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Isometric actions of compact connected Lie groups on globally hyperbolic Lorentz manifolds

By DÁVID SZEGHY (Budapest)

Abstract. Let G be a compact connected Lie group acting isometrically on a globally hyperbolic Lorentz manifold L. We will show that there are no isolated singular orbits in L. We will also show that if there is an orbit of co-dimension 1 then every orbit is principal moreover L is diffeomorphic to $G/G_x \times (\alpha, \beta)$ where $x \in L$ is arbitrary and G_x is the isotropy subgroup of x, and $\alpha, \beta \in \mathbb{R} \cup \{\pm\infty\}$ furthermore every orbit is a Cauchy hypersurface. Moreover a Lorentzian analogue of a theorem of J. Szenthe is given, namely we prove that: If L and G are as above and $G(x)$ is a principal orbit for which along the causal rays orthogonal to $G(x)$ a curvature property holds, then the singular orbits in the causal future of $G(x)$ correspond to first focal points along some causal geodesics orthogonal to $G(x)$. Finally the correspondence between the singular orbits and the focal points of maximal dimensional orbits is considered in a special situation.

1. Definitions, low dimensional cases

There are several well-known results about isometric actions of compact connected Lie groups on compact Riemannian manifolds, e.g. see S. Kobayashi [K]. Isometric actions of compact Lie groups on Lorentz manifolds were also studied by S. Adams [A]. In a compact time oriented Lorentz manifold the causality relations are not well-defined since there is always a time-like geodesic loop. Yet, in a globally hyperbolic Lorentz manifold the future and the past of a point are separated; moreover there is a compactness property included in the definition

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of global hyperbolicity which helps to get results analogous to the compact Riemannian case.

We will consider a Lorentzian manifold $(L, \langle,\rangle)$ which will be always time-oriented and a compact Lie group G acting isometrically on L, where we assume that the action $G \times L \to L$ is smooth. It is well known that in this case the orbits are smooth compact submanifolds of L. The following definitions will be used:

- $I^+(x),\ J^+(x)$ (respectively $I^-(x),\ J^-(x)$) will denote the time-like, causal future (respectively past) of the point x, see [B-E-E] p. 5.;
- L is *globally hyperbolic* if for every $x, y \in L$ the set $J^+(x) \cap J^-(y)$ is compact and L is strongly causal, i.e. every point has an arbitrary small open neighbourhood U such that there is no causal curve intersecting U in a disconnected set.

The above definition and concerning facts about global hyperbolicity can be found in [B-E-E] p. 65. By a neighbourhood we will mean always an open one. The following notations and definitions will be used throughout this paper:

- $G(x)$ is the *orbit* of the point x under the action of the group G;
- G_x is the *isotropy subgroup* of x that is $G_x = \{g \in G \mid g \cdot x = x\}$;
- $N_zG(x)$ is the *normal space* of $z \in G(x)$ i.e.
  $N_zG(x) = \{v \in T_zL \mid g(v, w) = 0,\ \forall w \in T_zG(x)\}$;
- $NG(x)$ is the *normal bundle* of $G(x)$ that is
  $NG(x) = \bigcup\{N_zG(x) \mid z \in G(x)\}$;
- $\widehat{I^+}(N_zG(x)),\ \widehat{J^+}(N_zG(x))$ will denote the set of future time-like, respectively causal vectors in $N_zG(x) \subset T_zL$
- $\widehat{I^+}(NG(x))$ is the *time-like future of the zero section in the normal bundle* that is $\widehat{I^+}(NG(x)) = \bigcup\left\{\widehat{I^+}(N_zG(x)) \mid z \in G(x)\right\}$
- ε is the exponential map, $\exp : TL \cap D \to L$, restricted to $NG(x) \cap D$, where D is the domain of the exponential map. For the sake of simplicity $\varepsilon|_{N_xG(x)}$ will mean $\varepsilon|_{N_xG(x)\cap D}$;
- $r_v : [0, \alpha) \to T_xL,\ v \in T_xL,\ x \in L$ is the *ray* in the direction of v that is $r_v(t) = t \cdot v$;
- $c_v : [0, \alpha) \to L,\ v \in T_xL,\ x \in L$ is the geodesic in the direction of v i.e. $c_v(t) = \exp \circ r_v(t)$;
- $\mathcal{L} : \Omega \to \mathbb{R}$ is the *length function* on the set of piecewise smooth curves, i.e. if $\sigma : [0, \alpha] \to \mathbb{R}$ is such a curve then $\mathcal{L}(\sigma) = \int_0^\alpha |\langle\sigma'(t), \sigma'(t)\rangle|^{\frac{1}{2}}\, dt$;

- $d : L \times L \to \mathbb{R} \cup \{\infty\}$ is the *Lorentzian distance* function, also called time separation, i.e. if Ω_x^y is the set of picewise smooth future directed causal curves from x to y then $d(x, y) = \sup\{\mathcal{L}(\sigma) \mid \sigma \in \Omega_x^y\}$, where we define $\sup \emptyset := 0$;
- if $\gamma : [0, \alpha] \to L$ is a geodesic between $\gamma(0)$ and $\gamma(\alpha)$ which is causal and $d(\gamma(0), \gamma(\alpha)) = \mathcal{L}(\gamma)$ holds then γ is called *maximal.*
- $v \in NG(x)$ is a *focal point* if $T\varepsilon$ is not injective on $T_vNG(x)$, thus $\ker T\varepsilon|_{T_vNG(x)} \neq \{0_v\}$. This definition is equivalent to the following: $v \in NG(x)$ is a *focal point* if in every neighbourhood of v the mapping ε fails to be a diffeomorphism. The *focal locus* is the set of the focal points in $NG(x)$. The image of a focal point under the mapping ε will be called focal point also.

