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A classification of some Finsler connections and their applications

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Abstract. Some general Finsler connections are defined. Emphasis is being made on the Cartan tensor and its derivatives. Vanishing of the hv-curvature tensors of these connections characterizes Landsbergian, Berwaldian as well as Riemannian structures. This viewpoint makes it possible to give a smart representation of connection theory in Finsler geometry and yields a classification of Finsler connections. Some practical applications of these connections are also considered.

1. Introduction

There is always a hope of finding a solution to some of the unsolved problems of Finsler geometry by developing a connection theory. This hope justifies the introduction of new connections [2]. The study of the hv-curvature of Finsler connections is by some authors thought to be rather urgent for theoretical physics, see for instance [7], [8] and [10]. Vanishing hv-curvatures of BERWALD and CARTAN connections characterize Berwaldian and Landsbergian structures respectively [4], [5]. The discovery of the SHEN connection whose hv-curvature characterizes the Riemannian structure, seems to complete their work and permits the classification of Finsler connections into three different categories [9].

In this paper, using the vanishing property of hv-curvatures, we define three general kinds of Finsler connections and extend the above property to a general family of Finsler connections. This point of view enables us to define a more

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general family of Finsler connections which contains some known Finsler connections as special cases. This characterization gives rise to the classification of some Finsler connections with respect to the Cartan tensor and its derivatives, which is a smart representation of Finsler connections (see table of Section 5). The distinguishing property of this connection is the flexibility of its reduced hv-curvature, which makes it very useful. In fact its reduced hv-curvature may be chosen to be equal to any linear differential equation formed in terms of the Cartan tensor and its derivatives. The above property makes the geometric interpretation of the solutions of these differential equations easy. As application of this connection, we consider some examples, especially those in which the flag curvature is constant.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. T_xM denotes the tangent space of M at x. The tangent bundle of M is the union of tangent spaces $TM := \cup_{x\in M} T_xM$. We will denote the elements of TM by (x,y) where $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM_0 \to M$ is given by $\pi(x,y) := x$.

A *Finsler structure* on M is a function $F : TM \to [0,\infty)$ with the following properties: (i) F is C^∞ on TM_0, (ii) F is positively 1-homogeneous on the fibers of the tangent bundle TM, and (iii) the Hessian of F^2 with elements $g_{ij}(x,y) := \frac{1}{2}[F^2(x,y)]_{y^iy^j}$ is positively defined on TM_0. The pair (M,F) is then called a *Finsler manifold.* F is Riemannian if the $g_{ij}(x,y)$ are independent of $y \neq 0$.

Let us consider the pull-back tangent bundle π^*TM over TM_0 defined by $\pi^*TM = \{(u,v) \in TM_0 \times TM_0 \mid \pi(u) = \pi(v)\}$. Take a local coordinate system (x^i) in M, then the local natural frame $\{\frac{\partial}{\partial x^i}\}$ of T_xM determines a local natural frame $\partial_i|_v$ for π_v^*TM the fibers of π^*TM, where $\partial_i|_v = \big(v, \frac{\partial}{\partial x^i}|_x\big)$, and $v = y^i\frac{\partial}{\partial x^i}|_x \in TM_0$. The fiber π_v^*TM is isomorphic to $T_{\pi(v)}M$ where $\pi(v) = x$. There is a canonical section ℓ of π^*TM defined by $\ell_v = (v,v)/F(v)$.

Let TTM be the tangent bundle of TM and ρ the canonical linear mapping $\rho : TTM_0 \to \pi^*TM$ defined by $\rho(\hat{X}) = (z, \pi_*(\hat{X}))$ where $\hat{X} \in T_zTM_0$ and $z \in TM_0$. The bundle map ρ satisfies $\rho\big(\frac{\partial}{\partial x^i}\big) = \partial_i$ and $\rho\big(\frac{\partial}{\partial y^i}\big) = 0$. Let V_zTM be the set of vertical vectors at z, that is, the set of vectors tangent to the fiber through z, or equivalently $V_zTM = ker\rho$, called the *vertical space.*

Let ∇ be a linear connection on π^*TM, that is $\nabla : T_zTM_0 \times \pi^*TM \to \pi^*TM$ such that $\nabla : (\hat{X}, Y) \to \nabla_{\hat{X}}Y$. Consider the linear mapping $\mu_z : T_zTM_0 \to T_{\pi z}M$ defined by $\mu_z(\hat{X}) = \nabla_{\hat{X}}F\ell$, where $\hat{X} \in T_zTM_0$. The connection ∇ is called a *Finsler connection* if for every $z \in TM_0$, μ_z defines an isomorphism of V_zTM_0

onto $T_{\pi z}M$. Therefore, the tangent space TTM_0 in z is decomposed as $T_zTM_0 = H_zTM \oplus V_zTM$, where $H_zTM = \ker \mu_z$ is called the *horizontal space* defined by ∇. Indeed, any tangent vector $\hat{X} \in T_zTM_0$ in z decomposes to $\hat{X} = H\hat{X} + V\hat{X}$ where $H\hat{X} \in H_zTM$ and $V\hat{X} \in V_zTM$. The structural equations of the Finsler connection ∇ are

$$\mathcal{T}_\nabla(\hat{X}, \hat{Y}) = \nabla_{\hat{X}} Y - \nabla_{\hat{Y}} X - \rho[\hat{X}, \hat{Y}], \tag{1}$$

