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On a class of projectively flat (α,β)-metrics

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Abstract. In this paper, we find a sufficient and necessary condition for an important class of (α,β)- metrics in the form $F=\alpha\phi(\beta/\alpha)$ to be locally projectively flat, where $\phi=\phi(s)$ is a positive C^∞ function satisfying certain conditions, characterized by a polynomial or a power series of s, α is a Riemannian metric and β is a 1-form.

1. Introduction

Hilbert's Fourth Problem in the regular case requires to study and characterize Finsler metrics $F=F(x,y)$ on an open domain $\mathbf{U}\subset R^n$ whose geodesics are straight lines [4]. Finsler metrics on $\mathbf{U}$ with this property are called *projectively flat* metrics. In [3], G. Hamel first found a simple system of partial differential equations that characterizes projectively flat Finsler metrics on an open domain $\mathbf{U}\subset R^n$. That is $F=F(x,y)$ on $\mathbf{U}$ is projectively flat if and only if the following PDE's hold:

$$F_{x^m y^i}y^m = F_{x^i}. \tag{1}$$

It is one of the important problems in Finsler geometry to characterize projectively flat metrics. According to Beltrami's Theorem, a Riemannian metric is projectively flat if and only if it is of constant sectional curvature[7], [8]. Further, it is known that a Randers metric $F=\alpha+\beta$ is projectively flat if and only if α is projectively flat and β is closed [1], where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_iy^i$ is a 1-form with $b:=\|\beta_x\|<1$ for $x\in M$.

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In this paper, we are going to consider a class of Finsler metrics on a manifold M which are expressed in the following form:

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\phi = \phi(s)$ is a C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(0) = 1, \quad \phi(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \tag{2}$$

where s and b are arbitrary numbers with $|s| \le b < b_0$. It is known that $F = \alpha\phi(s), s = \beta/\alpha$ is a Finsler metric if and only if the condition (2) holds. Finsler metrics in the above form are called (α, β)-metrics. The class of (α, β)-metrics contains all Riemannian metrics ($\phi = 1$) and all Randers metrics ($\phi = 1 + s$). In the past several years, various curvatures in Finsler geometry have been studied and their geometric meaning is better understood. This is partially due to the study of (α, β)-metrics. Thus, this motivates people to study (α, β)-metrics more deeply.

From now on, we are going to consider a special class of (α,β)-metrics $F = \alpha\phi(s)$, where $\phi = \phi(s)$ is a function satisfying (2) and

$$\phi - s\phi' = (p + rs^2)\phi''(s), \tag{3}$$

where p, r are constants. Recently, Z. Shen has proved the following

Theorem 1.1. ([11]) *Assume that $\phi = \phi(s)$ satisfies* (2) *and* (3). *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on a manifold M. If*

$$b_{i|j} = 2\tau\big\{(p + b^2)a_{ij} + (r - 1)b_i b_j\big\} \tag{4}$$

and the spray coefficients G^i_α of α are of the form:

$$G^i_\alpha = \xi y^i - \tau\alpha^2 b^i, \tag{5}$$

where $b := \sqrt{a^{ij} b_i b_j}$, $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α, $\tau = \tau(x)$ is a scalar function and $\xi = \xi_i(x)y^i$ is a 1-form on M, then F is locally projectively flat.

Unfortunately, we are not sure that the conditions (4) and (5) are necessary. A key step is to prove that β is closed. However, some progress has been made for certain types of functions ϕ recently. In [12], Z. Shen and G. C. Yildirim first showed that $F = (\alpha + \beta)^2/\alpha$ is projectively flat if and only if the conditions

(4) and (5) hold with $p=1/2$ and $r=-1/2$. Then they proved that the (α,β)-metrics $F=\alpha+\varepsilon\beta+k\beta^2/\alpha$ are projectively flat if and only if the conditions (4) and (5) hold with $p=1/(2k)$ and $r=-1/2$, where ε and $k\neq 0$ are constants. Further, Y. Shen and L. Zhao have shown that $\phi=1+\varepsilon s+2ks^2-\frac{k^2}{3}s^4$ satisfies (2) and (3) with $p=1/(4k)$ and $r=-1/4$, and the Finsler metrics in the form $F=\alpha\phi(\beta/\alpha)$ are projectively flat if and only if (4) and (5) hold [6]. In this paper we will show that, for a larger class of Finsler metrics $F=\alpha\phi(s)$ satisfying (2) and (3), they are projectively flat if and only if (4) and (5) hold.

