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Light-like homogeneous geodesics and the geodesic lemma for any signature

By ZDENĚK DUŠEK (Olomouc) and OLDŘICH KOWALSKI (Prague)

Abstract. Homogeneous geodesics on homogeneous Riemannian manifolds have been studied by many authors. The fundamental tool is the so-called *geodesic lemma*. On pseudo-Riemannian manifolds, a generalization of the geodesic lemma is necessary. Physicists already know and use the generalized version. However, the present authors are not aware of any reference to the detailed and correct mathematical proof. The aim of the present paper is to provide such a proof and illustrate the generalized lemma with an example. We also correct a minor error which occurred in the original proof of the standard geodesic lemma.

1. Introduction

Let M be a pseudo-Riemannian manifold. If there is a connected Lie group $G \subset I_0(M)$ which acts transitively on M as a group of isometries, then M is called a *homogeneous pseudo-Riemannian manifold*. Let $p \in M$ be a fixed point. If we denote by H the isotropy group at p, then M can be identified with the *homogeneous space* G/H. In general, there may exist more than one such group $G \subset I_0(M)$. For any fixed choice $M = G/H$, G acts effectively on G/H from

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the left. The pseudo-Riemannian metric g on M can be considered as a G-invariant metric on G/H. The pair $(G/H, g)$ is then called a *pseudo-Riemannian homogeneous space*.

If the metric g is positive definite, then $(G/H, g)$ is always a *reductive homogeneous space*: We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively and consider the adjoint representation $\mathrm{Ad} : H \times \mathfrak{g} \to \mathfrak{g}$ of H on $\mathfrak{g}$. There exists a direct sum decomposition (*reductive decomposition*) of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. If the metric g is indefinite, the reductive decomposition may not exist (see for instance [6] for an example of nonreductive pseudo-Riemannian homogeneous space). For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ there is a natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space T_pM via the projection $\pi : G \to G/H = M$. Using this natural identification and the scalar product g_p on T_pM we obtain a scalar product $\langle\, ,\rangle$ on $\mathfrak{m}$. This scalar product is obviously $\mathrm{Ad}(H)$-invariant.

The definition of a homogeneous geodesic is well-known in the Riemannian case (see, e.g., [10]). In the pseudo-Riemannian case it must be modified as follows:

Definition 1.1. The geodesic $\gamma(s)$ through the point p defined in an open interval J (where s is an affine parameter) is said to be homogeneous if there exists

1) a diffeomorphism $s = \varphi(t)$ between the real line and the open interval J;
2) a vector $X \in \mathfrak{g}$ such that $\gamma(\varphi(t)) = \exp(tX)(p)$ for all $t \in (-\infty, +\infty)$.

The vector X is then called a geodesic vector.

For results on homogeneous geodesics in homogeneous Riemannian manifolds we refer for example to [1], [2], [8], [10]–[14]. A homogeneous Riemannian manifold all of whose geodesics are homogeneous is called a Riemannian *g.o. manifold*. For some results and further references on Riemannian g.o. manifolds see for example [4], [5], [9]. The basic formula characterizing geodesic vectors in the Riemannian case was derived in [12]: The vector $X \in \mathfrak{g}$ is geodesic if and only if

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = 0 \quad \text{for all } Z \in \mathfrak{m}. \tag{1}$$

Homogeneous geodesics are interesting also in pseudo-Riemannian geometry and light-like homogeneous geodesics are of particular interest. In [6] and [15], the authors study plane-wave limits (Penrose limits) of homogeneous spacetimes along light-like homogeneous geodesics. In these papers, there is a characterization of the geodesic vector X by the formula

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = k \langle X_{\mathfrak{m}}, Z \rangle \text{ for all } Z \in \mathfrak{m}, \text{ where } k \in \mathbb{R} \text{ is some constant.} \tag{2}$$

In [3], this formula is also used for the computations. However, we are not aware of any work containing a detailed mathematical proof of this formula. We give a proof in Section 2 of this work. In Section 3, we give an example with an unimodular Lorentzian Lie group, which simply illustrates the interesting behaviour of light-like geodesics.

