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Almost everywhere convergence of a subsequence of the logarithmic means of quadratical partial sums of double Walsh–Fourier series

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Abstract. The main aim of this paper is to prove that the maximal operator of the logarithmic means of quadratical partial sums of double Walsh–Fourier series is of weak type $(1,1)$ provided that the supremum in the maximal operator is taken over special indicies. The set of Walsh polynomials is dense in $L_1(I\times I)$, so by the well-known density argument we have that $t_{2^n}f\left(x^1,x^2\right)\to f\left(x^1,x^2\right)$ a.e. for all integrable two-variable functions f.

1. Introduction

The partial sums $S_n(f)$ of the Walsh–Fourier series of a function $f\in L(I)$, $I=[0,1]$ converges in measure on I ([8], Ch. 5). The condition $f\in L\ln L(I\times I)$ provides convergence in measure on $I\times I$ of the rectangular partial sums $S_{n,m}(f)$ of double Fourier-Walsh series ([13], Ch. 3.) The first example of a function from classes wider than $L\ln L(I\times I)$ with $S_{n,n}(f)$ divergent in measure on $I\times I$ was obtained in [3]. Moreover, in each Orlicz space wider than $L\ln L(I\times I)$ the set of functions which quadratic Walsh–Fourier sums converge in measure on $I\times I$ is of first Baire category [11]. In [2] we proved that similar proposition is true also

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for logarithmic means of quadratical partial sums

$$t_n f\left(x^1, x^2\right) := \frac{1}{l_n} \sum_{i=1}^{n-1} \frac{S_{i,i}(f,x,y)}{n-i}$$

of double Walsh–Fourier series. We proved that for any Orlicz space, which is not a subspace of $L \ln L(I \times I)$, the set of the functions that these means converges in measure is of first Baire category. From this result follows that in classes wider than $L \ln L(I \times I)$ there exists functions f for which logarithmic means $t_n(f)$ of quadratical partial sums of double Walsh–Fourier series diverges in measure.

Besides, it is surprising that the two cases (the logarithmic means of quadratical and the two-dimensional partial sums) are not different in this point of view. Namely, for instance in the case of $(C,1)$ means we have a quite different situation. That is, it is well-known [13] that the Marcinkiewicz means $\sigma_n(f) = \frac{1}{n}\sum_{j=1}^{n} S_{j,j}(f)$, that is the $(C,1)$ means of quadratical partial sums of double trigonometric Fourier series of a function $f \in L$ converges in L-norm and a.e. to f. Analogical questions with respect to the Walsh, Vilenkin systems are discussed by Weisz [12], Goginava [5] and Gát [1].

Thus, in question of convergence in measure logarithmic means of quadratical partial sums of double Walsh–Fourier series differs from Marcinkiewicz means and like the usual quadratical partial sums of double Walsh–Fourier series. In spite of this in [7] it is proved the difference between Nörlund logarithmic summability and the usual convergence for Walsh–Fourier series.

The main aim of this paper is to prove that the maximal operator of the logarithmic means of quadratical partial sums of double Walsh–Fourier series is of weak type $(1,1)$ provided that the supremum in the maximal operator is taken over special indicies. The set of Walsh polynomials is dense in $L_1(I \times I)$, so by the well-known density argument we have that $t_{2^n} f\left(x^1, x^2\right) \to f\left(x^1, x^2\right)$ a.e. for all integrable two-variable function f.

2. Definitions and notation

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. Denote Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such a way that the measure of a singleton is $1/2$. Let I be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of I are of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0,1\} (k \in \mathbb{N})$. The group operation on I is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group I is called the Walsh group. A base for the neighborhoods of I can be given in the following way:

$$I_0(x) := I, \quad I_n(x) := \{y \in I : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\},$$
$$(x \in I,\ n \in \mathbb{N}).$$

These sets are called the dyadic intervals. Let $0 = (0 : i \in \mathbb{N}) \in I$ denote the null element of I, $I_n := I_n(0)$ $(n \in \mathbb{N})$. Set $\overline{I}_n := I \backslash I_n$.