Lemma 1. *Let L be a globally hyperbolic Lorentz manifold and $G \times L \to L$ a smooth isometric action of the compact connected Lie group G then every orbit $G(x), x \in L$ is a compact connected Riemannian submanifold of L.*

Proof. Let us take an orbit $G(x)$. First we prove that there are no causally related points in the orbit. If there are $y_1, y_2 \in G(x)$, such that $y_1 \gneqq y_2$, where $y_1 \gneqq y_2$ means that there is a future directed non trivial causal geodesic from y_1 to y_2, then for every $g \in G$, $g \cdot y_1 \gneqq g \cdot y_2$. The transitivity of G on $G(x)$ gives that for every $y \in G(x)$ we have a $z \in G(x)$ such that $y \gneqq z$, by the global hyperbolicity $y = z$ is not allowed, because there can not be causal loops in the manifold. The manifold L is globally hyperbolic so we have a partial ordering on $G(x)$ where every chain has an upper bound, because let $y_1 \gneqq y_2 \gneqq y_3 \gneqq \dots$ be a chain. Since G acts transitively on $G(x)$ we can write this chain in the following form: $y_1 \gneqq g_1 \cdot y_1 \gneqq g_2 \cdot y_1 \gneqq \dots$. Then the compactness of G yields that there is a subsequence $n_i \to \infty$ such that $g_{n_i} \to g \in G$. This yields that $g_{n_i} \cdot y_1 \to g \cdot y_1$. Since

$$J^+(y_1) \supset J^+(g_{n_1} \cdot y_1) \supset J^+(g_{n_2} \cdot y_1) \supset \cdots$$

are closed sets in L, by the global hyperbolicity, and for every $i \in \mathbb{Z}^+$ there is a $k \in \mathbb{Z}^+$ such that

$$J^+(g_{n_k} \cdot y_1) \supset J^+(y_i) \supset J^+(g_{n_{k+1}} \cdot y_1)$$

this gives with the fact $g_{n_i} \cdot y_1 \to g \cdot y_1$ that $g \cdot y_1 \in J^+(y_i)$ for every $i \in \mathbb{Z}^+$, because if $g \cdot y_1$ is not in the closed set $\bigcap_{i=1}^{\infty} J^+(y_{n_i})$ then $g \cdot y_1$ must be in the open set $L - \{\bigcap_{i=1}^{\infty} J^+(y_{n_i})\}$. But since $g_{n_i} \cdot y_1 \to g \cdot y_1$, for every point $p \in I^+(g \cdot y_1)$ which is suitably near to $g \cdot y_1$ there is an index n_l such that $p \in J^+ (g_{n_i} \cdot y_1)$ for every

$n_l \leq n_k$. This gives that $p \in \bigcap_{i=1}^{\infty} J^+(y_{n_i})$, but p can be chosen arbitrary near to $g \cdot y_1$ which contradicts to that $g \cdot y_1$ is in the open set $L - \{\bigcap_{i=1}^{\infty} J^+(y_{n_i})\}$. So every chain has an upper bound and the *Zorn lemma* gives that there is a maximal element w of $G(x)$ at this partial ordering. But such an element can not exist, because as we showed at the beginning of *this proof*, by the assumption that there are causally related points in $G(x)$, there is an element $v \in G(x)$ such that $w \lneqq v$, $v \neq w$ which contradicts the maximality of w.

Now we prove that the space $T_xG(x)$ is space-like. If there would be a time-like vector in the tangent space $T_xG(x)$ then there would be a point $y \in G(x)$ near to x for which $x \lneqq y$ but this can not be as we have seen. Otherwise there is a unique light-like vector in $T_xG(x)$, up to multiplying by a non-zero constant. So only one light-like line can be in the tangent space in this case. Then by the transitivity of G on $G(x)$, in every $T_yG(x)$, $y \in G(x)$ there is a unique light-like line. The smoothness yields that we can take a light-like curve in $G(x)$, an "integral curve", which would give causal related points in $G(x)$ but such points can not exist. So we have that for every point $y \in G(x)$ the tangent space of the orbit $T_yG(x)$ is space-like which proves the *lemma*. □

The above *lemma* can be proved without the axiom of choise.

The assumption of global hyperbolicity in the above *lemma* is necessary as the following not globally hyperbolic example shows.

Example 2. Let $L = \mathbb{R} \times \mathbb{S}^1_1$ be the product of the the real line $\mathbb{R}$, as a Riemannian manifold, and the 1-dimensional Lorentz space $\mathbb{S}^1_1$. If we take the canonical action of the Lie group $\mathbb{S}^1$ on $\mathbb{S}^1_1$ we get an action on L for which the orbits are time-like circles.

Corollary 3. *Let L be as in Lemma 1 and let G be a compact Lie group which acts isometrically on L then every orbit is a finite union of compact connected Riemannian submanifolds of L.*

Proof. We use *the proof of the above lemma* for each connected component of G. Since G is compact the number of the components of G is finite. □

Next we will prove the following geometric fact: To every point y in the time-like, (causal) future of an orbit we can find a time-like, (causal) curve starting orthogonally to the orbit and ending at y.

Lemma 4. *If G and L are as above then $\varepsilon(\widehat{I^+}(NG(x))) = I^+(G(x))$ and $\varepsilon(\widehat{J^+}(NG(x))) = J^+(G(x))$.*

Proof. It is clear that

$$\varepsilon(\widehat{I^+}(NG(x))) = \varepsilon\left(\bigcup\left\{\widehat{I^+}(N_zG(x)) \mid z \in G(x)\right\}\right) \subset I^+(G(x)).$$