$$\Omega(\hat{X}, \hat{Y})Z = \nabla_{\hat{X}} \nabla_{\hat{Y}} Z - \nabla_{\hat{Y}} \nabla_{\hat{X}} Z - \nabla_{[\hat{X}, \hat{Y}]} Z, \tag{2}$$

where $X = \rho(\hat{X})$, $Y = \rho(\hat{Y})$ and $Z = \rho(\hat{Z})$. The tensors $\mathcal{T}_\nabla$ and Ω are called respectively the *Torsion* and *Curvature* tensors of ∇. They determine two torsion tensors defined by $\mathcal{S}(X, Y) := \mathcal{T}_\nabla(H\hat{X}, H\hat{Y})$ and $\mathcal{T}(\dot{X}, Y) := \mathcal{T}_\nabla(V\hat{X}, H\hat{Y})$ and three curvature tensors defined by $R(X, Y) := \Omega(H\hat{X}, H\hat{Y})$, $P(X, \dot{Y}) := \Omega(H\hat{X}, V\hat{Y})$ and $Q(\dot{X}, \dot{Y}) := \Omega(V\hat{X}, V\hat{Y})$, where $\dot{X} = \mu(\hat{X})$ and $\dot{Y} = \mu(\hat{X})$.

Given a Finsler structure F on M, at each point $x \in M$, $F(v) = F\big(y^i \frac{\partial}{\partial x^i}|_x\big)$ is a function of $(y^i) \in \mathbb{R}^n$. The *fundamental tensor* g is defined by $g : \pi^*TM \otimes \pi^*TM \to [0, \infty)$ with the components $g(\partial_i|_v, \partial_j|_v) = g_{ij}(x, y)$. Thus (π^*TM, g) becomes a Riemannian vector bundle over TM_0. The *Cartan tensor* $A : \pi^*TM \otimes \pi^*TM \otimes \pi^*TM \to \mathbb{R}$ is defined by $A(\partial_i|_v, \partial_j|_v, \partial_k|_v) = A_{ijk}(x, y)$, where $A_{ijk}(x, y) = \frac{1}{2} F(x, y)[F^2(x, y)]_{y^i y^j y^k}$. If $A = 0$ then F is Riemannian.

Flag curvature. A flag curvature is a geometrical invariant that generalizes what in Riemannian geometry is called the sectional curvature. For all $x \in M$ and $0 \neq y \in T_xM$, $V := V^i \frac{\partial}{\partial x^i}$ is called the transverse edge. Flag curvature is obtained by carrying out the following computation at the point $(x, y) \in TM_0$, and viewing y and V as sections of π^*TM:

$$K(y, V) := \frac{V^i (y^j \ R_{jikl} \ y^l) V^k}{g(y, y)g(V, V) - [g(y, V)]^2}.$$

If K is independent of the transverse edge V, then (M, F) is called the *scalar flag curvature*. Denoting this scalar by $\lambda = \lambda(x, y)$, if it has no dependence on either x or y, then the Finsler manifold is said to be of *constant flag curvature.*

3. General-type Finsler connection

In this section we define a general family of Finsler connections which contains some known Finsler connections as special cases.

Definition 3.1. A tensor $S : \pi^*TM \otimes \pi^*TM \otimes \pi^*TM \to \mathbb{R}$ is called "compatible" if it has the following properties:

(1) $S(X, Y, Z)$ is symmetric with respect to X, Y, Z.

(2) $S(X, Y, \ell) = 0$.

(3) S is homogeneous, i.e., $S_{ijk}(x, ty) = S_{ijk}(x, y)$, $\forall\, t \in \mathbb{R}$, where $S_{ijk}(x, y) = S(\partial_i, \partial_j, \partial_k)$.

Definition 3.2. Consider a Finsler connection D on (M, F). Let S and T be two compatible tensors on π^*TM.

(i) The *torsion* tensor $\mathcal{T}_D$ of D, defined by (1), should satisfy

$$\mathcal{T}_D(\hat{X}, \hat{Y}) = F^{-1}T\big(\mu(\hat{X}), \rho(\hat{Y}))\big) - F^{-1}T(\mu(\hat{Y}), \rho(\hat{X})), \tag{3}$$

where $T(X, Y)$ is defined by $g(T(X, Y), Z) := T(X, Y, Z)$, $\hat{X}, \hat{Y} \in T_zTM_0$.

(ii) Let $(D_{\hat{Z}}g)(X, Y) := \hat{Z}g(X, Y) - g(D_{\hat{Z}}X, Y) - g(X, D_{\hat{Z}}Y)$. Then the connection D is called *almost-compatible* with the Finsler structure if for all $X, Y \in \pi^*TM$ and $\hat{Z} \in T_zTM_0$,

$$\begin{aligned}(D_{\hat{Z}}g)(X, Y) &= 2A(\rho(\hat{Z}), X, Y) + 2F^{-1}A(\mu(\hat{Z}), X, Y)\\ &\quad - 2S(\rho(\hat{Z}), X, Y) - 2F^{-1}T(\mu(\hat{Z}), X, Y).\end{aligned} \tag{4}$$

(iii) D is called *metric-compatible* with the Finsler structure if $(D_{\hat{Z}}g)(X, Y) = 0$.

