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Some results on the geometry of tangent bundle of Finsler manifolds

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Abstract. In this paper we study the geometry of tangent bundle of a Finsler manifold endowed with Sasaki metric and obtain some results. We characterize the Riemannian manifold as the Finsler manifold such that the vertical lift of any vector field is divergence-free or equivalently, such that the horizontal distribution is minimal in the tangent bundle of the slit tangent bundle, and prove that the almost complex structure on the slit tangent bundle is integrable if and only if the base manifold has zero flag curvature. In that case, the slit tangent bundle is Kählerian. We also prove that the slit tangent bundle is locally symmetric if and only if the base manifold is locally Euclidean. Our results generalize the corresponding results for the Riemannian setting in the literature.

1. Introduction

The geometry of tangent bundle or tangent sphere bundle of a Riemannian manifold has been well developed. In the general case, however, the geometry of tangent bundle or the indicatrix bundle of a Finsler manifold has not been studied at the same pace. Several people have made some fundamental contributions to this subject from various points of view. For instance, Hasegawa, Yamauchi, and Shimada proved that the indicatrix bundle of a Finsler manifold (M,F) with the induced almost contact metric structure is Sasakian if and only if (M,F) is of constant flag curvature 1 [8], while Bejancu and Farran obtained that (M,F)

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is of constant flag curvature 1 if and only if the unit horizontal Liouville vector field is a Killing vector field on the indicatrix bundle [3]. For other relative results on the geometry of tangent bundle or indicatrix bundle one is referred to see [1], [7], [10].

The purpose of this paper is to study the slit tangent bundle of a Finsler manifold further. As is well-known, the slit tangent bundle with the Sasaki metric is a Riemannian manifold, and moreover, it is also an almost complex manifold. We characterize the Riemannian manifold as the Finsler manifold such that the vertical lift of any vector field is divergence-free or equivalently, the horizontal distribution is minimal in the tangent bundle of the slit tangent bundle, and prove that the almost complex structure on the slit tangent bundle is integrable if and only if the base manifold has zero flag curvature. In that case, the slit tangent bundle is Kählerian. We also prove that the slit tangent bundle is locally symmetric if and only if the base manifold is locally Euclidean. Our results generalize the corresponding results for the Riemannian setting in the literature.

2. Finsler geometry

In this Section, we give a brief description of several geometric quantities in Finsler geometry, For more details one is referred to see [2]. Throughout this paper, we shall use the Einstein convention, that is, repeated indices with one upper index and one lower index denotes summation over their range.

Let (M, F) be a Finsler n-manifold with Finsler metric $F : TM \to [0, \infty)$. Let $(x, y) = (x^i, y^i)$ be the local coordinates on TM, and $\pi : TM\backslash 0 \to M$ the natural projection. Unlike in the Riemannian case, most Finsler quantities are functions of TM rather than M. Some fundamental quantities and relations:

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}, \quad \text{(positive definite fundamental tensor)}$$

$$C_{ijk}(x,y) := \frac{1}{4}\frac{\partial^3 F^2(x,y)}{\partial y^i \partial y^j \partial y^k}, \quad \text{(Cartan tensor)}$$

$$(g^{ij}) := (g_{ij})^{-1}, \quad C^i_{jk} = g^{il} C_{ljk},$$

$$\gamma^k_{ij} := \frac{1}{2} g^{km}\left(\frac{\partial g_{mj}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m}\right),$$

$$N^i_j := \gamma^i_{jk} y^k - C^i_{jk}\gamma^k_{rs} y^r y^s.$$

According to [4], the pulled-back bundle π^*TM admits a unique linear connection, called the *Chern connection*. Its connection forms are characterized by the structure equations:

