{2^{r+1}-1}{2^{r+2}+1}$.

Note that a similar formula is known for the divisor function τ, namely for any integer $r \ge 2$,

$$\sum_{n \le x} (\tau(n))^r = x Q_{2^r-1}(\log x) + O(x^{v_r+\varepsilon}), \tag{5}$$

valid for every $\varepsilon > 0$, where $v_r := \frac{2^r-1}{2^r+2}$ and Q_{2^r-1} is a polynomial of degree $2^r - 1$, this goes back to the work of S. Ramanujan, cf. [8].

Formula (4) is a direct consequence of a simple general result, given in Section 2 as Theorem, regarding certain multiplicative functions f such that $f(n)$ depends only on the ℓ-full kernel of n, where $\ell \ge 2$ is a fixed integer.

We also consider a generalization of the exponential divisor function, see Section 4.

Let $\phi^{(e)}(n)$ denote the number of divisors d of n such that d and n have no common exponential divisors. The function $\phi^{(e)}$ is multiplicative and for every prime power p^a $(a \ge 1)$, $\phi^{(e)}(p^a) = \phi(a)$, where ϕ is the Euler function.

As another consequence of our Theorem we obtain for every integer $r \ge 1$ that

$$\sum_{n \le x} (\phi^{(e)}(n))^r = B_r x + x^{1/3} R_{2^r-2}(\log x) + O(x^{t_r+\varepsilon}), \tag{6}$$

for every $\varepsilon > 0$, where $t_r := \frac{2^{r+1}-1}{3\cdot 2^{r+1}}$, R_{2^r-2} is a polynomial of degree $2^r - 2$ and

$$B_r := \prod_p \left(1 + \sum_{a=3}^{\infty} \frac{(\phi(a))^r - (\phi(a-1))^r}{p^a}\right). \tag{7}$$

In the case $r = 1$ formula (6) was proved in [6] with a better error term.

Our error terms depend on estimates for

$$D(1, \underbrace{\ell, \ell, \ldots, \ell}_{k-1}; x) := \sum_{ab_1^\ell b_2^\ell \cdot \ldots \cdot b_{k-1}^\ell \le x} 1,$$

where $k, \ell \ge 2$ are fixed and $a, b_1, b_2, \ldots, b_{k-1} \ge 1$ are integers.

2. A general result

We prove the following general result.

Theorem. *Let f be a complex valued multiplicative arithmetic function such that*

a) $f(p) = f(p^2) = \cdots = f(p^{\ell-1}) = 1$, $f(p^\ell) = f(p^{\ell+1}) = k$ *for every prime p, where ℓ, $k \ge 2$ are fixed integers and*

b) *there exist constants C, $m > 0$ such that $|f(p^a)| \le Ca^m$ for every prime p and every $a \ge \ell + 2$.*

Then for $s \in \mathbb{C}$

i) $$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s)\zeta^{k-1}(\ell s)V(s), \qquad \operatorname{Re} s > 1,$$

where the Dirichlet series $V(s) := \sum_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $\operatorname{Re} s > \frac{1}{\ell+2}$,

ii) $$\sum_{n \le x} f(n) = C_f x + x^{1/\ell} P_{f,k-2}(\log x) + O(x^{u_{k,\ell}+\varepsilon}),$$

for every $\varepsilon > 0$, where $P_{f,k-2}$ is a polynomial of degree $k-2$, $u_{k,\ell} := \frac{2k-1}{3+(2k-1)\ell}$ and

$$C_f := \prod_p \left(1 + \sum_{a=\ell}^{\infty} \frac{f(p^a) - f(p^{a-1})}{p^a}\right).$$

iii) *The error term can be improved for certain values of k and ℓ. For example in the case $k = 3$, $\ell = 2$ it is $O(x^{8/25} \log^3 x)$.*

3. Proofs

The proof of the Theorem is based on the following lemma. For an integer $\ell \geq 1$ let $\mu_\ell(n) = \mu(m)$ or 0, according as $n = m^\ell$ or not, where μ is the Möbius function. Note that function μ_ℓ is multiplicative and for any prime power p^a ($a \geq 1$),

$$\mu_\ell(p^a) = \begin{cases} -1, & \text{if } a = \ell, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Furthermore, for an integer $h \geq 1$ let the function $\mu_\ell^{(h)}$ be defined in terms of the Dirichlet convolution by

$$\mu_\ell^{(h)} = \underbrace{\mu_\ell * \mu_\ell * \cdots * \mu_\ell}_{h}.$$

The function $\mu_\ell^{(h)}$ is also multiplicative.