For torsion-free connections the bundle map μ satisfies $\mu\big(\frac{\partial}{\partial y^i}\big) = \partial_i$ and $\mu\big(\frac{\partial}{\partial x^i}\big) = N_i^k\partial_k$, where $N_i^k = F\Gamma_{ij}^k\ell^j$ and Γ_{ij}^k are Christoffel symbols of the torsion-free Finsler connection D.

We have the following general theorem of existence and uniqueness of linear connections in different versions.

Theorem A ([9])**.** *Let (M, F) be a Finsler manifold. Suppose S and T are two compatible tensors in π^*TM. Then there exists a unique almost-compatible linear connection D with torsion $\mathcal{T}_D$ on π^*TM satisfying* (i) *and* (ii).

Let $\bar{\ell}$ denote the unique vector field in HTM such that $\rho(\bar{\ell}) = \ell$. We define $\dot{A}, \ldots, \overset{m+1}{A}$ from $\pi^*TM \otimes \pi^*TM \otimes \pi^*TM$ to $\mathbb{R}$ as follows:

$$\dot{A}(X, Y, Z) := \bar{\ell}A(X, Y, Z) - A(D_{\bar{\ell}}X, Y, Z) - A(X, D_{\bar{\ell}}Y, Z) - A(X, Y, D_{\bar{\ell}}Z),$$

$$\begin{aligned}\overset{m+1}{A}(X, Y, Z) &:= \bar{\ell}\,\overset{m}{A}(X, Y, Z) - \overset{m}{A}(D_{\bar{\ell}}X, Y, Z) - \overset{m}{A}(X, D_{\bar{\ell}}Y, Z)\\ &\quad - \overset{m}{A}(X, Y, D_{\bar{\ell}}Z),\end{aligned} \tag{5}$$

where $\overset{0}{A} := A$, $\overset{1}{A} := \dot{A}$, $\overset{2}{A} := \ddot{A}$, ... and $m \in \mathbb{N}$. Obviously, $\forall\, m \in \mathbb{N}$, the tensors $\overset{m}{A}$ are symmetric with respect to X, Y and Z. Moreover, using $D_{\bar{\ell}}\, \ell = 0$ we have $\overset{m}{A}(X, Y, \ell) = 0$. A Finsler metric is called a *Berwald metric* if for any standard local coordinate system (x^i, y^i) in TM_0, the Christoffel symbols $\Gamma^k_{ij} = \Gamma^k_{ij}(x)$ are functions of $x \in M$ alone. A Finsler metric is called a *Landsberg metric* if $\dot{A} = 0$.

By means of Theorem A, we can define the general Finsler connection.

Definition 3.3. Let (M, F) be a Finsler manifold. A *general-type Finsler connection* is defined as a Finsler connection D on π^*TM such that its compatible tensors S and T can be defined as follows:

$$S := \kappa_0 A + \kappa_1 \dot{A} + \kappa_2 \ddot{A} + \cdots + \kappa_m \overset{m}{A} \quad \text{and} \quad T := rA, \tag{6}$$

where the coefficients κ_i, $i = 1, \ldots, m$ and r are real constants.

4. Curvature tensors

Let D be a Finsler connection defined on M. Let $\{e_i\}_{i=1}^n$ be a local (with respect to g) orthonormal frame field for the vector bundle π^*TM such that $g(e_i, e_n) = 0$, $i = 1, \ldots, n-1$ and $e_n = \ell^i \frac{\partial}{\partial x^i}$. Let $\{\omega^i\}_{i=1}^n$ be its dual co-frame field. One readily finds that $\omega^n := \frac{\partial F}{\partial y^i} dx^i = \omega$, which is called a *Hilbert form*, and $\omega(\ell) = 1$. Let $\rho = \omega^i \otimes e_i$, $De_i = \omega_i{}^j \otimes e_j$ and $\Omega e_i = 2\Omega_i{}^j \otimes e_j$, where $\{\Omega_i{}^j\}$ and $\{\omega_i{}^j\}$ are called, respectively, the *curvature forms* and *connection forms* of D with respect to $\{e_i\}$. We have $\mu := DF\ell = F\{\omega_n{}^i + d(\log F)\delta^i_n\} \otimes e_i$. Put $\omega^{n+i} := \omega_n{}^i + d(\log F)\delta^i_n$. It is easy to show that $\{\omega^i, \omega^{n+i}\}_{i=1}^n$ is a local basis for $T^*(TM_0)$. The equation (2) is equivalent to

$$d\omega_i{}^j - \omega_i{}^k \wedge \omega_k{}^j = \Omega_i{}^j. \tag{7}$$

Since the $\Omega_j{}^i$ are 2-forms on TM_0, they can be expanded as

$$\Omega_i{}^j = \frac{1}{2} R_i{}^j{}_{kl} \omega^k \wedge \omega^l + P_i{}^j{}_{kl} \omega^k \wedge \omega^{n+l} + \frac{1}{2} Q_i{}^j{}_{kl} \omega^{n+k} \wedge \omega^{n+l}. \tag{8}$$