- Torsion freeness:

$$dx^j \wedge \omega_j^i = 0;$$

- Almost g-compatibility:

$$dg_{ij} - g_{kj}\omega_i^k - g_{ik}\omega_j^k = 2C_{ijk}(dy^k + N_l^k dx^l).$$

It is easy to know that torsion freeness is equivalent to the absence of dy^k terms in ω_j^i; namely,

$$\omega_j^i = \Gamma_{jk}^i dx^k,$$

together with the symmetry

$$\Gamma_{jk}^i = \Gamma_{kj}^i.$$

Let

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^k \frac{\partial}{\partial y^k}.$$

The *Riemannian curvature tensor* $R_j{}^i{}_{kl}$ and the *Landsberg curvature tensor* L_{jk}^i can be expressed by

$$R_j{}^i{}_{kl} = \frac{\delta \Gamma_{jl}^i}{\delta x^k} - \frac{\delta \Gamma_{jk}^i}{\delta x^l} + \Gamma_{ks}^i \Gamma_{jl}^s - \Gamma_{jk}^s \Gamma_{ls}^i \tag{2.1}$$

and

$$L_{jk}^i = y^l \frac{\partial \Gamma_{lk}^i}{\partial y^j}, \tag{2.2}$$

respectively. Obviously, $R_j{}^i{}_{kl} = -R_j{}^i{}_{lk}$. The following Euler's lemma is very useful.

Lemma 2.1. *Suppose a real-valued function H on $\mathbb{R}^n$ of positively homogeneous of degree r is differentiable away from the origin of $\mathbb{R}^n$, then*

$$y^i \frac{\partial}{\partial y^i} H(y) = rH(y).$$

Let $L_{ijk} = g_{il}L_{jk}^l$, then both C_{ijk} and L_{ijk} are symmetric on all their indices, and by Lemma 2.1 we have

$$y^i C_{ijk} = y^i L_{ijk} = 0. \tag{2.3}$$

Let $R_{ijkl} = g_{js}R_i{}^s{}_{kl}$, $R^i{}_{kl} = y^j R_j{}^i{}_{kl}$, $R^i{}_j = y^k R^i{}_{jk}$ and $R_{ij} = g_{ik}R^k{}_j$, then

$$\begin{aligned} R_{klji} - R_{jikl} = {} & C_{jis}R^s{}_{kl} - C_{kls}R^s{}_{ji} - C_{kis}R^s{}_{lj} \\ & - C_{ljs}R^s{}_{ki} - C_{ils}R^s{}_{jk} - C_{jks}R^s{}_{il}, \end{aligned} \tag{2.4}$$

$$R_{ij} = R_{ji}, \tag{2.5}$$

$$R^i{}_{kl} = \frac{1}{3}\left\{\frac{\partial R^i{}_k}{\partial y^l} - \frac{\partial R^i{}_l}{\partial y^k}\right\}. \tag{2.6}$$

Thus, $R^i{}_k = 0$ if and only if $R^i{}_{kl} = 0$, and this is just the condition for (M, F) to have zero flag curvature.

3. The Sasaki metric on slit tangent bundle

For a given Finsler manifold (M, F), we can endow its slit tangent bundle $\widetilde{TM} = TM\backslash 0$ with a Riemannian metric, known as the *Sasaki metric*. It can be described in local coordinates as following. Let $(x, y) = (x^i, y^i)$ be the local coordinates on $\widetilde{TM}$. It is well known that the tangent space to $\widetilde{TM}$ at (x, y) splits into the direct sum of the *vertical subspace* $\mathcal{V}\widetilde{TM}_{(x,y)} = \text{span}\left\{\frac{\partial}{\partial y^i}\right\}$ and the *horizontal subspace* $\mathcal{H}\widetilde{TM}_{(x,y)} = \text{span}\left\{\frac{\delta}{\delta x^i}\right\}$:

$$T_{(x,y)}\widetilde{TM} = \mathcal{V}\widetilde{TM}_{(x,y)} \oplus \mathcal{H}\widetilde{TM}_{(x,y)}.$$

The Sasaki metric $\widetilde{G}$ on $\widetilde{TM}$ is defined by

$$\widetilde{G}\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = \widetilde{G}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g_{ij}(x, y), \widetilde{G}\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = 0. \tag{3.1}$$