2. The geodesic lemma

Let us formulate the generalization of the "geodesic lemma" from [12] (Proposition 2.1 there) to the pseudo-Riemannian setting. Let $M = G/H$ be a homogeneous pseudo-Riemannian space, denote by $\mathfrak{g}$, $\mathfrak{h}$ the corresponding Lie algebras. Let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be a reductive decomposition. Denote by p the basic point of G/H.

Lemma 2.1. *Let $X \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(tX)(p)$ (the orbit of a one-parameter group of isometries) is a geodesic curve with respect to some parameter s if and only if*

$$\langle [X,Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = k \langle X_{\mathfrak{m}}, Z \rangle \text{ for all } Z \in \mathfrak{m}, \text{ where } k \in \mathbb{R} \text{ is some constant.} \tag{3}$$

Further, if $k = 0$, then t is an affine parameter for this geodesic. If $k \neq 0$, then $s = e^{-kt}$ is an affine parameter for the geodesic. The second case can occur only if the curve $\gamma(t)$ is a light-like curve in a (properly) pseudo-Riemannian space.

Proof. For $k = 0$, the lemma was proved in [12], but with a small gap. For $k \neq 0$, the proof has to be modified. We shall give a correct proof valid for an arbitrary k. Let $X, Z \in \mathfrak{g}$. Denote $g_t = \exp(tX)$, $h_s = \exp(sZ)$ and denote by X^*, Z^* the corresponding fundamental vector fields on M, that is

$$X^*_x = \frac{d}{dt}\Big|_0 g_t(x), \qquad Z^*_x = \frac{d}{ds}\Big|_0 h_s(x) \tag{4}$$

for each $x \in M$. Let us consider the orbit $g_t(p)$ (which is the integral curve of the field X^* through p) and choose some value u of the parameter t. Then we get

$$(dg_u)\big|_p X^*_p = (dg_u)\big|_p \frac{d}{dt}\big|_0 g_t(p) = \frac{d}{dt}\big|_0 g_t(g_u(p)) = \frac{d}{dt}\big|_0 g_{u+t}(p) = X^*_{g_u(p)} \tag{5}$$

which means $X^*_{\gamma(u)} = \frac{d\gamma(t+u)}{dt}\big|_{t=0}$. Hence $X^*_{\gamma(t)} = \frac{d\gamma(t)}{dt}$ for all values of t. It is also well-known that the covariant derivative $\nabla_{X^*_{\gamma(u)}} X^*$ depends, for each u, only on the values of the vector field X^* along the curve $\gamma(t)$. Since g_u is an isometry, and hence an affine diffeomorphism with respect to the Levi–Civita connection ∇, we get easily

$$(dg_u)\big|_{g_t(p)}(\nabla_{X^*}X^*\big|_{g_t(p)}) = \nabla_{X^*}X^*\big|_{g_{t+u}(p)} \quad \text{for arbitrary } t. \tag{6}$$

Now we use the formulas

$$\begin{aligned} X^*_x g(X^*, Z^*) &= g(X^*, [X^*, Z^*])(x), \\ Z^*_x g(X^*, X^*) &= 2g(X^*, [Z^*, X^*])(x) \end{aligned} \tag{7}$$

derived in [12] (see the formulas on the top of the page 194) and valid also in the pseudo-Riemannian case. From the standard formula for the Riemannian connection (see for example [7], Chapter IV, Proposition 2.3)

$$2g(\nabla_{X^*}X^*, Z^*) = 2X^*g(X^*, Z^*) - Z^*g(X^*, X^*) + 2g([Z^*, X^*], X^*) \tag{8}$$

we obtain (by using (7)) the formula

$$g(\nabla_{X^*}X^*, Z^*) = g(X^*, [X^*, Z^*]) = -g(X^*, [X, Z]^*). \tag{9}$$

Suppose now that the curve $\gamma(t) = \exp(tX)(p)$ is the trajectory of a *geodesic* for which t is not necessarily an affine parameter. (An example of such a situation will be given below.) Then, at any t, it holds

$$\nabla_{X^*}X^*\big|_{g_t(p)} = c(t) \cdot X^*_{g_t(p)}. \tag{10}$$

From the formulas (5) and (6) it follows that $c(t)$ is a constant function. According to the formulas (9) and (10), at p it holds

$$g(cX^*_p, Z^*_p) = -g(X^*_p, [X, Z]^*_p) \tag{11}$$

and from the identification of $T_p(M)$ with $\mathfrak{m}$ we obtain

$$-c\langle X_{\mathfrak{m}}, Z\rangle = \langle X_{\mathfrak{m}}, [X, Z]_{\mathfrak{m}}\rangle. \tag{12}$$