For $k \in \mathbb{N}$ and $x \in I$ denote

$$r_k(x) := (-1)^{x_k} \qquad (x \in I, k \in \mathbb{N})$$

the k-th Rademacher function. If $n \in \mathbb{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0,1\}$ $(i \in \mathbb{N})$, i.e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh–Paley system is defined as the sequence of Walsh–Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \qquad (x \in I, n \in \mathbb{P}).$$

The Walsh–Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \in \overline{I}_n. \end{cases} \tag{1}$$

The rectangular partial sums of the 2-dimensional Walsh–Fourier series are defined as follows:

$$S_{M,N}(f; x^1, x^2) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \widehat{f}(i,j)\, w_i\left(x^1\right) w_j\left(x^2\right),$$

where the number

$$\widehat{f}(i,j) = \int_{I \times I} f\left(x^1, x^2\right) w_i\left(x^1\right) w_j\left(x^2\right) d\mu\left(x^1, x^2\right)$$

is said to be the (i,j)th Walsh–Fourier coefficient of the function f.

The norm of the space $L_p(I\times I)$ is defined by

$$\|f\|_p := \left(\int_{I\times I} \left|f\left(x^1,x^2\right)\right|^p d\mu\left(x^1,x^2\right)\right)^{1/p} \qquad (1\le p<\infty),$$

and $\|f\|_\infty := \operatorname{ess\,sup}|f(x^1,x^2)|$. The space weak-$L_1(I\times I)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak}-L_1(I\times I)} := \sup_{\lambda>0} \lambda\mu\left(|f|>\lambda\right) < +\infty.$$

The logarithmic means of cubical partial sums of the double Walsh–Fourier series are defined as follows

$$t_n f\left(x^1,x^2\right) = \frac{1}{l_n}\sum_{i=1}^{n-1}\frac{S_{i,i}\left(f,x^1,x^2\right)}{n-i},$$

where

$$l_n = \sum_{k=1}^{n-1}\frac{1}{k}.$$

Denote

$$F_n(x) = \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_k(x)}{n-k},$$

$$F_n\left(x^1,x^2\right) = \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_k\left(x^1\right)D_k\left(x^2\right)}{n-k},$$

$$K_n(x) = \frac{1}{n}\sum_{k=1}^{n} D_k(x),$$

$$K_n\left(x^1,x^2\right) = \frac{1}{n}\sum_{k=1}^{n} D_k\left(x^1\right)D_k\left(x^2\right).$$

For the function f we consider the maximal operators

$$t_\# f = \sup_{n\in\mathbb{N}} |t_{2^n} f|.$$

3. Formulation of the main results

Theorem 1. *Let $f \in L_1(I \times I)$. Then*

$$\mu\{t_{\#}f > \lambda\} \leq \frac{c}{\lambda}\|f\|_1.$$

Corollary 1. *Let $f \in L_1(I \times I)$. Then*

$$t_{2^n} f\left(x^1, x^2\right) \to f\left(x^1, x^2\right) \quad \text{a.e. as } n \to \infty.$$

4. Auxiliary propositions

Lemma 1 (Calderon–Zygmund decomposition [10])**.** *Let $f \in L_1(I \times I)$, $\lambda > \|f\|_1$. Then there exists $(u^{(i,1)}, u^{(i,2)}) \in I \times I$, $k_i \in \mathbb{N}\,(i = 1, 2, \ldots,)$ and a decomposition*

$$f = f_0 + \sum_{i=1}^{\infty} f_i,$$

where

1) $$\|f_0\|_1 \leq c\lambda, \qquad \|f_0\|_1 \leq c\|f\|_1;$$

2) $$\operatorname{supp} f_i \subset I_{k_i}\left(u^{i,1}\right) \times I_{k_i}\left(u^{i,2}\right), \quad \int_{I \times I} f_i = 0,\ i = 1, 2, \ldots;$$

3) $$\mu\left(\bigcup_{i=1}^{\infty}\left(I_{k_i}\left(u^{i,1}\right) \times I_{k_i}\left(u^{i,2}\right)\right)\right) \leq c\|f\|_1/\lambda.$$