For the inclusion $\varepsilon(\widehat{I^+}(NG(x))) \supset I^+(G(x))$ let $y \in I^+(G(x))$ then there is a time-like curve from a point in $G(x)$ to y, we can assume that this is from x to y. The global hyperbolicity of L gives a maximal future directed time-like geodesic from x to y, see [B-E-E], Theorem 6.1 p. 200. If this is also a maximal time-like geodesic from $G(x)$ to y then this must start orthogonally to $G(x)$, otherwise it cannot be a maximal one. If there is a sequence of time-like geodesics, $\gamma_1, \gamma_2, \dots$ from $G(x)$ to y whose Lorentzian length is increasing and

$$\mathcal{L}(\gamma_i) \to \sup\{d(z,y) \mid z \in G(x)\}$$

then by the compactness of $G(x)$ we can assume that $G(x) \ni \gamma_i(0) \to w \in G(x)$. This yields with the fact that $J^-(y)$ is closed that $w \in J^-(y)$. So we have a causal geodesic γ from w to y such that $\gamma_{n_i} \to \gamma$ holds, for a suitable subsequence $n_1 < n_2 < \dots$, (see [B-E-E] Corollary 3.32) for which $\mathcal{L}(\gamma_{n_i}) \to \mathcal{L}(\gamma)$. But then

$$\mathcal{L}(\gamma_{n_i}) \leq \mathcal{L}(\gamma_{n_{i+1}}) \leq \dots \leq \mathcal{L}(\gamma)$$

and this would give a maximal causal geodesic from $G(x)$ to y. So we proved the first equality of this lemma since $\gamma_1, \gamma_2, \dots$ are time-like geodesics so, by

$$0 < \mathcal{L}(\gamma_1) \leq \mathcal{L}(\gamma_2) \leq \dots \leq \mathcal{L}(\gamma),$$

γ is also time-like. In order to prove the second one consider an element $y \in J^+(G(x)) - I^+(G(x))$. Then there is a light-like geodesic $\gamma : [0,1] \to L$, with $\gamma(0) \in G(x), \gamma(1) = y$. If γ is not orthogonal to L then by taking a sufficiently small geodesically convex neighbourhood U of $\gamma(0)$, the orbit $G(x)$ would intersect $I^-(\gamma(\epsilon)) \cap U$ for a suitable small $\epsilon \in [0,1]$, which shows that we can replace γ by such a causal curve from y to $G(x)$ which is not geodesic. But this contradicts $y \in J^+(G(x)) - I^+(G(x))$. So γ is orthogonal to $G(x)$ but then this shows that $y \in \varepsilon(\bigcup\{\widehat{J^+}(N_zG(x)) \mid z \in G(x)\})$. This proves $J^+(G(x)) \subset \varepsilon(\widehat{J^+}(NG(x)))$. Since the inclusion $J^+(G(x)) \supset \varepsilon(\widehat{J^+}(NG(x)))$ is trivial the equality follows. $\square$

Definition 5. A subset $S \subset L$ is called *Cauchy hypersurface* if every inextendible causal curve intersects it exactly once.

The above definition can be found in [B-E-E] p. 65.

Corollary 6. *If G and L are as above moreover L and G are connected and there exists an orbit $G(x)$ of co-dimension 1 then $G(x)$ is a Cauchy hypersurface and every orbit is a Cauchy hypersurface.*

Proof. Since $G(x)$ is of co-dimension 1 the normal space $N_zG(x)$ is a time-like line for every $z \in G(x)$ thus $\widehat{I^+}(N_zG(x)) = \widehat{J^+}(N_zG(x))$ holds for every $z \in G(x)$. Moreover $\overline{I^+(G(x))} = J^+(G(x)) \cup G(x)$. So we have

$$\begin{aligned}
\overline{\varepsilon(NG(x))} &= \overline{\varepsilon(\widehat{I^+}(NG(x)))} \cup \overline{\varepsilon(\widehat{I^-}(NG(x)))}\\
&= \overline{I^+(G(x))} \cup \overline{I^-(G(x))} = J^+(G(x)) \cup J^-(G(x)) \cup G(x)\\
&= \varepsilon\left(\bigcup\left\{\widehat{J^+}(N_zG(x)) \mid z \in G(x)\right\}\right) \cup \varepsilon\left(\bigcup\left\{\widehat{J^-}(N_zG(x)) \mid z \in G(x)\right\}\right) \cup G(x)\\
&= \varepsilon\left(\bigcup\left\{\widehat{I^+}(N_zG(x)) \mid z \in G(x)\right\}\right) \cup \varepsilon\left(\bigcup\left\{\widehat{I^-}(N_zG(x)) \mid z \in G(x)\right\}\right) \cup G(x)\\
&= \varepsilon\big(\widehat{I^+}(NG(x))\big) \cup \varepsilon\big(\widehat{I^-}(NG(x))\big) \cup G(x) = \varepsilon(NG(x)).
\end{aligned}$$

But here the left hand side is closed and the right hand side is open so $\varepsilon(NG(x)) = L$.

Since the orbit $G(x)$ is of co-dimension 1 the future set $\widehat{I^+}(N_zG(x))$ is a time-like line for every $z \in G(x)$. But by our first *lemma* G_z acts trivial on this line. So if we take the unique future time-like unit vector in $\widehat{I^+}(N_zG(x))$ then by the action of G this extends to a unique future time-like unit vector field W on $G(x)$ which is G-invariant.

By our first *lemma* follows that from a point $y \in L$ there can be only future or only past directed causal curves to $G(x)$. The first two steps in *this proof* give that for every point $y \in L$ there is, let us assume, a past directed time-like geodesic $\gamma_1 : [0,\alpha] \to L, \gamma_1(0) = y$, $\gamma_1(\alpha) \in G(x)$ in unit speed parametrization which is orthogonal to $G(x)$. If there is an other one $\gamma_2 : [0,\beta] \to L, \gamma_2(0) = y$, $\gamma_2(\beta) \in G(x)$ in unit speed parametrization which is orthogonal to $G(x)$ then $\alpha = \beta$; In fact, let us assume that $\alpha < \beta$ then there is an element $g \in G$ such that $g \cdot \gamma_1(\alpha) = \gamma_2(\beta)$. By the *second step of this proof* $g \cdot \gamma_1|_{[0,\alpha]} = \gamma_2|_{[\beta-\alpha,\beta]}$, for the above $g \in G$. But then $\gamma_2|_{[0,\beta-\alpha]}$ would be a past directed non-trivial time-like geodesic from y to y which is a contradiction. So for every $t \in \mathbb{R}$ the set $\varepsilon(t \cdot W)$ is an orbit and for every orbit there is a unique $t \in \mathbb{R}$ such that the orbit is $\varepsilon(t \cdot W)$. We can say we have constructed a time function $f_{G(x)} : L \to \mathbb{R}$ such that the orbits are the constant levels of this function. And this function gives the maximal Lorentzian distance between a point and $G(x)$.