For $X = X^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(M)$, its *horizontal lift* X^h and *vertical lift* X^v are defined by

$$X^h = \left(X^i \circ \pi\right) \frac{\delta}{\delta x^i}$$

and

$$X^v = \left(X^i \circ \pi\right) \frac{\partial}{\partial y^i},$$

respectively. The Levi–Civita connection $\widetilde{\nabla}$ on $\widetilde{TM}$ with respect to $\widetilde{G}$ is given by the Koszul formula

$$\begin{aligned} 2\widetilde{G}\left(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}\right) &= \widetilde{X}\widetilde{G}\left(\widetilde{Y}, \widetilde{Z}\right) + \widetilde{Y}\widetilde{G}\left(\widetilde{Z}, \widetilde{X}\right) - \widetilde{Z}\widetilde{G}\left(\widetilde{X}, \widetilde{Y}\right) \\ &\quad + \widetilde{G}\left([\widetilde{X}, \widetilde{Y}], \widetilde{Z}\right) - \widetilde{G}\left([\widetilde{Y}, \widetilde{Z}], \widetilde{X}\right) + \widetilde{G}\left([\widetilde{Z}, \widetilde{X}], \widetilde{Y}\right), \end{aligned} \tag{3.2}$$

for any $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{X}(\widetilde{TM})$. By (2.1) and (2.2), we easily get

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = -R^k{}_{ij}\frac{\partial}{\partial y^k}, \tag{3.3}$$

$$\left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0, \tag{3.4}$$

$$\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = \left(\Gamma^k_{ij} + L^k_{ij}\right)\frac{\partial}{\partial y^k}, \tag{3.5}$$

which together with (3.1) and (3.2) yields

$$\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = L^k_{ij}\frac{\delta}{\delta x^k} + C^k_{ij}\frac{\partial}{\partial y^k}, \tag{3.6}$$

$$\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} = -L^k_{ij}\frac{\partial}{\partial y^k} + \left(C^k_{ij} + \frac{1}{2}y^l R_{lij}{}^k\right)\frac{\delta}{\delta x^k}, \tag{3.7}$$

$$\widetilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} = \Gamma^k_{ij}\frac{\partial}{\partial y^k} + \left(C^k_{ij} + \frac{1}{2}y^l R_{lji}{}^k\right)\frac{\delta}{\delta x^k}, \tag{3.8}$$

$$\widetilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = \left(-C^k_{ij} - \frac{1}{2}R^k{}_{ij}\right)\frac{\partial}{\partial y^k} + \Gamma^k_{ij}\frac{\delta}{\delta x^k}, \tag{3.9}$$

where $R_{lij}{}^k = g_{is}g^{kt}R_l{}^s{}_{jt}$ and $C^k_{ij} = g^{kl}C_{lij}$. Like in the submanifold case, we say that *the vertical distribution* $\mathcal{V}\widetilde{TM}$ *is totally geodesic* (resp. *minimal*) in $\widetilde{TM}$ if $\mathcal{H}\widetilde{\nabla}_{\frac{\partial}{\partial y_i}}\frac{\partial}{\partial y_j} = 0$ (resp. $g^{ij}\mathcal{H}\widetilde{\nabla}_{\frac{\partial}{\partial y_i}}\frac{\partial}{\partial y_j} = 0$), where $\mathcal{H}$ denotes the horizontal projection. Similarly, if we denote by $\mathcal{V}$ the vertical projection, then we say that *the horizontal distribution* $\mathcal{H}\widetilde{TM}$ *is totally geodesic* (resp. *minimal*) in $\widetilde{TM}$ if $\mathcal{V}\widetilde{\nabla}_{\frac{\delta}{\delta y_i}}\frac{\delta}{\delta y_j} = 0$ (resp. $g^{ij}\mathcal{V}\widetilde{\nabla}_{\frac{\delta}{\delta y_i}}\frac{\delta}{\delta y_j} = 0$). From (3.3) and (3.6) we have