Lemma. *For any integers $h, \ell \geq 1$ and any prime power p^a ($a \geq 1$),*

$$\mu_\ell^{(h)}(p^a) = \begin{cases} (-1)^j \binom{h}{j}, & \text{if } a = j\ell,\ 1 \leq j \leq h, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Proof of the Lemma. By induction on h. For $h = 1$ this follows from (8). We suppose that formula (9) is valid for h and prove it for $h+1$. Using the relation $\mu_\ell^{(h+1)} = \mu_\ell^{(h)} * \mu_\ell$ and (8) we obtain for $a < \ell$,

$$\mu_\ell^{h+1}(p^a) = \mu_\ell^{(h)}(p^a) = 0$$

and for $a \geq \ell$,

$$\mu_\ell^{(h+1)}(p^a) = \mu_\ell^{(h)}(p^a) - \mu_\ell^{h}(p^{a-\ell})$$

$$= \begin{cases} \mu_\ell^{(h)}(p^\ell) - 1 = -\binom{h}{1} - 1 = -\binom{h+1}{1}, & \text{if } a = \ell, \\ (-1)^j \binom{h}{j} - (-1)^{j-1}\binom{h}{j-1} = (-1)^j \binom{h+1}{j}, & \text{if } a = \ell j,\ 2 \leq j \leq h, \\ -\mu_\ell^{(h)}(p^{h\ell}) = -(-1)^h \binom{h}{h} = (-1)^{h+1}\binom{h+1}{h+1}, & \text{if } a = (h+1)\ell, \\ 0, & \text{otherwise,} \end{cases}$$

which proves the Lemma. $\square$

Proof of the Theorem. i) We can formally obtain the desired expression by taking $v = f * \mu * \mu_\ell^{(k-1)}$. Here v is multiplicative and easy computations show that $v(p^a) = 0$ for any $1 \le a \le \ell + 1$ and for $a \ge \ell + 2$,

$$v(p^a) = \sum_{j \ge 0} (-1)^j \binom{k-1}{j} \left(f(p^{a-j\ell}) - f(p^{a-j\ell-1}) \right),$$

where, according to the Lemma, the number of nonzero terms is at most k.

Let $M_k = \max_{0 \le j \le k-1} \binom{k-1}{j}$. We obtain that for every prime p and every $a \ge \ell + 2$,

$$|v(p^a)| \le 2kM_k C a^m.$$

For every $\varepsilon > 0$, $a^m \le 2^{a\varepsilon}$ for sufficiently large a, $a \ge a_0$ say, where $a_0 \ge \ell + 2$. For $\operatorname{Re} s > 1/(\ell+2)$ choose $\varepsilon > 0$ such that $\operatorname{Re} s - \varepsilon > 1/(\ell+2)$. Then

$$\sum_p \sum_{a \ge a_0} \frac{|v(p^a)|}{p^{as}} \le 2kM_kC \sum_p \sum_{a \ge a_0} \frac{2^{a\varepsilon}}{p^{as}} \le 2kM_kC \sum_p \sum_{a \ge a_0} \frac{1}{p^{a(s-\varepsilon)}}$$
$$= 2kM_kC \sum_p \frac{1}{p^{a_0(s-\varepsilon)}} \left(1 - \frac{1}{p^{s-\varepsilon}}\right)^{-1} \le 2kM_kC \left(1 - \frac{1}{2^{1/(\ell+2)}}\right)^{-1} \sum_p \frac{1}{p^{a_0(s-\varepsilon)}},$$

and obtain that $V(s)$ is absolutely convergent for Re $s > 1/(\ell+2)$.