On the other hand, let the formula (3) hold for some k. Then

$$g(X^*_p, [X, Z]^*_p) = kg(X^*_p, Z^*_p). \tag{13}$$

Using the formula (9) at p, we obtain

$$g(\nabla_{X^*}X^* + kX^*, Z^*)(p) = 0 \tag{14}$$

for every vector $Z \in \mathfrak{m}$. Hence we obtain

$$g((dg_t)(\nabla_{X^*}X^* + kX^*)_p, (dg_t)Z^*_p) = g((\nabla_{X^*}X^* + kX^*)_{g_t(p)}, (dg_t)Z^*_p) = 0 \quad (15)$$

for all Z. Here the vectors $(dg_t)Z^*_p$ fill in all the tangent space $T_{g_t(p)}M$, and the metric g is non-degenerate. Hence $(\nabla_{X^*}X^* + kX^*)_{g_t(p)} = 0$ for all t and the trajectory $\gamma(t) = \exp(tX)(p)$ is a geodesic with *some* parametrization. If $k = 0$, then t is an affine parameter. If $k \neq 0$, a routine calculation shows that $s = e^{-kt}$ is an affine parameter. Finally, if the curve $\gamma(t) = \exp(tX)(p)$ is a space-like or time-like geodesic with some parametrization, then the vector X^*_p has nonzero norm. From the formulas (5), (10) and the fact that g_t is an isometry, it follows that t is an affine parameter for γ and hence $k = 0$. This proves the last statement. □

Let us recall that the whole proof is a bit delicate because the fundamental vector fields on M are, in general, *not* invariant with respect to the isometries. In [12] this fact was not respected at the very end of the proof of Proposition 2.1. Our formula (15) gives now the correct way to finish the proof.

3. An unimodular Lorentzian Lie group

Let us consider the 3-dimensional unimodular Lie group $G = E(1,1)$ with a left-invariant Lorentzian metric. Its Lie algebra $\mathfrak{g}$ (named $\mathfrak{g}_1$ in [3]) admits the pseudo-orthonormal basis $\{E_1, E_2, E_3\}$ with E_3 timelike and the brackets are given by the formulas

$$[E_1, E_2] = E_1, \qquad [E_1, E_3] = -E_1, \qquad [E_2, E_3] = E_2 + E_3. \quad (16)$$

We consider G acting on itself as group of isometries by left translations. We see easily that the vectors $X = \frac{E_2+E_3}{2}$ and $Z = \frac{E_2-E_3}{2}$ are geodesic vectors, if we put, in the formula (3), $k = 0$ or $k = -1$, respectively. Both vectors are light-like (and, according to [3], each geodesic vector is proportional to one of them). We are going to investigate the orbits of the 1-parameter groups $\exp(tX)$ and $\exp(tZ)$.

First, let us change the basis of the Lie algebra in the following way:

$$X = \frac{E_3 + E_2}{2}, \qquad Y = E_1, \qquad Z = \frac{E_3 - E_2}{2}. \quad (17)$$

The matrix of the scalar product with respect to the basis $\{X, Y, Z\}$ is

$$\begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}. \tag{18}$$

Hence X and Z are light-like geodesic vectors. For the Lie bracket operation of the new vectors we obtain

$$[X, Y] = 0, \qquad [X, Z] = X, \qquad [Y, Z] = -Y. \tag{19}$$

We can identify the generators X, Y, Z with the matrices

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

The group G can thus be identified with the matrices of the form

$$\begin{pmatrix} e^{-c} & 0 & a \\ 0 & e^{c} & b \\ 0 & 0 & 1 \end{pmatrix}, \tag{21}$$

where (a, b, c) form a global coordinate system. The left-invariant vector fields (we denote it again by X, Y, Z) are given (see [8]) by the formulas

$$X = e^{-c}\frac{\partial}{\partial a}, \qquad Y = e^{c}\frac{\partial}{\partial b}, \qquad Z = \frac{\partial}{\partial c}. \tag{22}$$

