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Some remarks on CR–submanifolds of a locally conformal Kaehler manifold with parallel Lee form

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The study of the geometry of CR–submanifolds in locally conformal Kaehler (l.c.K.) manifolds is of recent interest and has been initiated by K. Matsumoto, [4], [5] and continued further by L. Ornea, [6] and S. Dragomir, [2], [3]. The main purpose of this note is to prove the nonexistence of some classes of proper CR–submanifolds of a l.c.K. manifold with parallel Lee form (Theorem 3).

Let $\left(\tilde{M}^{2n}, g, J\right)$ be a Hermitian manifold of complex dimension $n \geq 2$, where g denotes the Hermitian metric, while J stands for the complex structure. Let Ω be its fundamental 2–form, i.e. $\Omega(X,Y) = g(X, JY)$, where X, Y denote vector fields on $\tilde{M}^{2n}$. Then $\tilde{M}^{2n}$ is a l.c.K. manifold iff:

$$d\Omega = \omega \wedge \Omega \tag{1}$$

for some closed globally defined 1–form ω on $\tilde{M}^{2n}$ (see e.g. I. Vaisman, [8]). The 1–form ω is called the Lee form on $\tilde{M}^{2n}$. Suppose that $\omega \neq 0$ at every point and consider the corresponding unit 1–form $u = \omega/|\omega|$. Denote by U the unit vector field (called the Lee vector field) defined by

$$u(X) = g(U, X)\,. \tag{2}$$

Consider also the unit vector field V defined by $V = JU$ and the 1–form v defined by

$$v(X) = g(V, X)\,. \tag{3}$$

Because g is Hermitian we have

$$v = -u \circ J \quad , \quad u = v \circ J \quad , \quad u(V) = v(U) = 0\,. \tag{4}$$

Let $\tilde{M}^{2n}$ be a l.c.K. manifold and $\tilde{\nabla}$ be the Levi Civita connection of g. On $\tilde{M}^{2n}$ one also has another torsionless linear connection $\tilde{D}$, called the Weyl connection, expressed by:

$$\tilde{D}_X Y = \tilde{\nabla}_X Y - c\,[u(X)Y + u(Y)X - g(X,Y)U]\,, \tag{5}$$

where we denoted $c = |\omega|/2$.
It is known (see [7]) that the Hermitian manifold $(\tilde{M}^{2n}, g, J)$ is l.c.K. if and only if there exists a global closed 1–form ω satisfying

$$\tilde{D}J = 0\,, \tag{6}$$

where $\tilde{D}$ denotes the Weyl connection defined by (5).

Then we have

Proposition 1. *A Hermitian manifold $(\tilde{M}^{2n}, g, J)$ is a l.c.K. manifold if and only if there exists a global closed 1–form ω satisfying*

$$(\tilde{\nabla}_X J)Y = c\,[g(JX,Y)U + g(X,Y)V - u(Y)JX - v(Y)X] \tag{7}$$

for any vector fields X, Y tangent to $\tilde{M}^{2n}$, where $\tilde{\nabla}$ is the Levi Civita connection of g.

A l.c.K. manifold is called a $\mathcal{PK}$–manifold if its Lee form is absolutely parallel with respect to the Levi Civita connection $\tilde{\nabla}$ of the metric. If, moreover, the local conformal Kaehler metrics have vanishing curvature, the manifold is called a $\mathcal{P}_0\mathcal{K}$–manifold (see [8]).

It follows that a l.c.K. manifold $\tilde{M}^{2n}$ is a $\mathcal{PK}$–manifold if and only if $c =$ const. and the following relation holds

$$\tilde{\nabla}_X U = 0\,, \tag{8}$$

for any vector field X tangent to $\tilde{M}^{2n}$ (see also [8]).

Then, by using (7) and (8) we get

Lemma 2. *Let $\tilde{M}^{2n}$ be a $\mathcal{PK}$–manifold. Then we have*

$$\tilde{\nabla}_X V = c\,[u(X)V - v(X)U - JX]\,, \tag{9}$$

for any vector field X tangent to $\tilde{M}^{2n}$.

Now, let M be a real m–dimensional Riemannian manifold isometrically immersed in a l.c.K. manifold $\tilde{M}^{2n}$. The submanifold M is called a CR–submanifold if it is endowed with the pair of complementary orthogonal differentiable distributions $(D, D^\perp)$, such that $JD_x = D_x$, $JD_x^\perp \subset T_x^\perp M$ and $T_x M = D_x \oplus D_x^\perp$ for each $x \in M$, where $T_x M$ and $T_x^\perp M$ are the tangent and the normal space at x of M. Remark that the study of differential geometry of CR–submanifolds of a Kaehler manifold has been initiated by A. Bejancu in [1].
The distribution D from the definition of a CR–submanifold is called the holomorphic distribution and $D^\perp$ is called the totally real distribution. A CR–submanifold is said to be proper if both distributions D and $D^\perp$ have non–null dimensions.