Proposition 3.1. *Let* (M, F) *be a Finsler manifold. Then*

(i) (M, F) *is a Landsberg manifold, that is,* $L^i_{jk} = 0$ *if and only if the vertical distribution* $\mathcal{V}\widetilde{TM}$ *is totally geodesic in* $\widetilde{TM}$ [1];

(ii) (M, F) *is a weak Landsberg manifold, that is,* $L^i_{jk}g^{jk} = 0$ *if and only if the vertical distribution* $\mathcal{V}\widetilde{TM}$ *is minimal in* $\widetilde{TM}$ [11];

(iii) *The horizontal distribution* $\mathcal{H}\widetilde{TM}$ *is integrable if and only if* (M, F) *has zero flag curvature* [10].

4. The main results

It follows from the definition of Cartan tensor that a Finsler manifold (M, F) is Riemannian if and only if $C_{ijk} = 0$. According to [5], the condition $C_{ijk} = 0$ can be weakened to $C_{ijk}g^{jk} = 0$. Our first result characterize the Riemannian manifolds among Finsler manifolds from the viewpoint of the geometry of tangent bundle.

Theorem 4.1. *Let (M, F) be a Finsler manifold. Then the following statements are mutually equivalent:*

(i) *(M, F) is Riemannian;*

(ii) *$\widetilde{\operatorname{div}}(X^v) = 0$ for any $X \in \mathfrak{X}(M)$, where $\widetilde{\operatorname{div}}$ denotes the divergence operator of $(\widetilde{TM}, \widetilde{G})$;*

(iii) *the horizontal distribution $\mathcal{H}\widetilde{TM}$ is minimal in $T\widetilde{TM}$.*

Proof. The equivalence of (i) and (iii) follows from (3.9) and the Deicke's result [5]. By (3.6) and (3.8) we have

$$\widetilde{\operatorname{div}}\left(\frac{\partial}{\partial y^i}\right) = 2C_{ijk}g^{jk},$$

which implies that

$$\widetilde{\operatorname{div}}(X^v) = 2C_{ijk}X^i g^{jk}$$

for $X = X^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(M)$. Thus again by Deicke's result (ii) is equivalent to (i). $\square$

On the slit tangent bundle $(\widetilde{TM}, \widetilde{G})$, we can define an almost complex structure $\widetilde{J}$ as following:

$$\widetilde{J}\frac{\delta}{\delta x^i} = \frac{\partial}{\partial y^i}, \quad \widetilde{J}\frac{\partial}{\partial y^i} = -\frac{\delta}{\delta x^i}. \tag{4.1}$$

Then $(\widetilde{TM}, \widetilde{G}, \widetilde{J})$ is an almost Hermitian manifold. When (M, F) is Riemannian, it is known that $\widetilde{J}$ is integrable if and only if (M, F) is locally Euclidean [6], and in that case, $(\widetilde{TM}, \widetilde{G}, \widetilde{J})$ is Kählerian. We have the following theorem which is the analogue of Finsler setting.

Theorem 4.2. *Let (M, F) be a Finsler manifold. Then the following statements are mutually equivalent:*

(i) *(M, F) has zero flag curvature;*

(ii) *$\widetilde{J}$ is integrable;*

(iii) *$\widetilde{\nabla}\widetilde{J} = 0$;*

(iv) $(\widetilde{TM},\widetilde{G},\widetilde{J})$ *is Kählerian.*

Proof. By definition, the Nijenhuis torsion $[\widetilde{J},\widetilde{J}]$ of $\widetilde{J}$ is given by

$$[\widetilde{J},\widetilde{J}](\widetilde{X},\widetilde{Y})=[\widetilde{J}\widetilde{X},\widetilde{J}\widetilde{Y}]-[\widetilde{X},\widetilde{Y}]-\widetilde{J}[\widetilde{J}\widetilde{X},\widetilde{Y}]-\widetilde{J}[\widetilde{X},\widetilde{J}\widetilde{Y}], \tag{4.2}$$

for $\widetilde{X},\widetilde{Y}\in\mathfrak{X}(\widetilde{TM})$, which together with (3.3)-(3.5) and (4.1) yields

$$[\widetilde{J},\widetilde{J}]\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)=R^k{}_{ij}\frac{\partial}{\partial y^k}=-[\widetilde{J},\widetilde{J}]\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)=\widetilde{J}\left([\widetilde{J},\widetilde{J}]\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\right)\right).$$