If there is an orbit $G(y) = \varepsilon(t_0 \cdot W)$ then by our function $f_{G(x)}$ it is easy to see that the sets $\{y \in L \mid f_{G(x)}(y) < t_0\}$ and $\{y \in L \mid f_{G(x)}(y) > t_0\}$ are disjoint and

$$\{y \in L \mid f_{G(x)}(y) < t_0\} \cup \{y \in L \mid f_{G(x)}(y) > t_0\} = L - G(y).$$

But if $\dim G(y) < \dim L - 1$ then $L - G(y)$ is connected. So every orbit is a hypersurface.

Let us take an inextendible causal curve $\varphi : (\alpha, \beta) \to L$. By our *first lemma* $f_{G(x)} \circ \varphi : (\alpha, \beta) \to f_{G(x)}(L) \subset \mathbb{R}$ is injective. So we can assume that $f_{G(x)} \circ \varphi$ is a strictly monotone increasing function. We must only show that $f_{G(x)} \circ \varphi$ is surjective this gives that φ intersects every orbit exactly once. If $\lim_{t\to\beta} f_{G(x)} \circ \varphi(t) = t_0 \in f_{G(x)}(L)$, i.e. $\lim_{t\to\beta} f_{G(x)} \circ \varphi(t) \neq \sup f_{G(x)}(L)$, then since the orbit $f^{-1}_{G(x)}(t_0)$ is a hypersurface which is a smooth compact Riemannian submanifold it is easy to see that $\lim_{t\to\beta} \varphi(t)$ exists, because φ is causal and L is globally hyperbolic. But then we can continue the curve φ from the point $\lim_{t\to\beta} \varphi(t)$ which contradicts to the inextandiblity of φ. The same argument shows that $\inf f_{G(x)} \circ \varphi((\alpha, \beta)) = \inf f_{G(x)}(L)$. □

Corollary 7. *Let G and L be as above. If there is an orbit of co-dimension 2 then this is a maximal dimensional orbit.*

The above two corollaries show that there is a difference between the Riemannian and the Lorentz case. Because if we take the unit sphere, $\mathbb{S}^2$, in $\mathbb{R}^3$ and the rotations around an axis which goes through the origin we get an action of the Lie group $\mathbb{S}^1$ on the compact Riemannian manifold $\mathbb{S}^2$. In this example there are orbits with co-dimension 1 and 2.

There is a canonical partial ordering on the set of the orbits of a compact Lie group action. The orbit $G(x)$ has greater orbit type than the orbit $G(y)$, in notation $G(x) \succeq G(y)$, iff $G_x \leq g \cdot G_y \cdot g^{-1}$ for some $g \in G$. According to the *Principal orbit type theorem* there is a unique maximal orbit type under this partial ordering which is called *principal orbit*. An orbit which has maximal dimension but which is not principal is called *exceptional orbit* and an orbit which is not of maximal dimension is called *singular*. (see [B])

It is known that a globally hyperbolic space-time is diffeomorphic to $\mathbb{R} \times S$, where S is a Cauchy hypersurface, see [B-S]. But the proof in our case is much more simpler, as the following short outlined proof shows.

Proposition 8. *Let G, L be as above if there is an orbit $G(x)$ of co-dimension 1 then L is diffeomorphic to $G/G_x \times (\alpha, \beta)$, where $x \in L$ is arbitrary and $-\infty \leq \alpha < \beta \leq \infty$.*

Proof. From Corollary 6 we know that every orbit is a Cauchy hypersurface, so there are no singular orbits, and that the orbits are the level sets of the time function $f_{G(x)}$, for this function see also Corollary 6. From the properties of $f_{G(x)}$ it can be shown that $\varepsilon : NG(x) \to L$ is a homeomorphism. Since it can be proved that in our case the images by ε of the focal points in $NG(x)$ correspond to singular orbits, we have that there are no focal points in $NG(x)$ thus $\varepsilon : NG(x) \to L$ is the desired diffeomorphism. $\square$

From the above proof it is easy to see that every curve $(a,b) \ni t \mapsto z \times t$, $z \in G/G_x$ is a time-like geodesic on which the metric is the canonical metric multiplied by (-1) of (a,b). If an isometric action of a compact connected Lie group has an orbit of co-dimension 1 then in the Riemannian case the structure of the manifold can be more complicated, see [A-A].

2. Singular orbits

Definition 9. Let $(\mathbb{M}^n, \langle , \rangle)$ be the n-dimensional Minkowski space and $0_{\mathbb{M}^n}$ be its origin. Then the *semi-orthogonal group* of this space is:

$$MO\,(n) := \{\phi \in Iso\,(\mathbb{M}^n, \langle , \rangle) \mid \phi\,(0_{\mathbb{M}^n}) = 0_{\mathbb{M}^n}\},$$

where $Iso\,(\mathbb{M}^n, \langle , \rangle)$ is the group of isometries of the Minkowski space. The group of those orthogonal isometries which preserve the time orientation is denoted by $MSO\,(n)$ and it is called the *special semi-orthogonal group.*

Lemma 10. *For every compact Lie group $G \subset MSO\,(n)$ there is a non-zero time-like vector $v \in \mathbb{M}^n$ which is fixed under the action of G.*

Proof. Let us consider the submanifold $H \subset \mathbb{M}^n$, given by the end points of the future directed timelike vectors w defined by $\langle w, w\rangle = -1$. The compact group G acts isometrically on the Riemannian manifold $H \subset \mathbb{M}^n$. Moreover it is well-known that H is a model of the hyperbolic space so it has non-positive curvature. According to a theorem of E. Cartan (see [K-N] p. 111.) H has a fixed point v under the action of G which yields that $v \in H \subset \mathbb{M}^n$ remains fixed under the action of G. $\square$

In the following lemma the Lorentz manifold is not necessary globally hyperbolic.