Let $\{\bar{e}_i, \dot{e}_i\}_{i=1}^n$ be the local basis for $T(TM_0)$, which is dual to $\{\omega^i, \omega^{n+i}\}_{i=1}^n$, i.e., $\bar{e}_i \in HTM, \dot{e}_i \in VTM$ such that $\rho(\bar{e}_i) = e_i, \mu(\dot{e}_i) = Fe_i$. The objects R, P and Q are called, respectively, the hh-, hv- and vv-curvature tensors of the connection D with the components $R(\bar{e}_k, \bar{e}_l)e_i = R_i{}^j{}_{kl} e_j$, $P(\bar{e}_k, \dot{e}_l)e_i = P_i{}^j{}_{kl} e_j$

and $Q(\dot{e}_k, \dot{e}_l)e_i = Q_i{}^j{}_{kl}e_j$. From (8) we see that $R_i{}^j{}_{kl} = -R_i{}^j{}_{kl}$ and $Q_i{}^j{}_{kl} = -Q_i{}^j{}_{kl}$. Let us put

$$dg_{ij} - g_{kj}\omega_i{}^k - g_{ik}\omega_j{}^k g_{ij|k}\omega^k + g_{ij.k}\omega^{n+k}, \tag{9}$$

$$dA_{ijk} - A_{ljk}\omega_i{}^l - A_{ilk}\omega_j{}^l - A_{ijl}\omega_k{}^l = A_{ijk|l}\omega^l + A_{ijk.l}\omega^{n+l}, \tag{10}$$

where the slash " | " and point "." are horizontal and vertical covariant derivatives with respect to the Finsler connection. In a similar way, for $\forall\, m \in \mathbb{N}$ we have

$$d\overset{m}{A}_{ijk} - \overset{m}{A}_{ljk}\omega_i{}^l - \overset{m}{A}_{ilk}\omega_j{}^l - \overset{m}{A}_{ijl}\omega_k{}^l = \overset{m}{A}_{ijk|l}\omega^l + \overset{m}{A}_{ijk.l}\omega^{n+l}, \tag{11}$$

where $\overset{m}{A}_{ijk} = \overset{m}{A}(e_i, e_j, e_k)$ and $\overset{m}{A}{}^k{}_{ij} = g^{kl}\overset{m}{A}_{ijk}$. From (10) and (11) we see that $A_{ijk|l}$, $A_{ijk.l}$, $\overset{m}{A}_{ijk|l}$ and $\overset{m}{A}_{ijk.l}$, $(\forall\, m, l \in \mathbb{N})$ are all symmetric with respect to i, j and k. By definition of the Landsberg tensor, we have $A_{ijk|n} = \dot{A}_{ijk}$. Here we use the notation $\overset{m}{A}_{ijk|n} = \overset{m}{A}_{ijk|s}\ell^s$ and $\overset{m}{A}_{ijk|n} = \overset{m+1}{A}_{ijk}$. From (10) and (11) we get

$$A_{njk|l} = 0, \quad A_{njk.l} = -A_{jkl}, \quad \overset{m}{A}_{njk|l} = 0 \quad \text{and} \quad \overset{m}{A}_{njk.l} = -\overset{m}{A}_{jkl}. \tag{12}$$

Remark 4.1. In general-type connection, the horizontal and vertical covariant derivatives of the metric tensor are given by

$$g_{ij|k} = 2\big((1-\kappa_0)A_{ijk} - \kappa_1\dot{A}_{ijk} + \cdots + \kappa_m\overset{m}{A}_{ijk}\big) \quad \text{and} \quad g_{ij.k} = 2(1-r)A_{ijk}.$$

5. A classification of some Finsler connections

The following results due to Berwald, Cartan and Shen determine the relation between hv-curvature and special Finsler spaces. These results enable us to classify some non-Riemannian Finsler connections and distinguish three different categories.

Theorem B ([4], [6]). *Let (M, F) be a Finsler manifold. Then for the Berwald connection (or Chern connection), hv-curvature vanishes if and only if F is a Berwald metric.*

Theorem C ([5]). *Let (M, F) be a Finsler manifold. Then for the Cartan connection (or Hashiguchi connection), hv-curvature vanishes if and only if F is a Landsberg metric.*

Theorem D ([9]). *Let (M, F) be a Finsler manifold. Then for the Shen connection hv-curvature vanishes if and only if F is Riemannian.*

The remarkable property of Shen connection, proved by Theorem D, comes from the fact that vanishing of its hv-curvature singles out Riemannian metric. In contrast, Cartan, Berwald, Chern and Hashiguchi connections do not possess this property. Thus we have three different types of Finsler connections. Theorems 5.1, 5.2 and 5.3 of this paper deal with a more general case and give rise to new families of Finsler connections that we call Berwald-type, Cartan-type and Shen-type connections and which are defined according to the behavior of their hv-curvature.

Definition 5.1. Let (M, F) be a Finsler manifold. A Finsler connection is called of *Berwald-type* (resp. *of Cartan-type* or *Shen-type*) if and only if vanishing of its hv-curvature reduces the Finsler structure to the Berwaldian (resp. Landsbergian or Riemannian) one.

From this viewpoint one can compare some of the non-Riemannian Finsler connections according to the compatibility of the tensors S and T.