We will firstly show that, if $p=0$ then the solutions of (3) will not satisfy (2). Hence, we always assume that $p\neq 0$ in this paper. In this case, the solutions of (3) are analytic near the origin and the power series of the solutions are of the form

$$\phi(s)=C_0+C_1s+C_2s^2+C_4s^4+\cdots+C_{2n}s^{2n}+\cdots, \tag{6}$$

where C_0 and C_1 are arbitrary constants and the coefficients of ϕ satisfy

$$C_{2n+2}=\left(-\frac{1}{p}\right)\frac{(2n-1)(2nr+1)}{(2n+2)(2n+1)}C_{2n}.$$

By a simple analysis, we can see that $\phi(s)$ satisfies (2) and (3) if and only if $C_0=1$ and $b\in(0,\sqrt{|p|}\,)$ is sufficiently small. In particular, if we take $r=-\frac{1}{2k}$ and k is a positive integer, then the $\phi(s)$ in (6) are polynomials of the following form

$$\phi(s)=C_0+C_1s+C_2s^2+C_4s^4+\cdots+C_{2k}s^{2k}. \tag{7}$$

Furthermore, if we take $k=1$ and then $k=2$, then the (α,β)-metrics $F=\alpha\phi(s)$, $s=\beta/\alpha$, given by (7) have the form

$$F=\alpha+C_1\beta+\frac{1}{2p}\frac{\beta^2}{\alpha}, \tag{8}$$

and

$$F=\alpha+C_1\beta+\frac{1}{2p}\frac{\beta^2}{\alpha}-\frac{1}{48p^2}\frac{\beta^4}{\alpha^3} \tag{9}$$

respectively. These are just the (α,β)-metrics discussed in [12] and [6] respectively.

By using the form (6) of ϕ, we have the following main

Theorem 1.2. *Assume that $\phi=\phi(s)$ is a function in the form* (6) *satisfying* (2) *and* (3). *If $C_1\neq 0$ and $r\neq 1$ or $C_1=0$ but $r=-1/(2k)$, where k is any positive integer, then the (α,β)-metric $F=\alpha\phi(\beta/\alpha)$ is projectively flat if and only if* (4) *and* (5) *hold.*

According to the discussion above, Theorem 1.2 generalizes the results in [12] and [6].

2. The analysis of the solutions

In this section, we will study the second order linear ODE (3) by the power series method. If $p = 0$, then equation (3) has explicit solutions as follows:

$$\phi(s) = \begin{cases} Cs & \text{if } r = 0 \\ Cs + \widetilde{C} s \ln s & \text{if } r = -1 \\ Cs + \widetilde{C} s^{-\frac{1}{r}} & \text{if } r \neq 0, -1. \end{cases}$$

In this case, $\phi(0) \neq 1$. Now, the resulting $\phi(s)$ does not satisfy (2) and the $F = \alpha\phi(\beta/\alpha)$ defined by these $\phi(s)$ are not Finsler metrics. Therefore, we always assume that $p \neq 0$.