Lemma 11. *Let L be a Lorentz manifold and G a compact Lie group which acts isometrically on L. Assume that $G(x)$ is a singular orbit which is space-like, that is its tangent space $T_xG(x)$ is space-like. Then the orbit $G(x)$ is not an isolated singular orbit, i.e. in every neighbourhood of $G(x)$ there is an other singular orbit.*

Proof. Consider the isotropy subgroup G_x of the point x and let G_x^0 be its unit component. The group G_x^0 acts isometrically on the normal space $N_xG\,(x)$ which dimension $n \geq 2$, because $G(x)$ is a singular orbit. Since the orbit $G\,(x)$ is space-like the normal space $N_xG\,(x)$ can be considered as a Minkowski space where the metric on $N_xG\,(x)$ is the metric induced by the semi-Euclidean metric $\langle,\rangle|_{N_xG(x)}$ in the following way: Let $A_z : T_zN_xG\,(x) \to N_xG\,(x)$ be the canonical isomorphism where $z \in N_xG\,(x)$; this extends the semi-Euclidean metric $\langle,\rangle|_{N_xG(x)}$ to a semi-Riemannian metric $\widetilde{\langle,\rangle}$ on $N_xG\,(x)$ such that $\left(N_xG\,(x)\,, \widetilde{\langle,\rangle}\right)$ is a Minkowski space. On this Minkowski space the compact *connected* Lie group G_x^0 acts isometrically so $G_x^0 \subset MSO\,(n)$. By the *above lemma* we have a non-zero future directed time-like vector $v \in N_xG\,(x)$ which is fixed under the action of G_x^0. Since $N_xG\,(x) \subset NG(x)$ and the exponential mapping ε on $NG\,(x)$ is a diffeomorphism in a neighbourhood of the zero section $G\,(x) \subset NG\,(x)$ we have that the geodesic c_v is fixed under the action of G_x^0. This means that for every $\delta \in \mathbb{R}$ we have $G_x^0 \leq G_{c_v(\delta)}$ for the isotropy subgroups. Since $G\,(x) \approx G/G_x$ and $G\,(c_v\,(\delta)) \approx G/G_{c_v(\delta)}$ we have that

$$\begin{gathered}\dim\left(G\,(x)\right) = \dim\left(G/G_x\right)\\ = \dim\left(G/G_x^0\right) \geq \dim\left(G/G_{c_v(\delta)}\right) = \dim\left(G\,(c_v\,(\delta))\right).\end{gathered}$$

So the orbit $G\,(c_v\,(\delta))$ is singular. □

The lemma is not necessarily true for exceptional orbits as the following example shows.

Example 12. Let $\mathbb{S}^1 := \{\alpha \in \mathbb{C} \mid \|\alpha\| = 1\}$ be the unit circle endowed with the canonical Riemannian metric and $\mathbb{R}_1^1$ the real line with the canonical Riemannian metric multiplied by -1. Furthermore let $G = \mathbb{S}_I^1 \cup \mathbb{S}_{II}^1$ be the disjoint union of two unit circles, the elements of the unit circles are labelled by α_I and α_{II}. Let consider the product manifold

$$L := \mathbb{S}^1 \times \mathbb{R}_1^1$$

which is a globally hyperbolcic Lorentz manifold on which the action of the group $\theta : G \times L \to L$ is the following: $\alpha_I \times (\alpha, x) \mapsto (\alpha_I \cdot \alpha, x)$ and $\alpha_{II} \times (\alpha, x) \mapsto (\alpha_{II} \cdot \alpha, -x)$. Here the principal orbits are $\mathbb{S}^1 \times \{x\} \cup \mathbb{S}^1 \times \{-x\}$ where $x \in \mathbb{R}_1^1 - \{0\}$ and the only exceptional orbit is the isolated $\mathbb{S}^1 \times \{0\}$.

Theorem 13. *Let L be a globally hyperbolic Lorentz manifold and G a compact connected Lie group which acts isometrically on L. Then there are no isolated singular orbits.*

Proof. Since in a globally hyperbolic Lorentz manifolds all the orbits are space-like we can apply the *above lemma*. $\square$

3. Singular orbits and focal points of maximal dimensional orbits

If we want to prove the Lorentzian analogue of Riemannian Theorem 1 in [Sz] we have to define the Lorentzian analogues of the cut points and of a special set in $NG(x)$ which were used in the Riemannian case in the above paper of Szenthe.

Let us fix an orbit $G(x)$. By the definition of the global hyperbolicity and the fact that the mapping $\varepsilon : NG(x) \to L$ is a diffeomorphism in a neighbourhood of the zero section it is easy to prove, that if we take a causal ray $r_v,\ v \in NG(x)$ then its image is locally maximal, that is there is a $t_0 > 0$ for which the geodesic segment $c_v|_{[0,t_0]}$ is maximal. For every future directed ray there is also a maximal value t_v such that the geodesic $c_v|_{[0,t_v]}$ is maximal, where $t_v = \infty$ is also allowed. The point $r_v(t_v)$ is a *future causal cut point*. More precisely we have the following definition, which is a natural generalization of Definition 9.9 [B-E-E], p. 302, see also [B-E-E], Definition 9.3 p. 299. and p. 305.