Note that $v(p^{\ell+2}) = f(p^{\ell+2}) - k$ for every $\ell \ge 3$, $k \ge 2$ and for $\ell = 2$, $k \ge 2$ it is $v(p^4) = f(p^4) - \binom{k+1}{2}$.

ii) Consider the k-dimensional generalized divisor function

$$d(1, \underbrace{\ell, \ell, \ldots, \ell}_{k-1}; n) = \sum_{ab_1^\ell b_2^\ell \ldots b_{k-1}^\ell = n} 1.$$

According to i),

$$f(n) = \sum_{ab=n} d(1, \underbrace{\ell, \ell, \ldots, \ell}_{k-1}; a) v(b).$$

One has, see [3], Ch. 6,

$$\sum_{n \le x} d(1, \underbrace{\ell, \ell, \ldots, \ell}_{k-1}; n)$$
$$= K_1 x + x^{1/\ell} \left(K_2 \log^{k-2} x + K_3 \log^{k-3} x + \cdots + K_{k-1} \log x + K_k \right)$$
$$+ O(x^{u_{k,\ell} + \varepsilon}), \tag{10}$$

for every $\varepsilon > 0$, where $u_{k,\ell} = \frac{2k-1}{3+(2k-1)\ell}$ (see [3], Theorem 6.10), $K_1, K_2, \dots, K_{k-1}, K_k$ are absolute constants depending on k and ℓ and $K_1 = \zeta^{k-1}(\ell)$. For example for $k = 2$ one has $K_2 = \zeta(\frac{1}{\ell})$, and for $k = 3$: $K_2 = \frac{1}{\ell}\zeta(\frac{1}{\ell})$, $K_3 = (2\gamma - 1)\zeta(\frac{1}{\ell}) + \frac{1}{\ell}\zeta'(\frac{1}{\ell})$, where γ is Euler's constant.

We obtain

$$\sum_{n\le x} f(n) = \sum_{ab\le x} d(1, \underbrace{\ell, \ell, \dots, \ell}_{k-1}; a)v(b) = \sum_{b\le x} v(b) \sum_{a\le x/b} d(1, \underbrace{\ell, \ell, \dots, \ell}_{k-1}; a)$$
$$= \sum_{b\le x} v(b)\Big(K_1(x/b) + (x/b)^{1/\ell}\big(K_2 \log^{k-2}(x/b) + K_3 \log^{k-3}(x/b) + \dots$$
$$+ K_{k-1}\log(x/b) + K_k\big) + O((x/b)^{u+\varepsilon})\Big),$$

and obtain the desired result by partial summation and by noting that $u_{k,\ell} > 1/(\ell + 2)$.

iii) For $k = 3$, $\ell = 2$ the errror term of (10) is $O(x^{8/25}\log^3 x)$, cf. [3], Theorem 6.4. □

4. Applications

1. In case $f(n) = (\tau^{(e)}(n))^r$, where $r \ge 1$ is an integer, we obtain formula (4) applying the Theorem for $\ell = 2$, $k = 2^r$.

2. For $k \ge 2$ consider the multiplicative function $f(n) = \tau_k^{(e)}(n)$, where for every prime power p^a $(a \ge 1)$, $\tau_k^{(e)}(p^a) := \tau_k(a)$ representing the number of ordered k-tuples of positive integers $(x_1, \dots, x_k)$ such that $a = x_1 \cdot \ldots \cdot x_k$. Here $\tau_k(p^b) = \binom{b+k-1}{k-1}$ for every prime power p^b $(b \ge 1)$. In case $k = 2$, $\tau_2^{(e)}(n) = \tau^{(e)}(n)$.

Taking $\ell = 2$ and $k := k$ we obtain that $v(p^4) = \tau_k(4) - k(k+1)/2 = 0$ and $V(s)$ is absolutely convergent for $\operatorname{Re} s > \frac{1}{5}$ (and not only for $\operatorname{Re} s > \frac{1}{4}$ given by the Theorem),

$$\sum_{n\le x} \tau_k^{(e)}(n) = C_k x + x^{1/2} S_{k-2}(\log x) + O(x^{w_k+\varepsilon}), \tag{11}$$

for every $\varepsilon > 0$, where S_{k-2} is a polynomial of degree $k - 2$, $w_k := \frac{2k-1}{4k+1}$ and

$$C_k = \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{\tau_k(a) - \tau_k(a-1)}{p^a}\right).$$

For $k = 3$ the error term of (11) can be improved into $O(x^{8/25}\log^3 x)$.

A similar formula can be obtained for $\sum_{n\le x}(\tau_k^{(e)}(n))^r$.

3. For the function $\phi^{(e)}(n)$ defined in the Introduction we obtain formula (6) by choosing $\ell = 3$, $k = 2^r$.

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