Lemma 2. [6] *Let $A \geq k$, $A, k \in \mathbb{N}$. Then*

$$\int_{\overline{I}_k} \sup_{n \geq 2^A} |K_n(x)|\, d\mu(x) \leq c\frac{A - k + 1}{2^{A-k}}.$$

Lemma 3. [1] *Let $k \in \mathbb{N}$. Then*

$$\int_{\overline{I_k \times I_k}} \sup_{n \geq 2^k} \left|K_n\left(x^1, x^2\right)\right| d\mu\left(x^1, x^2\right) \leq c < \infty.$$

5. Proof of the main results

PROOF OF THEOREM 1. Since

$$D_{2^n-j} = D_{2^n} - w_{2^n-1} D_j,$$

we can write

$$\begin{aligned}
F_{2^n}\left(x^1, x^2\right) &= \frac{1}{l_{2^n}} \sum_{j=1}^{2^n-1} \frac{D_{2^n-j}\left(x^1\right) D_{2^n-j}\left(x^2\right)}{j} \\
&= D_{2^n}\left(x^1\right) D_{2^n}\left(x^2\right) - \frac{D_{2^n}\left(x^1\right) w_{2^n-1}\left(x^2\right)}{l_{2^n}} \sum_{j=1}^{2^n-1} \frac{D_j\left(x^2\right)}{j} \\
&\quad - \frac{D_{2^n}\left(x^2\right) w_{2^n-1}\left(x^1\right)}{l_{2^n}} \sum_{j=1}^{2^n-1} \frac{D_j\left(x^1\right)}{j} \\
&\quad + \frac{w_{2^n-1}\left(x^1\right) w_{2^n-1}\left(x^2\right)}{l_{2^n}} \sum_{j=1}^{2^n-1} \frac{D_j\left(x^1\right) D_j\left(x^2\right)}{j} \\
&= F_n^{(1)}\left(x^1, x^2\right) - F_n^{(2)}\left(x^1, x^2\right) - F_n^{(3)}\left(x^1, x^2\right) + F_n^{(4)}\left(x^1, x^2\right). \qquad (2)
\end{aligned}$$

Denote

$$t_n^{(i)} f := f * F_n^{(i)}, \quad i = 1, 2, 3, 4.$$

Since the operator

$$\sup_{n \in \mathbb{N}} 2^{2n} \left| \int_{I_n(x^1) \times I_n(x^2)} f\left(u^1, u^2\right) d\mu\left(u^1, u^2\right) \right|$$

is of weak type $(1, 1)$ and

$$t_{\#}^{(1)} f := \sup_{n \in \mathbb{N}} \left| t_n^{(1)} f \right| = \sup_{n \in \mathbb{N}} 2^{2n} \left| \int_{I_n(x^1) \times I_n(x^2)} f\left(u^1, u^2\right) d\mu\left(u^1, u^2\right) \right|,$$

we obtain that

$$\left\| t_{\#}^{(1)} f \right\|_{\text{weak}-L_1(I \times I)} \le c \left\| f \right\|_1. \qquad (3)$$

We prove that

$$\int_{\overline{I_N \times I_N}} \sup_{n \ge N} \left| F_n^{(4)}\left(x^1, x^2\right) d\mu\left(x^1, x^2\right) \right| \le c < \infty. \qquad (4)$$

Using Abel's transformation we can write that

$$\sum_{j=1}^{2^n-1}\frac{D_j\left(x^1\right)D_j\left(x^2\right)}{j}=\sum_{j=1}^{2^n-2}\frac{K_j\left(x^1,x^2\right)}{j+1}+K_{2^n-1}\left(x^1,x^2\right).$$