Definition 14. Let $TL(-1)$ be the set of time-like vectors in TL which have length -1. We define the function $s_{G(x)} : TL(-1) \cap NG(x) \to \mathbb{R} \cup \{\infty\}$ as

$$s_{G(x)}(v) := \sup\{t \geq 0 \mid d(G(x), c_v(t)) = t\}.$$

A point $v \in NG(x)$ is a *future time-like cut point* if $s_{G(x)}\big(\frac{v}{\|v\|}\big) = \|v\|$ and v is future directed, where $\|v\| = |\langle v, v\rangle|^{\frac{1}{2}}$.

Let $TL(0)$ be the set of the light-like vectors in TL. Moreover let $l_{G(x)} : TL(0) \cap NG(x) \to \mathbb{R} \cup \{\infty\}$ be defined for a light-like vector $v \in NG(x)$ as

$$l_{G(x)}(v) := \sup\{t \geq 0 \mid d(G(x), \varepsilon(t \cdot v)) = 0\}.$$

A point $v \in NG(x)$ is a *future light-like cut point* if $l_{G(x)}(v) = 1$ and v is future directed.

So if the meaning of t_v is such as above, that is for a point $v \in NG(x)$ the geodesic $c_v([0, t_v])$ is maximal and if $t > t_v$ then the geodesic $c_v([0, t])$ is not maximal, where $t_v = \infty$ is also allowed, then we can consider the following set:

$$\bigcup \big\{r_v([0, t_v)) \mid v \in \widehat{I^+}(NG(x))\big\}.$$

The image of this set under the mapping ε roughly covers the timelike future of the orbit $G(x)$, see Lemma 17, so we will call this set, the *regular time-like future of the orbit* $G(x)$. This set can be given by the above functions in the following form:

Definition 15. The *regular time-like future* is

$$T^+_{\mathrm{reg}}(G(x)) := \left\{ v \in \widehat{I^+}(NG(x)) \mid s_{G(x)}\left(\frac{v}{\|v\|}\right) > \|v\| \right\}.$$

The *regular light-like future* is

$$N^+_{\mathrm{reg}}(G(x)) := \left\{ v \in \widehat{J^+}(N_zG(x)) - \widehat{I^+}(N_zG(x)) \mid z \in G(x),\ l_{G(x)}(v) > 1 \right\},$$

and the *regular causal future* is

$$TN^+_{\mathrm{reg}}(G(x)) := T^+_{\mathrm{reg}}(G(x)) \cup N^+_{\mathrm{reg}}(G(x)).$$

First let us show some elementary properties of the regular future and cut points of an orbit $G\,(x)$.

Lemma 16. *If L is a globally hyperbolic Lorentz manifold and the compact connected Lie group G is acting on it isometrically, then for every orbit $G(x)$ there is a future (and a past) directed inextendible causal geodesic $\gamma : [0, \alpha) \to L$, with $\gamma(0) \in G(x)$ such that $d(\gamma(t), G(x)) = \mathcal{L}(\gamma|_{[0,t]})$, for every $t \in [0, \alpha)$*

Proof. The proof goes along the lines of the proof of Theorem 8.10 [B-E-E] which is a special case of this *lemma* considering the trivial action of G on L, thus $G(x) = \{x\}$. In the proof of Theorem 8.10 [B-E-E] it was shown that we can take such a sequence of causal geodesics for which the length of their maximal segments is monotone increasing, and this sequence is converging to an inextedible maximal geodesic. Now the difference is only that there the initial points of the geodesics were fixed, but now, if we use the technique of [B-E-E] for choosing a geodesic sequence as above, with monotone increasing length of maximal segments, then the initial point can vary, but it remains on $G(x)$ which is compact. So by the compactness this sequence will also converge to a maximal inextendible geodesic which starts from $G(x)$. We will also need in this analogous proof that if $q \in I^+(G(x))$ then there is a geodesic $c : [0, 1] \to L$, with $c(0) \in G(x)$, $c(1) = q$ such that $\mathcal{L}(c) = d(c(0), c(1))$. □

The above *lemma* gives that there is a causal ray in $N_xG(x)$ for every $x \in L$ which does not have cut points, because there is such a ray in $NG(x)$ and by the transitivity of G on $G(x)$ and by the isometry, we can assume that this ray is starting at $0_x \in N_xG(x)$, thus it is in $N_xG(x)$. The following *lemma* yields that for every point in the causal future of an orbit there is a maximal causal geodesic joining the orbit and the point.

Lemma 17. *The following equalities are true*

$$I^+(G(x)) = \varepsilon\big(\widehat{I^+}(NG(x))\big) = \varepsilon\big(\overline{T^+_{\mathrm{reg}}(G(x))} \cap \widehat{I^+}(NG(x))\big),$$

$$J^+(G(x)) \cup G(x) = \varepsilon\big(\widehat{J^+}(NG(x))\big) \cup G(x) = \varepsilon\big(\overline{TN^+_{\mathrm{reg}}(G(x))}\big).$$

Proof. The first equality is Lemma 4. For the second one we must only prove that $\varepsilon(\widehat{I^+}(NG(x))) \subset \varepsilon(\overline{T^+_{\mathrm{reg}}(G(x))} \cap \widehat{I^+}(NG(x)))$. If we take an element $y \in \varepsilon(\widehat{I^+}(NG(x)))$ then there is a geodesic $\gamma : [0,\alpha] \to L$, such that $\gamma(0) \in G(x), \gamma(\alpha) = y, \gamma'(0) \perp T_{\gamma(0)}G(x), \mathcal{L}(\gamma) = d(y, G(x))$. This gives that $r_{\gamma'(0)}(\alpha) \in \overline{T^+_{\mathrm{reg}}(G(x))}$. The equalities for J^+ can be proved the same way. □

Theorem 18. *Let L be a globally hyperbolic Lorentz manifold with non-positive sectional curvature along the non-space-like geodesics, that is $\langle R(Z,\gamma'(t))\gamma'(t), Z\rangle \le 0$ for every γ non-space-like geodesic and for every (non-time-like) vector $Z \in T^\perp\gamma(t)$ orthogonal to γ at $\gamma(t)$. Moreover let G be a compact connected Lie group whose elements are isometries of L and such that its action is smooth. Let $G(x)$ be a principal orbit such that the isotropy subgroup G_x is of maximal rank. Then the orbit $G(z)$ of a point $z \in J^+(G(x))$ is singular if and only if z is a first focal point of $G(x)$ along a causal geodesic.*

Proof. The proof goes along the lines of the proof of [Sz] Theorem 1, where the preparatory propositions can be proved for $J^+(G(x))$. □

In a special situation we can prove something similar without assuming that the isotropy subgroup G_x is of maximal rank.