A classification of Finsler connections according to their compatible tensors S and T

	Compatible tensors			
Connection	**S**	**T**	**Metric compatibility**	**Torsion**
1. Berwald	$A + \overset{\bullet}{A}$	0	almost compatible	free
2. Chern–Rund	A	0	almost compatible	free
3. Berwald-type	$A + \kappa_1 \overset{\bullet}{A} + \cdots + \kappa_m \overset{m}{A}$	0	almost compatible	free
4. Cartan	A	A	metric compatible	not free
5. Hashiguchi	$A + \overset{\bullet}{A}$	A	almost compatible	not free
6. Cartan-type	$A + \kappa_1 \overset{\bullet}{A} + \cdots + \kappa_m \overset{m}{A}$	A	depends on κ_i	not free
7. Shen	0	0	almost compatible	free
8. Shen-type	$\kappa_1 \overset{\bullet}{A} + \cdots + \kappa_m \overset{m}{A}$	0	almost compatible	free
9. General-type	$\kappa_0 A + \kappa_1 \overset{\bullet}{A} + \cdots + \kappa_m \overset{m}{A}$	rA	depends on κ_i and r	depends on r

In this table A, $\dot{A}$, $\ddot{A}$,..., $\overset{m}{A}$ are Cartan tensors and their covariant derivatives, κ_i and r are arbitrary real constants. The connections 1, 2 and 3 belong to the Berwald-type category. The connections 4, 5 and 6 are Cartan-type connections. The connections 7 and 8 belong to the Shen-type Category. The connection 9 contains all other connections. From at the freeness of torsion point of view, the Shen connection is the one most similar to the Levi–Civita connection. But from

the metric compatibility viewpoint, it is the Cartan connection which is closest to the Levi–Civita connection.

Now we extend Theorem C to Cartan-type connections and show that the hv-curvature tensor of this type of connections characterizes Landsbergian structures.

Theorem 5.1. *Let (M, F) be a Finsler manifold. Then for Cartan-type connections hv-curvature vanishes if and only if F is Landsbergian.*

To prove Theorem 5.1, we need the following

Lemma 5.1. *Let (M, F) be a Finsler manifold. Then for Cartan-type connections we have*

1) $R_j{}^i{}_{kl} + R_k{}^i{}_{lj} + R_l{}^i{}_{jk} C^i_{jm} R_n{}^m{}_{kl} + C^i_{km} R_n{}^m{}_{lj} + C^i_{lm} R_n{}^m{}_{jk}$,
2) $P_j{}^i{}_{kl} = P_k{}^i{}_{jl} + C^i_{kl|j} - C^i_{jl|k} + C^i_{jr} P_n{}^r{}_{kl} - C^i_{kr} P_n{}^r{}_{jl}$,
3) $Q_j{}^i{}_{kl} = Q_j{}^i{}_{lk} + 2(C^i_{jk.l} - C^i_{jl.k}) + 2(C^i_{mk} C^m_{jl} - C^i_{ml} C^m_{jk}) + C^i_{jm}(Q_n{}^m{}_{kl} - Q_n{}^m{}_k)$,

where $R_{ijkl} = g_{sj} R_i{}^s{}_{kl}$, $P_{ijkl} = g_{sj} P_i{}^s{}_{kl}$, $Q_{ijkl} = g_{sj} Q_i{}^s{}_{kl}$, $C^i_{jk} = F^{-1} g^{im} A_{mjk}$.

Proof. Let us consider the Cartan-type connection with compatible tensors $S = A + \kappa_1 \dot{A} + \cdots + \kappa_m \overset{m}{A}$ and $T = A$. By (3) and (4), there exist connection 1-forms $\{\omega^i_j\}$ satisfying the following torsion and almost compatibility conditions:

$$d\omega^i = \omega^j \wedge \omega_j{}^i - C^i_{kl}\ \omega^k \wedge \omega^{n+l}, \tag{13}$$

$$dg_{ij} = g_{kj}\omega_i{}^k + g_{ik}\omega_j{}^k - 2\kappa_1 \dot{A}_{ijk}\omega^k - \cdots - 2\kappa_m \overset{m}{A}_{ijk}\omega^k. \tag{14}$$

Differentiating (13) and using (7) and (10), we get:

$$\omega^j \wedge \Omega^i_j = (C^i_{kl|j}\omega^j + C^i_{kl.j}\omega^{n+j}) \wedge \omega^k \wedge \omega^{n+l} - C^i_{lm} C^m_{jk} \omega^j \wedge \omega^{n+k} \wedge \omega^{n+l} - C^i_{kl} \omega^k \wedge \Omega^l_n.$$

Replacing Ω^i_j by (8), we prove the Lemma. □

Proof of Theorem 5.1. Let (M, F) be a Finsler manifold with Cartan-type connection and compatible tensors $S = A + \tilde{S}$ and $T = A$, where $\tilde{S} = \kappa_1 \dot{A} + \cdots + \kappa_m \overset{m}{A}$. Then the almost compatibility condition (14) becomes

$$dg_{ij} = g_{kj}\omega_i{}^k + g_{ik}\omega_j{}^k - 2\widetilde{S}_{ijk}\omega^k. \tag{15}$$

Differentiating this relation leads to

$$g_{kj}\Omega_i{}^k + g_{ik}\Omega_j{}^k = 2(\widetilde{S}_{ijk|s}\omega^s + \widetilde{S}_{ijk.s}\omega^{n+s}) \wedge \omega^k + 2C^i_{uv}\widetilde{S}_{ijk}\omega^u \wedge \omega^{n+v}.$$