Now, we state

Theorem 3. *Let M be an m–dimensional CR–submanifold of a $\mathcal{PK}$–manifold $\tilde{M}^{2n}$. If the distinguished vector field V is normal to M, then M is a totally real submanifold of $\tilde{M}^{2n}$ and $m \leq n$.*

Proof. Suppose V is normal to M. Then, by using Lemma 2 we obtain

$$
\begin{aligned}
g([X,Y],V) &= g(\tilde{\nabla}_X Y, V) - g(\tilde{\nabla}_Y X, V) = \\
&= g(X, \tilde{\nabla}_Y V) - g(Y, \tilde{\nabla}_X V) = \\
&= cg(X, u(Y)V - JY) - cg(Y, u(X)V - JX) = \\
&= c[g(JX,Y) - g(X,JY)] = 2cg(Y,JX) = 2c\Omega(Y,X)\ ,
\end{aligned}
\tag{10}
$$

for all vector fields X, Y tangent to M.
Now, if we suppose that $D \neq \{0\}$, for a unit vector field $X \in \Gamma(D)$ we take $Y = JX \in \Gamma(D)$ and obtain $\Omega(Y,X) = 1$. Because $c = |\omega|/2 \neq 0$, the assertion of Theorem 3 follows from (10).

From Theorem 3 we have

Corollary 4. *There does not exist proper CR–submanifolds M of a $\mathcal{PK}$–manifold $\tilde{M}^{2n}$ such that V is normal to M. In particular, there does not exist proper CR–submanifolds of a $\mathcal{PK}$–manifold such that $U \in \Gamma(D^{\perp})$.*

Corollary 5. *There does not exist complex submanifolds M of a $\mathcal{PK}$–manifold $\tilde{M}^{2n}$ such that the Lee vector field U is normal to M.*

Therefore, if $\tilde{M}^{2n}$ is a $\mathcal{PK}$–manifold we can consider two classes of proper CR–submanifolds M of $\tilde{M}^{2n}$, according to the position of the Lee vector field U with respect to the tangent bundle of the submanifold.

Let M be proper CR–submanifold of a $\mathcal{PK}$–manifold $\tilde{M}^{2n}$. We say that M is a L–CR–submanifold (resp. a $\mathrm{L}^{\perp}$–CR–submanifold) if the Lee vector field on $\tilde{M}^{2n}$ is tangent (resp. normal) to the submanifold M.

Denote by $\{U\}$ and $\{V\}$ the 1–dimensional distributions defined by U and V, respectively. Then, if M is a proper L–CR–submanifold of a $\mathcal{PK}$–manifold $\tilde{M}^{2n}$, the holomorphic distribution D of M has the decomposition

$$
D = \bar{D} \oplus \{U\} \oplus \{V\}\,,
\tag{11}
$$

where $\bar{D}$ is the holomorphic distribution complementary orthogonal of $\{U\} \oplus \{V\}$ in D. Therefore, the tangent bundle to M has the decomposition

$$
TM = D \oplus \{U\} \oplus \{V\} \oplus D^{\perp}\,.
\tag{12}
$$

Concerning the integrability of all distributions which are involved in the decomposition (12) of the tangent bundle of a proper L–CR–submanifold of a $\mathcal{PK}$–manifold we have

Theorem 6. i). *The distributions $D^\perp, D^\perp \oplus \{U\}$, $D^\perp \oplus \{V\}$, $D^\perp \oplus \{U\} \oplus \{V\}$ and $\bar{D} \oplus D^\perp \oplus \{V\}$ are always integrable.*

ii). *If $\bar{D} \neq \{0\}$, the distributions $\bar{D} \oplus \{V\}$ and $\bar{D} \oplus \{U\} \oplus \{V\}$ are integrable if and only if the following relation holds*

$$h(X, JY) = h(JX, Y)$$

for all $X, Y \in \Gamma(\bar{D})$, where h denotes the second fundamental form of the submanifold M in $\tilde{M}^{2n}$.

iii). *If $\bar{D} \neq \{0\}$, the distributions $\bar{D}, \bar{D} \oplus \{U\}$ and $\bar{D} \oplus D^\perp \oplus \{U\}$ are never integrable.*

Now suppose M is a proper L$^\perp$–CR–submanifold of a $\mathcal{PK}$–manifold $\tilde{M}^{2n}$. Then the tangent bundle to M has the decomposition

$$TM = D \oplus \bar{D}^\perp \oplus \{V\}\,, \tag{13}$$

i.e. the totally real distribution $D^\perp$ can be write $D^\perp = \bar{D}^\perp \oplus \{V\}$, where $\bar{D}^\perp$ is the complementary orthogonal totally real distribution of $\{V\}$ in $D^\perp$.

Concerning the integrability of distributions which are involved in the decomposition (13) of the tangent bundle of a proper L$^\perp$–CR–submanifold of a $\mathcal{PK}$–manifold we have

Theorem 7. i). *The distributions $\bar{D}^\perp$ and $\bar{D}^\perp \oplus \{V\} = D^\perp$ are always integrable.*

ii). *The distribution $D \oplus \{V\}$ is integrable if and only if we have $h(X, JY) = h(JX, Y)$ for all $X, Y \in \Gamma(D)$.*

iii). *The distributions D and $D \oplus \bar{D}^\perp$ are never integrable.*

Remark that by $\Gamma(D)$ (resp. $\Gamma(D^\perp)$) we have denoted the module of differentiable sections of D (resp. $D^\perp$).

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