Theorem 19. *Let L be a globally hyperbolic Lorentz manifold and G a compact connected Lie group. Moreover let $G(x)$ be an orbit with co-dimension 2. Furthermore let $c_v : [0,\alpha) \to L$, $v \in NG(x)$, be a light-like geodesic for which $c_v(0) = x$. If $c_v(t_0)$ is a focal point along c_v then the orbit $G(c_v(t_0))$ is singular.*

Proof. Let $v \in NG(x)$ then the geodesic c_v starts orthogonal to the orbit $G(x)$. First we prove that this geodesic remains orthogonal to the other orbits, that is $c_v'(t) \perp G(c_v(t))$. Let $X \in \mathfrak{g}$ be a Lie algebra element and X_ψ the vector field along c_v generated by the infinitesimal isometry of X. This vector field is a $G(x)$-Jacobi field. From the well-known equality for Jacobi fields

$$\langle(\gamma')', X_\psi\rangle - \langle\gamma', X'_\psi\rangle \equiv \mathrm{const}$$

we have that $-\langle\gamma', X'_\psi\rangle \equiv \text{const}$ since $(\gamma')' = 0$. But

$$\text{const} \equiv -\langle\gamma', X'_\psi\rangle = \langle\gamma', A_{X_\psi}(\gamma')\rangle,$$

where A_{X_ψ} is the constant tensor field of X_ψ. Since the vector field X_ψ was generated by an infinitesimal isometry this is a Killing field, but then the constant tensor field A_{X_ψ} is antisymmetric with respect to the metric, that is $\langle\gamma', A_{X_\psi}(\gamma')\rangle = -\langle A_{X_\psi}(\gamma'), \gamma'\rangle$ which gives that the constant in the above equality is zero. So we have that

$$0 = \langle\gamma', X'_\psi\rangle = \frac{d}{dt}\langle\gamma', X_\psi\rangle,$$

which yields that $\langle\gamma', X_\psi\rangle \equiv \text{const}$. Since $\langle\gamma'(0), X_\psi(0)\rangle = 0$ this constant is 0. But $\{X_\psi(t), \mid X \in \mathfrak{g}\} = T_{c_v(t)}G(c_v(t))$ gives that the geodesic c_v is orthogonal to the orbits.

Assume that $c_v(t_0)$ is a focal point of $G(x)$. In this case there is a vector $w \in T_{t_0 \cdot v}NG(x), w \neq 0$ for which $T\varepsilon(w) = 0$. Note that on the normal bundle $NG(x)$ there is an action of the group G, and since the orbit $G(x)$ has maximal dimension all the orbits in $NG(x)$ are of maximal dimension under the action of G. So the space $T_{t_0 \cdot v}NG(x)$ has a canonical decomposition

$$T_{t_0 \cdot v}NG(x) = T_{t_0 \cdot v}N_xG(x) \oplus T_{t_0 \cdot v}G(t_0 \cdot v).$$

So we can write $w = w_N + w_G$, where $w_N \in T_{t_0 \cdot v}N_xG(x)$, $w_G \in T_{t_0 \cdot v}G(t_0 \cdot v)$. If $w_N = \text{const} \cdot r'_v(t_0)$ then $T\varepsilon(w_N) = \text{const} \cdot c'_v(t_0)$. If $w_N \neq \text{const} \cdot r'_v(t_0)$ then by applying the Gauss lemma, (see [B-E-E] p. 338,) to $r'_v(t_0)$ and w_N we have that $\langle c'_v(t_0), T\varepsilon(w_N)\rangle \neq 0$ because the space $N_xG(x)$ is of dimension 2 which gives that the lightlike vector $r'_v(t_0)$ is not orthogonal to any $w_N \neq \text{const} \cdot r'_v(t_0)$, also note that if $w_N \neq \text{const} \cdot r'_v(t_0)$ then $T\varepsilon(w_N) \neq 0$ so $T\varepsilon(w_N) = 0$ iff $w_N = 0$. Since $T_{c_v(t_0)}G(c_v(t_0)) \perp c'_v(t_0)$ we have that $T\varepsilon(w_N) \notin T_{c_v(t_0)}G(c_v(t_0))$ if $w_N \neq 0$. Since ε is G equivariant we have $T\varepsilon(w_G) \in T_{\gamma(t_0)}G(\gamma(t_0))$ which yields that if $T\varepsilon(w) = 0$ then $w_N = 0$. So there is an element A of the Lie algebra $\mathfrak{g}$ such that the vector $w = w_G$ is generated by the infinitesimal isometry corresponding to A. This gives that $A \notin \mathfrak{g}_x, A \in \mathfrak{g}_{c_v(t_0)}$ holds because $w \neq 0$ but $T\varepsilon(w) = 0$. Since $\mathfrak{g}_x \subset \mathfrak{g}_{c_v(t_0)}$ in our case this involves that the dimension of $G(x)$ is greater than that of $G(c_v(t_0))$ which proves the *theorem.* □

Corollary 20. *If the terms $L, G, G(x)$ are as above then an orbit $G(z)$ in $J^+(G(x)) - I^+(G(x))$ is singular if and only if z is a first focal point of $G(x)$ along a light-like geodesic which is orthogonal to $G(x)$.*

Proof. If z is a first focal point along a light-like geodesic which is orthogonal to $G(x)$ then $G(z)$ is singular, by *the above theorem.*