By a theorem on the power series method of ODE, we know that the solutions of (3) are analytic near the origin. Let the power series expressions of $\phi(s)$ be

$$\phi(s) = \sum_{n=0}^{\infty} C_n s^n.$$

The first and the second order derivatives of $\phi(s)$ are

$$\phi'(s) = \sum_{n=0}^{\infty} (n+1) C_{n+1} s^n, \quad \phi''(s) = \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} s^n.$$

Plugging them into (3) yields

$$(2pC_2 - C_0) + 6pC_3 s + \sum_{n=0}^{\infty} \big\{ p(n+4)(n+3) C_{n+4} + (n+1)\big[r(n+2)+1\big] C_{n+2} \big\} s^{n+2} = 0. \tag{10}$$

From (10) we know that the coefficients of $\phi(s)$ must satisfy

$$C_{2n+1} = 0, \quad n \geq 1, \tag{11}$$

$$C_{2n+2} = \left(-\frac{1}{p}\right) \frac{(2n-1)(2nr+1)}{(2n+2)(2n+1)} C_{2n}, \quad n \geq 0. \tag{12}$$

Hence the power series expression of $\phi(s)$ is

$$\phi(s) = C_0 + C_1 s + C_2 s^2 + C_4 s^4 + \cdots + C_{2n} s^{2n} + \cdots. \tag{13}$$

We assert that the $F=\alpha\phi(\beta/\alpha)$ defined by $\phi(s)$ in the form (13) are Finsler metrics if and only if $C_0=1$ and $b\in(0,\sqrt{|p|}\,)$ is sufficiently small. Because of $\phi(0)=C_0$ and (2), we know that $C_0=1$. Then $\phi(s)>0$ in a sufficiently small neighborhood of the origin. Note that

$$[\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)]_{s=0}=\frac{p+b^2}{p}.$$

Thus, if $b\in(0,\sqrt{|p|}\,)$ is sufficiently small and $|s|\le b$, then

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0.$$

This proves our assertion. In this case,

$$C_2=\frac{1}{2p}\neq 0. \tag{14}$$

3. (α,β)-metrics

In this section, for a function $\phi=\phi(s)$ satisfying (2), we will recall some well-known properties of an (α,β)-metric $F=\alpha\phi(\beta/\alpha)$. Let $b_{i|j}dx^i\otimes dx^j$ denote covariant derivatives of β with respect to α. Let

$$r_{ij}:=\frac{1}{2}(b_{i|j}+b_{j|i}),\qquad s_{ij}:=\frac{1}{2}(b_{i|j}-b_{j|i}),$$
$$s^i{}_j:=a^{il}s_{lj},\qquad s_j:=b^is_{ij}$$

and

$$r_{00}:=r_{ij}y^iy^j,\quad s_0:=s_iy^i,\quad s^i{}_0:=s^i{}_jy^j,\quad s_{l0}:=s_{lj}y^j.$$

Clearly, β is closed if and only if $s_{ij}=0$.

The geodesic coefficients G^i of $F=\alpha\phi(\beta/\alpha)$ are given by

$$G^i=G^i_\alpha+\alpha Qs^i{}_0+H(-2\alpha Qs_0+r_{00})\left\{\chi\frac{y^i}{\alpha}+b^i\right\}, \tag{15}$$

where

$$\chi:=\frac{(\phi-s\phi')\phi'}{\phi\phi''}-s,$$
$$Q:=\frac{\phi'}{\phi-s\phi'},$$
$$H:=\frac{\phi''}{2[(\phi-s\phi')+(b^2-s^2)\phi'']}.$$

The formula (15) is given in [2], [10], [11]. Further, by (1) and (15) we have the following

Lemma 3.1 ([12]). *An (α,β)-metric $F=\alpha\phi(\beta/\alpha)$ is projectively flat on an open domain $\mathbf{U}\subset R^n$ if and only if*

$$(a_{ml}\alpha^2-y_my_l)G^m_\alpha+\alpha^3Qs_{l0}+H(-2\alpha Qs_0+r_{00})(b_l\alpha^2-\beta y_l)=0, \tag{16}$$

where $y_l:=a_{il}y^i$.

4. Proof of Theorem 1.2

In this section, we are going to prove Theorem 1.2 using Lemma 3.1.