Consequently, (i) is equivalent to (ii). If (M,F) has zero flag curvature, it is easy to verify that the Kähler form of $(\widetilde{TM},\widetilde{G},\widetilde{J})$ is closed, and hence (i) is also equivalent to (iv). Finally, combining (3.6)–(3.9) and (4.1) we get

$$\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^i}}\widetilde{J}\right)\left(\frac{\delta}{\delta x^j}\right)=\frac{1}{2}y^l\left(R_{lji}{}^k+R_l{}^k{}_{ij}\right)\frac{\delta}{\delta x^k}=\widetilde{J}\left(\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^i}}\widetilde{J}\right)\left(\frac{\partial}{\partial y^j}\right)\right),$$

$$\left(\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\widetilde{J}\right)\left(\frac{\partial}{\partial y^j}\right)=-\frac{1}{2}y^lR_{lij}{}^k\frac{\delta}{\delta x^k}=-\widetilde{J}\left(\left(\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\widetilde{J}\right)\left(\frac{\delta}{\delta x^j}\right)\right),$$

and consequently, (i) is equivalent to (iii). □

In 1971, Kowalski [9] proved that that the tangent bundle of a Riemannian manifold with Sasaki metric is locally symmetric if and only if the base manifold is locally Euclidean. The following theorem indicates that this result still holds for Finsler manifold.

Theorem 4.3. *Let (M,F) be a Finsler manifold. Then $(\widetilde{TM},\widetilde{G})$ is locally symmetric if and only if (M,F) is locally Euclidean.*

Proof. We need only to prove the necessity. Using (3.6) and (3.7), a direct computation shows that

$$\begin{aligned}\widetilde{R}\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k}&=\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\widetilde{\nabla}_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^k}-\widetilde{\nabla}_{\frac{\partial}{\partial y^j}}\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^k}-\widetilde{\nabla}_{[\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}]}\frac{\partial}{\partial y^k}\\&=\left(\frac{\partial L^l_{jk}}{\partial y^i}-\frac{\partial L^l_{ik}}{\partial y^j}+L^s_{jk}C^l_{si}+C^s_{jk}L^l_{si}-L^s_{ik}C^l_{sj}-C^s_{ik}L^l_{sj}\right.\\&\quad\left.+\frac{1}{2}L^t_{jk}y^sR_{sit}{}^l-\frac{1}{2}L^t_{ik}y^sR_{sjt}{}^l\right)\frac{\delta}{\delta x^l}+\left(L^s_{ik}L^l_{js}-L^s_{jk}L^l_{si}\right)\frac{\partial}{\partial y^l},\end{aligned}\tag{4.3}$$

where $\widetilde{R}$ denotes the Riemannian curvature of $(\widetilde{TM},\widetilde{G})$. Let $\eta\in\mathfrak{X}(\widetilde{TM})$ such that $\eta_{(x,y)}=y^v=y^i\frac{\partial}{\partial y^i}$, then it is easy to see that

$$\widetilde{\nabla}_\eta\frac{\partial}{\partial y^i}=\widetilde{\nabla}_\eta\frac{\delta}{\delta x^i}=0. \tag{4.4}$$