From this relation and (8) we have

$$R_{ijkl} + R_{jikl} = 2(\widetilde{S}_{ijl|k} - \widetilde{S}_{ijk|l}), \tag{16}$$

$$P_{ijkl} + P_{jikl} = -2(\widetilde{S}_{ijk.l} - C^u_{kl}\widetilde{S}_{uij}), \tag{17}$$

$$Q_{ijkl} + Q_{jikl} = 0. \tag{18}$$

Permuting i, j and k in (17) and using Lemma 1 yields

$$\begin{aligned} P_{ijkl} = &-\widetilde{S}_{ijk.l} + (C^u_{kl}\widetilde{S}_{uij} + C^u_{il}\widetilde{S}_{ujk} - C^u_{jl}\widetilde{S}_{uki}) + (C_{kjl|i} - C_{ijl|j}) \\ &+ (C_{kiv}P^v_{njl} - C_{jkv}P^v_{nil}). \end{aligned} \tag{19}$$

Multiplying this relation by y^i and replacing $\widetilde{S} = \kappa_1\dot{A} + \cdots + \kappa_m \overset{m}{A}$, we get

$$P_{njkl} = \dot{C}_{ijk} + \{\kappa_1\dot{A}_{ijk} + \cdots + \kappa_m \overset{m}{A}_{ijk}\}. \tag{20}$$

If F is a Landsbergian manifold, then from the above relation we have $P_{njkl} = 0$. Therefore by replacing this value in (19) we find $P_{ijkl} = C_{kjl|i} - C_{kil|j}$. In the case of Landsbergian manifolds, $C_{ijk|l}$ is totally symmetric in all of its four indices and we have $P_{ijkl} = 0$. Conversely, let the hv-curvature be zero. Then by Lemma 1 we have $C^i_{kl|j} = C^i_{jl|k}$, therefore M is Landsbergian. □

Theorem 5.2. *Let (M, F) be a Finsler manifold. Then for Berwald-type connections the hv-curvature vanishes if and only if F is a Berwaldian metric.*

Proof. The complete proof of this theorem will not be given, only a sketch of the proof will be presented. For a Berwald-type connection, the hv-curvature is

$$\begin{aligned} P_{ijkl} = &-\{\kappa_1\dot{A}_{ijk.l} + \cdots + \kappa_m \overset{m}{A}_{ijk.l}\} - (A_{ijl|k} + A_{jkl|i} - A_{kil|j}) \\ &+ A_{kis}P_n{}^s{}_{jl} - A_{jks}P_n{}^s{}_{il} - A_{ijs}P_n{}^s{}_{kl}. \end{aligned} \tag{21}$$

Therefore, we have

$$P_{njkl} = \{\kappa_1\dot{A}_{jkl} + \cdots + \kappa_m \overset{m}{A}_{jkl}\} - \dot{A}_{jkl}. \tag{22}$$

Using these relations, the theorem will follow. □

Theorem 5.3. *Let (M, F) be a Finsler manifold. Then for Shen-type connections, the hv-curvature vanishes if and only if F is Riemannian.*

Proof. The proof of this theorem is analogous to that of Theorem 5.1 and is not presented here. □

Theorem 5.4. *Let (M, F) be a Finsler manifold. Then the hv-curvature of general-type (respectively Berwald-type, Cartan-type or Shen-type) connections vanishes if and only if F is Berwaldian, Landsbergian or Riemannian.*

6. Some applications of general-type connections

Much of the practical importance of this kind of connections results from the fact that it is adaptable, in the sense that it is useful for getting a geometric interpretation for a given system of differential equations formed by the Cartan tensor and its derivatives. Suppose that we are given a differential equation of this kind and we want to find a geometric meaning for its solutions. It would suffice to consider a Finsler connection – by fixing the compatible tensors S and T – for which the reduced hv-curvature coincides with the differential equation in question. We then apply one of the Theorems 5.1, 5.2 or 5.3 as applicable.

6.1. Application of Shen-type connections. Here we define Shen-type connection D as $S_{ijk} = (1-k)A_{ijk} + k\dot{A}_{ijk} - \ddot{A}_{ijk}$ and $T_{ijk} = 0$ for which the reduced hv-curvature $P_{jkl} := \ell^i\ P_{ijkl}$ is equal to the given differential equation $P_{jkl} = \ddot{A}_{jkl} + kA_{jkl}$.

Theorem 6.1. *Let (M,F) be a Finsler manifold with constant flag curvature λ such that $P_{jkl} = 0$. Then F is Riemannian.*

Proof. Let us consider the Shen-type connection D with $S_{ijk}=(1-k)A_{ijk}+k\dot{A}_{ijk} - \ddot{A}_{ijk}$, $k \neq \lambda$ and $T = 0$. Replacing S and T in (4) and by an argument similar to the one used in the proof of Theorem 1, we get

$$P_{ijkl} + P_{jikl} = -2\{\dot{A}_{ijk.l} + \ddot{A}_{ijk.l}\} - 2k\{A_{ijk.l} - A_{ijl|k}\} - 2A_{ijm}P_n{}^m{}_{kl}. \quad (23)$$

From (23) we have

$$\begin{aligned} P_{ijkl} = &-\{\dot{A}_{ijk.l} + \ddot{A}_{ijk.l}\} - kA_{ijk.l} + k\{-A_{ijl|k} + A_{jkl|i} - A_{kil|j}\} \\ &- 2A_{ijm}P_n{}^m{}_{kl} + 2A_{kim}P_n{}^m{}_{jl} - 2A_{jkm}P_{nil}^{\ \ m}. \end{aligned} \quad (24)$$