From (3) and (15) we have

$$Q=\frac{\phi'}{(p+rs^2)\phi''},\qquad H=\frac{1}{2[(p+b^2)+(r-1)s^2]}. \tag{17}$$

If (4) holds, then β is closed and $r_{00}=2\tau\{(p+b^2)\alpha^2+(r-1)\beta^2\}$. Hence (15) reduces to

$$G^i=G^i_\alpha+\tau\left\{\chi\frac{y^i}{\alpha}+b^i\right\}\alpha^2.$$

In addition, if (5) holds, we may obtain

$$G^i=(\xi+\tau\chi\alpha)y^i.$$

Therefore F is projectively flat.

Conversely, assume that $\phi=\phi(s)$ is a function satisfying (2) and (3) and the (α,β)-metric $F=\alpha\phi(\beta/\alpha)$ is projectively flat.

If β is closed, i.e. $s_{ij}=0$, (16) reduces to

$$(a_{ml}\alpha^2-y_my_l)G^m_\alpha+\frac{\alpha^2r_{00}}{2[(p+b^2)\alpha^2+(r-1)\beta^2]}(b_l\alpha^2-\beta y_l)=0,$$

that is,

$$2(a_{ml}\alpha^2-y_my_l)\big[(p+b^2)\alpha^2+(r-1)\beta^2\big]G^m_\alpha=-\alpha^2r_{00}(b_l\alpha^2-\beta y_l). \tag{18}$$

Contracting (18) with b^l, we get

$$2\big[(p+b^2)\alpha^2+(r-1)\beta^2\big](b_m\alpha^2-y_m\beta)G^m_\alpha=-\alpha^2(b^2\alpha^2-\beta^2)r_{00}.$$

Note that the polynomial $(p+b^2)\alpha^2+(r-1)\beta^2$ is not divisible by α^2 and $b^2\alpha^2-\beta^2$ as $r\neq 1$. Thus $(b_m\alpha^2-y_m\beta)G^m_\alpha$ is divisible by $\alpha^2(b^2\alpha^2-\beta^2)$. Therefore, there is a function $\tau=\tau(x)$ such that

$$r_{00}=2\tau\{(p+b^2)\alpha^2+(r-1)\beta^2\}. \tag{19}$$

Note that (19) is equivalent to (4) since $s_{ij}=0$. Now the formula (15) for G^i can be simplified to

$$G^i=G^i_\alpha+\tau\chi\alpha y^i+\tau\alpha^2 b^i. \tag{20}$$

On the other hand, it is well-known that F is projectively flat if and only if there is a scalar function $P(x,y)$ satisfying $P(x,\lambda y)=\lambda P(x,y)$ for $\lambda>0$, such that $G^i=Py^i$. Thus, by (20), we have

$$G^i_\alpha=(P-\tau\chi\alpha)y^i-\tau\alpha^2 b^i.$$

Because both of G^i_α and $\tau\alpha^2 b^i$ are quadratic forms of (y^i), we assert that $\xi:=P-\tau\chi\alpha y^i$ must be a 1-form and we get (5). Thus the key step of the proof is to prove that β is closed.

It is easy to see by (17) that (16) can be rewritten as

$$\begin{aligned}2[(p+b^2)&+(r-1)s^2](p+rs^2)(a_{ml}\alpha^2-y_m y_l)G^m_\alpha\phi''\\&+2[(p+b^2)+(r-1)s^2]\alpha^3 s_{l0}\phi'-2\alpha s_0(b_l\alpha^2-\beta y_l)\phi'\\&+r_{00}(p+rs^2)(b_l\alpha^2-\beta y_l)\phi''=0.\end{aligned} \tag{21}$$

Contracting (21) with b^l yields

$$\begin{aligned}2[(p+b^2)&+(r-1)s^2](p+rs^2)(b_m\alpha^2-y_m\beta)G^m_\alpha\phi''\\&+2[(p+b^2)+(r-1)s^2]\alpha^3 s_0\phi'-2\alpha s_0(b^2\alpha^2-\beta^2)\phi'\\&+r_{00}(p+rs^2)(b^2\alpha^2-\beta^2)\phi''=0.\end{aligned} \tag{22}$$