Let us consider the coordinates a, b, c on G and express the Lorentzian metric. At any point (a, b, c), the matrix of the scalar product of vector fields X, Y, Z is given by the formula (18) and hence the matrix of the scalar product of the coordinate vector fields $\frac{\partial}{\partial a}$, $\frac{\partial}{\partial b}$, $\frac{\partial}{\partial c}$ is

$$\begin{pmatrix} 0 & 0 & -\frac{1}{2}e^{c} \\ 0 & e^{-2c} & 0 \\ -\frac{1}{2}e^{c} & 0 & 0 \end{pmatrix}. \tag{23}$$

Hence the pseudo-Riemannian metric is

$$ds^2 = e^{-2c}db^2 - e^{c}\, da\, dc, \tag{24}$$

so the nonzero components of the Levi–Civita connection are

$$\Gamma^1_{22} = -2e^{-3x^3}, \qquad \Gamma^2_{23} = \Gamma^2_{32} = -\Gamma^3_{33} = -1. \tag{25}$$

Let us come back to the geodesic vectors X and Z. The 1-parameter subgroups of G generated by the vectors X and Z are the following:

$$\exp(tX) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \exp(tZ) = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

The unit matrix corresponds to the origin $p = (0,0,0)$ of the coordinate system $(x^1, x^2, x^3) = (a, b, c)$. The orbits of the subgroups (26) starting at the origin are

$$\gamma_1(t) = \exp(tX)(p) = (t, 0, 0), \qquad \gamma_2(t) = \exp(tZ)(p) = (0, 0, t). \tag{27}$$

For the tangent vectors $\gamma_1'(t) = \frac{d\gamma_1}{dt} = \frac{\partial}{\partial x^1}$ and $\gamma_2'(t) = \frac{d\gamma_2}{dt} = \frac{\partial}{\partial x^3}$ we obtain

$$\begin{aligned} \nabla_{\gamma_1'(t)}\gamma_1'(t) &= \nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^1} = \Gamma^k_{11} \frac{\partial}{\partial x^k} = 0 \\ \nabla_{\gamma_2'(t)}\gamma_2'(t) &= \nabla_{\frac{\partial}{\partial x^3}} \frac{\partial}{\partial x^3} = \Gamma^k_{33} \frac{\partial}{\partial x^k} = \frac{\partial}{\partial x^3} = \gamma_2'(t). \end{aligned} \tag{28}$$

We see that the curve γ_1 is a geodesic with the affine parameter t. Now we shall find the corresponding affine parameter for the geodesic γ_2. Let us define the curve $\gamma_3(t)$ by the formula

$$\gamma_3(t) = (0, 0, \log(t)). \tag{29}$$

For the tangent vector field we have $\gamma_3'(t) = \frac{d\gamma_3}{dt} = \frac{1}{t}\frac{\partial}{\partial x^3}$ and for the covariant derivative we obtain

$$\begin{aligned} \nabla_{\gamma_3'(t)}\gamma_3'(t) &= \nabla_{\frac{d\gamma_3}{dt}} \frac{1}{t}\frac{\partial}{\partial x^3} = \frac{d}{dt}\left(\frac{1}{t}\right)\frac{\partial}{\partial x^3} + \frac{1}{t}\nabla_{\frac{d\gamma_3}{dt}} \frac{\partial}{\partial x^3} \\ &= -\frac{1}{t^2}\frac{\partial}{\partial x^3} + \frac{1}{t}\nabla_{\frac{1}{t}\frac{\partial}{\partial x^3}} \frac{\partial}{\partial x^3} = \left(-\frac{1}{t^2} + \frac{1}{t^2}\right)\frac{\partial}{\partial x^3} = 0. \end{aligned} \tag{30}$$

We see that the curve γ_3 is the same as the curve γ_2 reparametrized by an affine parameter. It is defined on the open interval $(0, \infty)$.

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ZDENĚK DUŠEK
DEPARTMENT OF ALGEBRA AND GEOMETRY
PALACKY UNIVERSITY
TOMKOVA 40
77900 OLOMOUC
CZECH REPUBLIC

OLDŘICH KOWALSKI
FACULTY OF MATHEMATICS AND PHYSICS
CHARLES UNIVERSITY
SOKOLOVSKÁ 83
18675 PRAGUE
CZECH REPUBLIC

E-mail: kowalski@karlin.mff.cuni.cz

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