Therefore $P_{jkl} = \ddot{A}_{jkl} + kA_{jkl}$. The equation $P_{jkl} = 0$ holds, from which we have

$$\ddot{A} + kA = 0. \quad (25)$$

Since (M, F) is a Finsler manifold with constant flag curvature λ,

$$\ddot{A} + \lambda A = 0. \quad (26)$$

From (25) and (26) one has $(\lambda - k)A = 0$, which means that F is a Riemannian metric. □

Using the above special Shen-type connection again together with a hypothesis on the topology of M, we have the following

Theorem 6.2. *Let (M,F) be a complete Finsler manifold with bounded Cartan tensor. Then (M, F) is a Riemannian manifold if and only if $P_{jkl} = 0$.*

PROOF. Let us consider the above Shen-type connection on the complete Finsler manifold (M, F). Then from the last theorem we have that the hv-curvature of this connection reduces to $P_{jkl} = \ddot{A}_{jkl} + kA_{jkl}$. Fix any $X, Y, Z \in \pi^*TM$ at $v \in I_xM = \{w \in T_xM,\ F(w) = 1\}$. Let $c : \mathbb{R} \to M$ be the unit speed geodesic in (M, F) with $\frac{dc}{dt}(0) = v$ and $\hat{c} := \frac{dc}{dt}$ be the canonical lift of c to TM_0. Let $X(t), Y(t)$ and $Z(t)$ denote the parallel sections along $\hat{c}$ with $X(0) = X$, $Y(0) = Y$ and $Z(0) = Z$. Put $A(t) = A(X(t), Y(t), Z(t))$, $\dot{A}(t) = \dot{A}(X(t), Y(t), Z(t))$ and $\ddot{A}(t) = \ddot{A}(X(t), Y(t), Z(t))$. Now along geodesics we have $\frac{d\dot{A}}{dt} = \ddot{A}$ and from $\ddot{A}_{jkl} + kA_{jkl} = 0$ we get

$$A(t) = \big(c_1 \sinh \sqrt{k}t + c_2 \cosh \sqrt{k}t\big) A(0). \tag{27}$$

For $v \in TM_0$, let us define $\|A\|_v := \sup A(X, Y, Z)$ where the supremum is taken over all unit vectors of π_v^*TM. Let us put $\|A\| = \sup_{v \in IM} \|A\|_v$ where $IM = \bigcup_{x \in M} I_xM$. Since M is complete and $\|A\| < \infty$, by letting $t \to +\infty$ and $t \to -\infty$, we have $c_1 = 0$ and $c_2 = 0$. Therefore $A = 0$, and F is Riemannian. □

6.2. Application of Berwald-type connections. Here we consider a special Berwald-type connection for which the hv-curvature is equal to the given differential equation.

Theorem 6.3. *Let (M, F) be a complete Finsler manifold with bounded Landsberg tensor. Then F is a Landsberg metric if and only if $P_{jkl} = 0$.*

PROOF. If we put $\kappa_1 = \kappa_3 = \cdots = \kappa_m = 0$ and $\kappa_2 \neq 0$ in (22), then we find a special Berwald-type connection for which the hv-curvature is equal to $P_{jkl} = \kappa_2 \ddot{A}_{jkl} - \dot{A}_{jkl}$. Let F be a Landsberg metric, then from the above equation we get $P_{jkl} = 0$. Conversely, if $P_{jkl} = 0$ we will have

$$\kappa_2 \ddot{A}_{jkl} - \dot{A}_{jkl} = 0. \tag{28}$$

By an argument like the one presented in the proof of the last theorem, we have along the geodesics

$$\dot{A}(t) = e^{k_2 t} \dot{A}(0). \tag{29}$$

For $v \in TM_0$, let us define $\|\dot{A}\|_v := \sup \dot{A}(X, Y, Z)$ and $\|\dot{A}\| = \sup_{v \in IM} \|\dot{A}\|_v$. Using completeness of M, $\|\dot{A}\| < \infty$ and letting $t \to +\infty$ we have $\dot{A}(0) = \dot{A}(X, Y, Z) = 0$. From (29) we get $\dot{A} = 0$, that is, F is a Landsberg metric. □

Corollary 6.1. *Every compact Finsler manifold is Landsbergian if and only if P_{jkl} vanishes.*

Next we consider another special Berwald-type connection and give a proof of the following well-known result due to Akbar-Zadeh [1].

Corollary 6.2. *Let (M, F) be a complete Finsler manifold with negative constant flag curvature λ and bounded Cartan tensor. Then F is Riemannian.*

Proof. Let us put $\kappa_2 = \kappa_4 = \cdots = \kappa_m = 0$, $\kappa_1 = 2$ and $\kappa_3 = \frac{1}{\lambda} \neq 0$ in (21). We obtain a connection for which the hv-curvature becomes

$$P_{ijkl} = -\left\{2\dot{A}_{ijk.l} + \frac{1}{\lambda}\dddot{A}_{ijk.l}\right\} - (A_{ijl|k} + A_{jkl|i} - A_{kil|j}) + A_{kis}P_n{}^s{}_{jl} - A_{jks}P_n{}^s{}_{il} - A_{ijs}P_{nkl}^{\ s}. \tag{30}$$

From this $P_{njkl} = \frac{1}{\lambda}\,\dddot{A}_{jkl} + \dot{A}_{jkl}$. As M has constant flag curvature we have $\ddot{A} + \lambda A = 0$. So by the same argument as in the above theorem we find

$$A(t) = \left(c_1 + c_2 e^{\sqrt{-\lambda}t} + c_3 e^{-\sqrt{-\lambda}t}\right)A(0). \tag{31}$$

Using the boundary assumption on the Cartan tensor and letting $t \to \infty$ and $t \to -\infty$, we get $c_2 = c_3 = 0$. Therefore $A = c_1$ and $\dot{A} = 0$. It is easy to see that $A = 0$. □

7. Relation between some connections

There is a well-known result which can be used as a definition for Landsberg spaces, see for example [3].