Then we have

$$\begin{aligned}\big\{2\big[(p+b^2)+(r-1)s^2\big](b_m\alpha^2-y_m\beta)G^m_\alpha+r_{00}(b^2\alpha^2-\beta^2)\big\}(p+rs^2)\phi''\\+2\alpha s_0(p\alpha^2+r\beta^2)\phi'=0.\end{aligned} \tag{23}$$

Case 1: $C_1\neq 0$ and $r\neq 1$. In this case, we have

$$\phi(s)=1+C_1 s+C_2 s^2+C_4 s^4+\cdots+C_{2n}s^{2n}+\cdots,$$

$$\phi'(s)=C_1+2C_2 s+4C_4 s^3+\cdots+2nC_{2n}s^{2n-1}+\cdots,$$

$$\phi''(s)=2C_2+4\cdot 3C_4 s^2+\cdots+2n\cdot(2n-1)C_{2n}s^{2n-2}+\cdots$$

Let

$$\psi(s) = 2C_2 + 4C_4 s^2 + \cdots + 2nC_{2n} s^{2n-2} + \cdots .$$

Then $\phi'(s) = C_1 + s\psi(s)$ and (23) becomes

$$\begin{aligned}\big\{2\big[(p+b^2)+(r-1)s^2\big](p+rs^2)(b_m\alpha^2 - y_m\beta)G^m_\alpha \\ + r_{00}(p+rs^2)(b^2\alpha^2-\beta^2)\big\}\phi'' + 2\beta s_0(p\alpha^2+r\beta^2)\psi \\ + 2\alpha s_0(p\alpha^2+r\beta^2)C_1 = 0. \end{aligned} \tag{24}$$

Note that, when we change y into $-y$ in (24), only $2\alpha s_0(p\alpha^2 + r\beta^2)C_1$ changes its sign. Hence

$$2\alpha s_0(p\alpha^2 + r\beta^2)C_1 = 0. \tag{25}$$

Because of $p \neq 0$, we assert that $p\alpha^2 + r\beta^2 \neq 0$. Thus

$$s_0 = 0.$$

Substituting it back into (21), by a similar discussion and by paying attention to $\phi'(-s) = C_1 - s\psi(s)$, we get $C_1 s_{l0} = 0$. By assumption, we obtain

$$s_{l0} = 0.$$

That is, β is closed.

Case 2: $C_1 = 0$ but $r = -\frac{1}{2k}$, where k is a positive integer.

When $k = 1$, the metrics are just those of the form (8). In [12], Z. Shen and G. C. Yidirim have proved that such metrics are projectively flat if and only if (4) and (5) hold. So, in the following, we will always assume that $k \geq 2$. In this case, we have

$$\begin{aligned}\phi(s) &= 1 + C_2 s^2 + C_4 s^4 + \cdots + C_{2k} s^{2k}, \\ \phi'(s) &= 2C_2 s + 4C_4 s^3 + \cdots + 2kC_{2k} s^{2k-1}, \\ \phi''(s) &= 2C_2 + 4\cdot 3C_4 s^2 + \cdots + 2k\cdot(2k-1)C_{2k} s^{2k-2}.\end{aligned}$$

Let

$$\psi(s) = 2C_2 + 4C_4 s^2 + \cdots + 2kC_{2k} s^{2k-2}.$$

Then $\phi'(s) = s\psi(s)$ and (23) can be rewritten as

$$\begin{aligned}\big\{2\big[(p+b^2)+(r-1)s^2\big](b_m\alpha^2 - y_m\beta)G^m_\alpha + r_{00}(b^2\alpha^2-\beta^2)\big\}(p+rs^2)\phi'' \\ + 2\beta s_0(p\alpha^2+r\beta^2)\psi = 0. \end{aligned} \tag{26}$$