Then we have

$$\begin{aligned}&\int_{\overline{I_N\times I_N}}\sup_{n\geq N}\left|F_n^{(4)}\left(x^1,x^2\right)d\mu\left(x^1,x^2\right)\right|\\ &\quad\leq\int_{\overline{I_N\times I_N}}\sup_{n\geq N}\frac{1}{l_{2^n}}\sum_{j=1}^{2^n-2}\frac{\left|K_j\left(x^1,x^2\right)\right|}{j+1}d\mu\left(x^1,x^2\right)\\ &\quad\quad+\int_{\overline{I_N\times I_N}}\sup_{n\geq N}\left|K_{2^n-1}\left(x^1,x^2\right)\right|d\mu\left(x^1,x^2\right)=I+II.\end{aligned}\tag{5}$$

Since [4]

$$\sup_n\int_{I\times I}\left|K_n\left(x^1,x^2\right)\right|d\mu\left(x^1,x^2\right)<\infty,$$

from Lemma 3 we get

$$II\leq c<\infty,\tag{6}$$

and also

$$\begin{aligned}I&\leq\int_{\overline{I_N\times I_N}}\sup_{n\geq N}\frac{1}{l_{2^n}}\sum_{j=1}^{2^N-1}\frac{\left|K_j\left(x^1,x^2\right)\right|}{j+1}d\mu\left(x^1,x^2\right)\\ &\quad+\int_{\overline{I_N\times I_N}}\sup_{n\geq N}\frac{1}{l_{2^n}}\sum_{j=2^N}^{2^n-2}\frac{\left|K_j\left(x^1,x^2\right)\right|}{j+1}d\mu\left(x^1,x^2\right)\\ &\leq\frac{1}{l_N}\sum_{j=1}^{2^N-1}\frac{1}{j}\int_{I\times I}\left|K_j\left(x^1,x^2\right)\right|d\mu\left(x^1,x^2\right)\\ &\quad+\sup_{n\geq N}\frac{1}{l_{2^n}}\sum_{j=2^N}^{2^n-2}\frac{1}{j+1}\int_{\overline{I_N\times I_N}}\sup_{j\geq 2^N}\left|K_j\left(x^1,x^2\right)\right|d\mu\left(x^1,x^2\right)\leq c<\infty.\end{aligned}\tag{7}$$

Combining (5)–(7) we obtain the proof of (4).

Hence, we can write that (see Gát [1])

$$\left\|t_{\#}^{(4)}f\right\|_{\text{weak}-L_1(I\times I)}\leq c\left\|f\right\|_1.\tag{8}$$

Finally, we prove that

$$\left\|t_{\#}^{(2)} f\right\|_{\text{weak}-L_1(I\times I)} \leq c\,\|f\|_1 . \tag{9}$$

Since

$$\sum_{j=1}^{2^n-1} \frac{D_j(u)}{j} = \sum_{j=1}^{2^n-2} \frac{K_j(u)}{j+1} + K_{2^n-1}(u),$$

we have

$$\begin{aligned}
& t_n^{(2)} f\left(y^1, y^2\right) \\
&= \int_{I\times I} f\left(x^1, x^2\right) \frac{D_{2^n}\left(x^1+y^1\right) w_{2^n-1}\left(x^2+y^2\right)}{l_{2^n}} \sum_{j=1}^{2^n-1} \frac{D_j\left(x^2+y^2\right)}{j} d\mu\left(x^1, x^2\right) \\
&= \int_{I\times I} f\left(x^1, x^2\right) \frac{D_{2^n}\left(x^1+y^1\right) w_{2^n-1}\left(x^2+y^2\right)}{l_{2^n}} \sum_{j=1}^{2^n-2} \frac{K_j\left(x^2+y^2\right)}{j+1} d\mu\left(x^1, x^2\right) \\
&\quad + \int_{I\times I} f\left(x^1, x^2\right) \frac{D_{2^n}\left(x^1+y^1\right) w_{2^n-1}\left(x^2+y^2\right)}{l_{2^n}} K_{2^n-1}\left(x^2+y^2\right) d\mu\left(x^1, x^2\right) \\
&= t_n^{(2,1)} f\left(y^1, y^2\right) + t_n^{(2,2)} f\left(y^1, y^2\right).
\end{aligned} \tag{10}$$