Theorem E. *Let (M, F) be a Finsler manifold. Then M is a Landsberg manifold if and only if the Berwald connection coincides with the Chern connection.*

In this context we prove the following

Theorem 7.1. *Let (M, F) be a complete Finsler manifold with bounded Cartan tensor. Then M is a Riemannian manifold if and only if the Berwald connection coincides with the Shen connection.*

PROOF. Simple calculation shows that ${}^b\Gamma^i_{jk} = {}^s\Gamma^i_{jk} + A^i_{jk} + \dot{A}^i_{jk}$, where ${}^b\Gamma^i_{jk}$ and ${}^s\Gamma^i_{jk}$ are the Christoffel coefficients of the Berwald and Shen connections respectively. If ${}^b\Gamma^i_{jk} = {}^s\Gamma^i_{jk}$, then $A^i_{jk} + \dot{A}^i_{jk} = 0$. By the same argument as in the above theorems, we find $A + \dot{A} = 0$ whose solution is $A(t) = e^{-t}A(0)$. Completeness of M and the bounded Cartan tensor hypothesis imply that A=0. □

Lemma 7.1. *The Christoffel symbols for Berwald-type, Cartan-type and Shen-type connections denoted by ${}^B\Gamma$, ${}^C\Gamma$ and ${}^S\Gamma$ respectively, are given by:*

$$
\begin{aligned}
{}^B\Gamma^i_{jk} &= \frac{g^{is}}{2}\left\{\frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j}\right\} + A^i{}_{jk} + \big(\kappa_1 \dot{A}^i{}_{jk} + \cdots + \kappa_m \overset{m}{A}{}^i{}_{jk}\big),\\
{}^C\Gamma^i_{jk} &= \frac{g^{is}}{2}\left\{\frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j}\right\} + C^i_{js}N^s_k + \big(\kappa_1 \dot{A}^i{}_{jk} + \cdots + \kappa_m \overset{m}{A}{}^i{}_{jk}\big),\\
{}^S\Gamma^i_{jk} &= \frac{g^{is}}{2}\left\{\frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j}\right\} + \big(\kappa_1 \dot{A}^i{}_{jk} + \cdots + \kappa_m \overset{m}{A}{}^i{}_{jk}\big),
\end{aligned}
$$

where $\frac{\delta}{\delta x^j} := \frac{\partial}{\partial x^j} - N^i_j \frac{\partial}{\partial y^i}$.

PROOF. We prove this lemma for Cartan-type connections only. In local coordinates (x^i, y^i) for TM_0, we write $D_{\frac{\partial}{\partial x^i}}\partial_j = {}^C\Gamma^k_{ij}\partial_k$ and $D_{\frac{\partial}{\partial y^i}}\partial_j = F^k_{ij}\partial_k$. Put $N^k_i = {}^C\Gamma^k_{ij}y^j = F\{\gamma^k_{ij}\ell^j - A^k_{il}\gamma^l_{ab}\ell^a\ell^b\}$ where $\gamma^k_{ij} = \frac{1}{2}g^{kl}\{\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\}$. For Cartan-type connections we consider the compatible tensors S and T defined by $S = A + \tilde{S}$ and $T = A$, where $\tilde{S} = \kappa_1\dot{A} + \cdots + \kappa_m \overset{m}{A}$. From (3) and (4) we have

$$
{}^C\Gamma^k_{ij} = {}^C\Gamma^k_{ji} + N^s_i C^k_{sj} - N^s_j C^k_{si}, \tag{32}
$$

$$
F^k_{ij} = C^k_{ij} + y^l F^s_{jl} C^k_{si}, \tag{33}
$$

$$
\frac{\partial}{\partial x^k}(g_{ij}) = g_{li}\,{}^C\Gamma^l_{kj} - g_{lj}\,{}^C\Gamma^l{}_{ki} + 2\tilde{S}_{ijk}, \tag{34}
$$

$$
\frac{\partial}{\partial y^k}(g_{ij}) = g_{jl}F^l_{ik} - g_{li}F^l{}_{kj}. \tag{35}
$$

Permuting i, j and k in (34) and using (32), one obtains

$$
{}^C\Gamma^k_{ij} = \gamma^k_{ij} + N^s_j C^k_{is} - g^{km}N^s_m C_{ijs} + \tilde{S}^k_{ij}. \tag{36}
$$

Since $\frac{g^{is}}{2}\{\frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j}\} = \gamma^i_{jk} - g^{im}N^s_m C_{jks}$, we get the desired Christoffel symbols. For other connections the same method can be used. □

Corollary 7.1. *Let (M, F) be a Finsler manifold. The Berwald-type connection coincides with the Shen-type connection if and only if F is Riemannian.*

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