If $G(z)$ is a singular orbit in $J^+(G(x)) - I^+(G(x))$ then using Lemma 14 we have that there is a light-like geodesic $\gamma : [0,1] \to L$ such that $\gamma(0) \in G(x)$, $\gamma(1) = z$, $\gamma \perp G(x)$, moreover let $r_{\gamma'(0)}$ be the ray defined by $\gamma'(0)$ then $r_{\gamma'(0)}[0,1) \subset N^+_{\mathrm{reg}}(G(x))$, where $N^+_{\mathrm{reg}}(G(x))$ is the regular light-like future. As we have seen in the proof of Theorem 29, $\gamma|_{[0,1]}$ is orthogonal to each of the orbits $G(\gamma(t)), t \in [0,1]$ and the orbit $G(\gamma(1))$ is singular. In this case since $G(x)$ is maximal and $G(\gamma(t_0))$ is singular, there is an element $A \in \mathfrak{g}_{\gamma(t_0)}, A \notin \mathfrak{g}_{\gamma(0)}$ for which the infinitesimal deformation corresponding to this element gives vector fields $X(t)$ along $r_v(t)$ in $TNG(x)$ and $Y(t)$ along $\gamma(t)$ in TL such that $T\varepsilon(X(t)) = Y(t)$ and Y is a $G(x)$-Jacobi-field along γ. Moreover the properties of G and the maximality of the dimension of $G(x)$ show that $X(t_0) \neq 0$ but $T\varepsilon(X(t_0)) = Y(t_0) = 0$. Thus $\gamma(t_0)$ is a focal point of $G(x)$ since $r_{\gamma'(0)}|_{[0,1)} \in N^+_{\mathrm{reg}}(G(x))$ there are no focal points on $\varepsilon(r_{\gamma'(0)}|_{[0,1)})$. □

The following example shows that if the isotropy subgroup G_x of a principal orbit is not of maximal rank then the first focal point along a time-like or light-like geodesic, corresponding to a time-like or light-like ray in $N_xG(x)$, does not give necessarily a singular orbit.

Example 21. We take the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ and the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$ endowed with the canonical Riemannian metric. We also take the real line $\mathbb{R}^1_1$ with the canonical Riemannian metric multiplied by -1. If we take the product space

$$\mathbb{S}^2 \times \mathbb{S}^1 \times \mathbb{R}^1_1,$$

where we take the canonical action of $\mathbb{S}^1$ on itself then all the orbits are principal but for each orbit there are time-like and light-like rays in the normal bundle on which there are focal points, also conjugate points.

The next example shows that a cut point is not necessarily corresponding to a singular orbit, it can correspond to an exceptional orbit.

Example 22. Let us take $\mathbb{R}^1 \times [0,1]$ endowed with the canonical Riemannian metric and glue $\mathbb{R}^1 \times \{0\}$ and $\mathbb{R}^1 \times \{1\}$ together such that we get a Mobius strip and $\{0\} \times [0,1]$ is a closed circle. This Riemannian manifold will be denoted by M. If we take the real line $\mathbb{R}^1_1$ with the canonical Riemannian metric multiplied by -1 and the product space

$$M \times \mathbb{R}^1$$

we have the wanted manifold. The Lie group will be $\mathbb{S}^1$ and the action will be the canonical action on M. In M the principal orbits are $\{a\}\times[0,1]\cup\{-a\}\times[0,1]$, $a\in\mathbb{R}^1-\{0\}$ and there is only one exceptional orbit $\{0\}\times[0,1]$. If we take a past directed light-like geodesic in $\mathbb{R}^1\times\{0\}\times\mathbb{R}^1_1$ from $0\times 0\times 0$ and a point x on it different from $\{0\}\times\{0\}\times\{0\}$ this will correspond to a principal orbit for which on a future directed light-like geodesic, the above light-like geodesic in the other direction, there will be a cut point, $\{0\}\times\{0\}\times\{0\}$, which corresponds to an exceptional orbit. It is easy to see that there will be time-like rays in $N_xG(x)$ on which there is a cut point which corresponds to an exceptional orbit.

References

[A] S. Adams, Dynamics on Lorentz manifolds, *World Scientific Publications*, *NJ*, 2001.

[A-A] A. V. Alekseevsky and D. V. Alekseevsky, Riemannian G-Manifolds with One-Dimensional Orbit Space, *Annals of Glob. Anal. and Geo.* **11** (1993), 197–211.

[B] G. E. Bredon, Introduction to compact transformation groups, *Academic Press*, *New York – London*, 1972.

[B-S] Antonio N. Bernal and Miguel Sanchez, On Smooth Cauchy Hypersurfaces and Geroch's Splitting Theorem, *Commun. Math. Phys.* **243** (2003), 461–470.

[B-E-E] J. K. Beem, P. E. Ehrlich and K. L. Easley, Global Lorentzian Geometry, *Marcel Dekker Inc.*, *New York*, 1996.

[K] S. Kobayashi, Transformation groups in differential geometry, *Springer-Verlag*, *New York – Heidelberg*, 1972.

[K-N] S. Kobayashi and K. Nomizu, Fundation of differential geometry Vol. II., *Interscience Tracts in Pure and Applied Mathematics*, *New York – London – Sydney*, 1969.

[O'N] B. O'Neill, Semi-Riemannian geometry with applications to relativity, *Academic Press*, *New York*, 1983.

[Sz] J. Szenthe., On the cut locus of a principal orbit in a Riemannian manifold of non-positive sectional curvature, *Annales Univ. Sci. Budapest. Eötvös Sect. Math.* **24** (1981), 227–240.

[W] F. W. Warner, Extension of the Rauch comparison theorem to submanifolds, *Am. Math. Soc.* **122** (1966), 341–356.

DÁVID SZEGHY
DEPARTMENT OF GEOMETRY
EÖTVÖS LORÁND UNIVERSITY
H-1518 BUDAPEST P. O. BOX 120
HUNGARY

E-mail: szeghy@cs.elte.hu

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