Denote (use the notation of Lemma 1)

$$g(t) := \sum_{i=1}^{\infty} \frac{|f_i(t)|}{k_i}, L(t) := \sum_{i=1}^{\infty} \frac{|K_i(t)|}{i+1}.$$

Let

$$\left(y^1, y^2\right) \in \overline{\bigcup_{i=1}^{\infty} \left(I_{k_i}\left(u^{i,1}\right) \times I_{k_i}\left(u^{i,2}\right)\right)}. \tag{11}$$

Since $\int f_i = 0$ we have

$$t_n^{(2,1)} f_i\left(y^1, y^2\right) = 0 \quad \text{ for } n \leq k_i. \tag{12}$$

Let $y^1 \in \overline{I_{k_i}\left(u^{i,1}\right)}$. Then from (1) we can write that $t_n^{(2,1)} f_i\left(y^1, y^2\right) = 0$ for $n > k_i$. Hence $t_n^{(2,1)} f_i\left(y^1, y^2\right) \neq 0$ implies that $y^1 \in I_{k_i}\left(u^{i,1}\right)$. Consequently, from (11) we can suppose that

$$y^2 \in \bigcap_{i=1}^{\infty} \overline{I_{k_i}\left(u^{i,2}\right)}.$$

Then we write

$$D := \mu\left\{\left(y^1,y^2\right) \in I \times \left(\bigcap_{i=1}^{\infty} \overline{I_{k_i}\left(u^{i,2}\right)}\right) : t_{\#}^{(2,1)} f\left(y^1,y^2\right) > c\lambda\right\}$$
$$\leq \int_{\bigcap_{i=1}^{\infty} \overline{I_{k_i}(u^{i,2})}} \mu\left\{y^1 \in I : t_{\#}^{(2,1)}\left(\sum_{i=1}^{\infty} f_i\right)\left(y^1,y^2\right) > c\lambda\right\} d\mu\left(y^2\right). \tag{13}$$

From (12), we have

$$\left|t_n^{(2,1)}\left(\sum_{i=1}^{\infty} f_i\right)\left(y^1,y^2\right)\right| \leq \sum_{i=1}^{\infty}\left|\int_{I_{k_i}(u^{i,1})\times I_{k_i}(u^{i,2})} f_i\left(x^1,x^2\right)\right.$$
$$\left.\times \frac{D_{2^n}\left(x^1+y^1\right) w_{2^n-1}\left(x^2+y^2\right)}{l_{2^n}} \sum_{j=1}^{2^n-2} \frac{K_j\left(x^2+y^2\right)}{j+1} d\mu\left(x^1,x^2\right)\right|$$
$$\leq \int_I\left(\int_I \sum_{i=1}^{\infty} \frac{\left|f_i\left(x^1,x^2\right)\right|}{k_i} \sum_{j=1}^{2^n-2} \frac{\left|K_j\left(x^2+y^2\right)\right|}{j+1} d\mu\left(x^2\right)\right) D_{2^n}\left(x^1+y^1\right) d\mu\left(x^1\right)$$
$$= \int_I\left(\int_I g\left(x^1,x^2\right) L\left(x^2+y^2\right) d\mu\left(x^2\right)\right) D_{2^n}\left(x^1+y^1\right) d\mu\left(x^1\right).$$

The one-dimensional operator $\sup_{n\in\mathbb{N}}|S_{2^n} f|$ is of weak type $(1,1)$. We apply this fact for the one-dimensional function $h(x^1) := \int_I g(x^1,x^2)L(x^2+y^2)d\mu(x^2)$ for every fixed $y^2 \in I$. Consequently, from (13) and by the above we can write