Assume that $\widetilde{\nabla}\widetilde{R}=0$, then by (2.3), (4.3), (4.4) and Lemma 2.1 we have

$$0=y^j\left(\widetilde{\nabla}_\eta\widetilde{R}\right)\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k}=y^j\widetilde{\nabla}_\eta\left(\widetilde{R}\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k}\right)$$
$$=-y^j\frac{\partial L^l_{jk}}{\partial y^i}\frac{\delta}{\delta x^l}=L^l_{ik}\frac{\delta}{\delta x^l},$$

and hence, $L^i_{jk}=0$, which together with (4.3) yields $\widetilde{R}\big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\big)\frac{\partial}{\partial y^k}=0$.
Now let $\xi\in\mathfrak{X}(\widetilde{TM})$ such that $\xi_{(x,y)}=y^h=y^i\frac{\delta}{\delta x^i}$, then

$$\widetilde{\nabla}_\xi\frac{\partial}{\partial y^i}=N^j_i\frac{\partial}{\partial y^j}-\frac{1}{2}R^j{}_i\frac{\delta}{\delta x^j}. \tag{4.5}$$

We have by (4.5),

$$\begin{aligned}0&=\left(\widetilde{\nabla}_\xi\widetilde{R}\right)\left(\eta,\frac{\partial}{\partial y^i}\right)\frac{\partial}{\partial y^j}=-\widetilde{R}\left(\eta,\widetilde{\nabla}_\xi\frac{\partial}{\partial y^i}\right)\frac{\partial}{\partial y^j}-\widetilde{R}\left(\eta,\frac{\partial}{\partial y^i}\right)\widetilde{\nabla}_\xi\frac{\partial}{\partial y^j}\\&=\frac{1}{2}R^k{}_i\widetilde{R}\left(\eta,\frac{\delta}{\delta x^k}\right)\frac{\partial}{\partial y^j}+\frac{1}{2}R^k{}_j\widetilde{R}\left(\eta,\frac{\partial}{\partial y^i}\right)\frac{\delta}{\delta x^k}.\end{aligned}\tag{4.6}$$

It can be directly verified that

$$\widetilde{R}\left(\eta,\frac{\delta}{\delta x^k}\right)\frac{\partial}{\partial y^j}=\left(-C^l_{kj}+\frac{1}{2}y^sR_{sjk}{}^l\right)\frac{\delta}{\delta x^l}, \tag{4.7}$$

$$\widetilde{R}\left(\eta,\frac{\partial}{\partial y^i}\right)\frac{\delta}{\delta x^k}=y^sR_{sik}{}^l\frac{\delta}{\delta x^l}. \tag{4.8}$$

Combining (4.6)–(4.8) it follows that

$$R^k{}_i\left(-C^l_{kj}+\frac{1}{2}y^sR_{sjk}{}^l\right)+R^k{}_jy^sR_{sik}{}^l=0. \tag{4.9}$$

Multiply (4.9) by $y_l=g_{li}y^i$ and then sum up, we get

$$\frac{1}{2}R^k{}_iR_{jk}+R^k{}_jR_{ik}=0. \tag{4.10}$$

It is easy to deduce from (4.10) that $R_{ij}=0$, or equivalently, $R^k{}_{ij}=0$. Since $L^k_{ij}=R^k{}_{ij}=0$, it follows that

$$0=\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^k}}\widetilde{R}\right)\left(\eta,\frac{\partial}{\partial y^i}\right)\frac{\partial}{\partial y^j}=-\widetilde{R}\left(\eta,\widetilde{\nabla}_{\frac{\delta}{\delta x^k}}\frac{\partial}{\partial y^i}\right)\frac{\partial}{\partial y^j}=C^l_{ki}C^s_{lj}\frac{\delta}{\delta x^s}. \tag{4.11}$$

Therefore, $C^l_{ki}C^s_{lj}=0$ or equivalently, $C_{ikl}C_{sjt}g^{lt}=0$ for any i, k, s, j. Since (g^{ij}) is positive definite, we get $C_{ijk}=0$ and consequently, (M,F) is Riemannian. Since $R^k{}_{ij}=0,(M,F)$ is a flat Riemannian manifold, thus it is locally Euclidean, and the theorem is proved. □

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