Multiplying (26) by α^{2k+2} yields

$$\big\{2\big[(p+b^2)\alpha^2+(r-1)\beta^2\big](b_m\alpha^2-y_m\beta)G^m_\alpha+r_{00}\alpha^2(b^2\alpha^2-\beta^2)\big\}\eta \\ +2\alpha^4\beta s_0\theta=0, \quad (27)$$

where

$$\eta = 2C_2\alpha^{2k-2}+4\cdot 3C_4\alpha^{2k-4}\beta^2+\cdots+2k\cdot(2k-1)C_{2k}\beta^{2k-2},$$
$$\theta = 2C_2\alpha^{2k-2}+4C_4\alpha^{2k-4}\beta^2+\cdots+2kC_{2k}\beta^{2k-2}.$$

When $k \geq 2$, it is clear that θ and η are relatively prime polynomials of (y^i). Hence $\alpha^4\beta s_0$ must be divisible by η. Because both of α^2 and β are irreducible polynomials of (y^i), $\alpha^4\beta$ and η are obviously relatively prime polynomials of (y^i). Then s_0 must be divisible by η, which implies that

$$s_0 = 0.$$

Substituting this back into (21) yields that

$$\big\{2\big[(p+b^2)+(r-1)s^2\big](a_{ml}\alpha^2-y_my_l)G^m_\alpha+r_{00}(b_l\alpha^2-\beta y_l)\big\}(p+rs^2)\phi'' \\ +2[(p+b^2)+(r-1)s^2]\alpha^3 s_{l0}\phi'=0,$$

that is,

$$\big\{2\big[(p+b^2)+(r-1)s^2\big](a_{ml}\alpha^2-y_my_l)G^m_\alpha+r_{00}(b_l\alpha^2-\beta y_l)\big\}(p+rs^2)\phi'' \\ +2[(p+b^2)+(r-1)s^2]\alpha^2\beta s_{l0}\psi=0. \quad (28)$$

Multiplying (28) by α^{2k+2} yields

$$\big\{2\big[(p+b^2)\alpha^2+(r-1)\beta^2\big](a_{ml}\alpha^2-y_my_l)G^m_\alpha \\ +r_{00}\alpha^2(b_l\alpha^2-\beta y_l)\big\}(p\alpha^2+r\beta^2)\eta \\ +2[(p+b^2)\alpha^2+(r-1)\beta^2]\alpha^4\beta s_{l0}\theta=0. \quad (29)$$

Hence $[(p+b^2)\alpha^2+(r-1)\beta^2]s_{l0}$ must be divisible by η.

If $k=2$, we know from (12) and (14) that

$$\eta=\frac{1}{p^2}\left(p\alpha^2-\frac{1}{12}\beta^2\right),$$
$$(p+b^2)\alpha^2+(r-1)\beta^2=15\left[\left(\frac{p+b^2}{15}\right)\alpha^2-\frac{1}{12}\beta^2\right].$$

It is clear that $(p+b^2)\alpha^2+(r-1)\beta^2$ and η are relatively prime polynomials of (y^i). Hence s_{l0} must be divisible by η. This yields that $s_{l0}=0$.

If $k\geq 3$, the degrees of $[(p+b^2)\alpha^2+(r-1)\beta^2]s_{l0}$ and η show that $s_{l0}=0$.

Thus we may conclude that β is closed when $k\geq 2$. This completes the proof of Theorem 1.2.

From Theorem 1.2 and (12), (14) and the discussion in section 2, we have the following

Corollary 4.1. *Assume that*

$$\phi(s)=1+C_1s+\frac{1}{2p}s^2+\cdots+C_{2n}s^{2n}+\cdots$$

and $b\in(0,\sqrt{|p|}\,)$ *is sufficiently small, where*

$$C_{2n+2}=(-1)^n\frac{(2n-1)!!}{(2n+2)!p^{n+1}}\prod_{i=1}^{n}(2ir+1),\quad n\geq 1.$$

If $C_1\neq 0$ *and* $r\neq 1$ *or* $C_1=0$ *but* $r=-1/(2k)$ *where* k *is any positive integer, then the* (α,β)*-metric* $F=\alpha\phi(\beta/\alpha)$ *is projectively flat if and only if* (4) *and* (5) *hold.*

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