$$D \leq \int_{\bigcap_{i=1}^{\infty} \overline{I_{k_i}(u^{i,2})}} \mu\left\{y^1 \in I : \sup_n \int_I \left(\int_I g\left(x^1,x^2\right) L\left(x^2+y^2\right) d\mu\left(x^2\right)\right)\right.$$
$$\left. \cdot D_{2^n}\left(x^1+y^1\right) d\mu\left(x^1\right) > c\lambda\right\} d\mu(y^2)$$
$$\leq \frac{c}{\lambda} \int_{\bigcap_{i=1}^{\infty} \overline{I_{k_i}(u^{i,2})}} \left[\int_I \left(\int_I g\left(x^1,x^2\right) L\left(x^2+y^2\right) d\mu\left(x^2\right)\right) d\mu\left(x^1\right)\right] d\mu\left(y^2\right)$$
$$= \frac{c}{\lambda} \int_{\bigcap_{i=1}^{\infty} \overline{I_{k_i}(u^{i,2})}} \left[\int_I \left(\int_I g\left(x^1,x^2\right) d\mu\left(x^1\right)\right) L\left(x^2+y^2\right) d\mu\left(x^2\right)\right] d\mu\left(y^2\right)$$
$$\leq \frac{c}{\lambda} \sum_{i=1}^{\infty} \frac{1}{k_i} \int_{\overline{I_{k_i}(u^{i,2})}} \left[\int_{I_{k_i}(u^{i,2})} \left(\int_{I_{k_i}(u^{i,1})} |f_i(x^1,x^2)| d\mu(x^1)\right) L(x^2+y^2) d\mu(x^2)\right] d\mu(y^2)$$

$$= \frac{c}{\lambda}\sum_{i=1}^{\infty}\frac{1}{k_i}\int\limits_{\overline{I_{k_i}(u^{i,2})}}\left[\int\limits_{I_{k_i}(u^{i,2})}\left(\int\limits_{I_{k_i}(u^{i,1})}\left|f_i\left(x^1,x^2\right)\right|d\mu\left(x^1\right)\right)\right.$$

$$\left.\cdot\sum_{j=1}^{2^{k_i}-1}\frac{\left|K_j\left(x^2+y^2\right)\right|}{j+1}d\mu(x^2)\right]d\mu\left(y^2\right)$$

$$+\frac{c}{\lambda}\sum_{i=1}^{\infty}\frac{1}{k_i}\int\limits_{\overline{I_{k_i}(u^{i,2})}}\left[\int\limits_{I_{k_i}(u^{i,2})}\left(\int\limits_{I_{k_i}(u^{i,1})}\left|f_i\left(x^1,x^2\right)\right|d\mu\left(x^1\right)\right)\right.$$

$$\left.\cdot\sum_{j=2^{k_i}}^{\infty}\frac{\left|K_j\left(x^2+y^2\right)\right|}{j+1}d\mu\left(x^2\right)\right]d\mu\left(y^2\right)=S+M. \tag{14}$$

Since [10]

$$\int_I\left|K_j\left(x\right)\right|d\mu\left(x\right)\leq c<\infty,$$

we have

$$S\leq\frac{c}{\lambda}\sum_{i=1}^{\infty}\frac{1}{k_i}\int_{I_{k_i}(u^{i,2})}\left[\int_{I_{k_i}(u^{i,1})}\left|f_i\left(x^1,x^2\right)\right|d\mu\left(x^1\right)\right.$$

$$\left.\int_{\overline{I_{k_i}(u^{i,2})}}\sum_{j=1}^{2^{k_i}-1}\frac{\left|K_j\left(x^2+y^2\right)\right|}{j+1}d\mu\left(y^2\right)\right]d\mu\left(x^2\right)$$

$$\leq\frac{c}{\lambda}\sum_{i=1}^{\infty}\left\|f_i\right\|_1\leq\frac{c}{\lambda}\left\|f\right\|_1 \tag{15}$$

Using Lemma 2 for M we have

$$M\leq\frac{c}{\lambda}\sum_{i=1}^{\infty}\frac{1}{k_i}\int_{I_{k_i}(u^{i,2})}\left[\int_{I_{k_i}(u^{i,1})}\left|f_i\left(x^1,x^2\right)\right|d\mu\left(x^1\right)\right.$$

$$\left.\int_{\overline{I_{k_i}(u^{i,2})}}\sum_{j=2^{k_i}}^{\infty}\frac{\left|K_j\left(x^2+y^2\right)\right|}{j+1}d\mu\left(y^2\right)\right]d\mu\left(x^2\right)$$

$$\leq\frac{c}{\lambda}\sum_{i=1}^{\infty}\frac{1}{k_i}\int_{I_{k_i}(u^{i,2})}\left[\int_{I_{k_i}(u^{i,1})}\left|f_i\left(x^1,x^2\right)\right|d\mu\left(x^1\right)\right.$$

$$\left.\sum_{r=k_i}^{\infty}\sum_{j=2^r}^{2^{r+1}-1}\frac{1}{j}\int_{\overline{I_{k_i}(u^{i,2})}}\left|K_j\left(x^2+y^2\right)\right|d\mu\left(y^2\right)\right]d\mu\left(x^2\right)$$

$$\leq \frac{c}{\lambda}\sum_{i=1}^{\infty}\left(\sum_{r=k_i}^{\infty}\frac{r-k_j+1}{2^{r-k_i}}\right)\int_{I_{k_i}(u^{i,2})}\int_{I_{k_i}(u^{i,1})}\left|f_i\left(x^1,x^2\right)\right|d\mu\left(x^1,x^2\right)$$

$$\leq \frac{c}{\lambda}\sum_{i=1}^{\infty}\|f_i\|_1\leq\frac{c}{\lambda}\|f\|_1\,. \tag{16}$$

Combining (14)–(16) we obtain

$$\mu\left\{\left(y^1,y^2\right)\in\overline{\bigcup_{i=1}^{\infty}\left(I_{k_i}\left(u^{i,1}\right)\times I_{k_i}\left(u^{i,2}\right)\right)}:t_{\#}^{(2,1)}f\left(y^1,y^2\right)>c\lambda\right\}\leq\frac{c}{\lambda}\|f\|_1\,. \tag{17}$$

From Lemma 1, we get

$$\mu\left\{\left(y^1,y^2\right)\in\bigcup_{i=1}^{\infty}\left(I_{k_i}\left(u^{i,1}\right)\times I_{k_i}\left(u^{i,2}\right)\right):t_{\#}^{(2,1)}f\left(y^1,y^2\right)>c\lambda\right\}$$
$$\leq\mu\left(\bigcup_{i=1}^{\infty}\left(I_{k_i}\left(u^{i,1}\right)\times I_{k_i}\left(u^{i,2}\right)\right)\right)\leq\frac{c}{\lambda}\|f\|_1\,, \tag{18}$$

and consequently from (17) and (18) we have

$$\mu\left\{\left(y^1,y^2\right)\in I\times I:t_{\#}^{(2,1)}f\left(y^1,y^2\right)>c\lambda\right\}\leq\frac{c}{\lambda}\|f\|_1\,. \tag{19}$$

Analogously, we can prove that

$$\mu\left\{\left(y^1,y^2\right)\in I\times I:t_{\#}^{(2,2)}f\left(y^1,y^2\right)>c\lambda\right\}\leq\frac{c}{\lambda}\|f\|_1\,. \tag{20}$$

Combining (10), (19) and (20) we obtain

$$\mu\left\{\left(y^1,y^2\right)\in I\times I:t_{\#}^{(2)}f\left(y^1,y^2\right)>c\lambda\right\}\leq\frac{c}{\lambda}\|f\|_1\,. \tag{21}$$

The estimation of $\mu\{(y^1,y^2)\in I\times I:t_{\#}^{(3)}f(y^1,y^2)>c\lambda\}$ is analogous to the estimation of $\mu\{(y^1,y^2)\in I\times I:t_{\#}^{(2)}f(y^1,y^2)>c\lambda\}$ and we have

$$\mu\left\{\left(y^1,y^2\right)\in I\times I:t_{\#}^{(3)}f\left(y^1,y^2\right)>c\lambda\right\}\leq\frac{c}{\lambda}\|f\|_1\,. \tag{22}$$

Combining (2), (3), (8), (21) and (22) we complete the proof of Theorem 1.

By making use of the well-known density argument due to Marcinkiewicz and Zygmund [9] we can show that Corollary 1 follows from Theorem 1.

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