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# Invariance of weighted quasi–arithmetic means with continuous generators

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**Abstract.** Let $I \subset \mathbb{R}$ be an open interval and $p, q, r \in (0,1)$. We find all continuous and strictly monotonic functions $\alpha, \beta, \gamma : I \to \mathbb{R}$ satisfying the functional equation

$$\lambda\alpha(\beta^{-1}(\mu\beta(x) + (1-\mu)\beta(y))) + (1-\lambda)\alpha(\gamma^{-1}(\nu\gamma(x) + (1-\nu)\gamma(y)))$$
$$= \lambda\alpha(x) + (1-\lambda)\alpha(y)$$

generalizing the Matkowski–Sutô equation. In the proof we adopt a method elaborated by Z. Daróczy and Zs. Páles when solving the Matkowski–Sutô equation, some results of A. Járai on improving regularity of solutions and an extension theorem by Z. Daróczy and G. Hajdu. We also use a theorem giving the form of all twice continuously differentiable solutions of the above equation proved jointly with J. Matkowski.

## 1. Introduction

One of the most important classes of means consists of weighted quasi–arithmetic ones, that is means of the form

$$A_\lambda^{[\alpha]}(x,y) := \alpha^{-1}(\lambda\alpha(x) + (1-\lambda)\alpha(y)),$$

where $\alpha$ is a continuous strictly monotonic function defined on a real interval and $\lambda \in (0,1)$. In this paper we give a complete solution of the following invariance problem: find all means $A_\lambda^{[\alpha]}$, $A_\mu^{[\beta]}$, $A_\nu^{[\gamma]}$ such that

$$A_\lambda^{[\alpha]} \circ \left(A_\mu^{[\beta]}, A_\nu^{[\gamma]}\right) = A_\lambda^{[\alpha]}, \tag{1}$$

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that is $A_\lambda^{[\alpha]}$ is invariant with respect to the pair $\left(A_\mu^{[\beta]}, A_\nu^{[\gamma]}\right)$. We consider this problem assuming that $\alpha, \beta, \gamma$ are continuous functions defined on a real interval. In the class $C^2$ of twice continuously differentiable functions it was solved jointly with J. Matkowski in [10]. A special case of (1), namely

$$A_{1/2}^{[\alpha]} \circ \left(A_{1/2}^{[\beta]}, A_{1/2}^{[\gamma]}\right) = A_{1/2}^{[\alpha]},$$

with $\alpha$ being the identity function, was solved by O. Sutô [15] in the class of analytic generators and then by J. Matkowski [12] in the class $C^2$. All continuously differentiable solutions were found by Z. Daróczy and Zs. Páles in [4]. In the proof of Theorem 1 below we adopt a method elaborated by them in [5], [6]. In [5] they presented a complete solution of the Matkowski–Sutô problem with continuous generators. The same method was applied by them in [6] to determine all solutions of (1) in the class of continuous generators in the case $\lambda = \mu = \nu$.

The following result is the main one of the paper and plays a fundamental role in solving our problem.

**Theorem 1.** *Let $I \subset \mathbb{R}$ be an open interval. Continuous strictly monotonic functions $\varphi, \psi : I \to \mathbb{R}$ and numbers $\lambda, \mu, \nu \in (0,1)$ satisfy*

$$\lambda\varphi^{-1}(\mu\varphi(x) + (1-\mu)\varphi(y)) + (1-\lambda)\psi^{-1}(\nu\psi(x) + (1-\nu)\psi(y)) \\ = \lambda x + (1-\lambda)y \quad (2)$$

*for all $x, y \in I$ if and only if the following two conditions are fulfilled:*

(i) $\lambda = \dfrac{\nu}{1-\mu+\nu}$,

(ii) *there exist $a, c \in \mathbb{R} \setminus \{0\}$ and $b, d \in \mathbb{R}$ such that*

$$\varphi(x) = ax + b \quad \text{and} \quad \psi(x) = cx + d, \qquad x \in I,$$

*or $\lambda = 1/2$ and*

$$\varphi(x) = ae^{px} + b \quad \text{and} \quad \psi(x) = ce^{-px} + d, \qquad x \in I,$$

*with some $p \in \mathbb{R} \setminus \{0\}$.*

Given an interval $I$, functions $f, g : I \to \mathbb{R}$ and an interval $J \subset I$ we say that $f$ and $g$ are equivalent on $J$, in notation $f \sim g$ on $J$, if there are $a, b \in \mathbb{R}$, $a \neq 0$, such that

$$g(x) = af(x) + b, \qquad x \in J.$$

*Remark 1.* Using the relation $\sim$ and defining $\chi_p : I \to \mathbb{R}$ by

$$\chi_p(x) = \begin{cases} x, & \text{if } \ x \in I, \quad p = 0, \\ e^{px}, & \text{if } \ x \in I, \quad p \neq 0, \end{cases}$$

for every $p \in \mathbb{R}$, the assertion (ii) of Theorem 1 can be rewritten as follows:

(ii') $$\varphi \sim \chi_0 \quad \text{and} \quad \psi \sim \chi_0 \quad \text{on } I$$

or $\lambda = 1/2$ and

$$\varphi \sim \chi_p \quad \text{and} \quad \psi \sim \chi_{-p} \quad \text{on } I$$

with some $p \in \mathbb{R} \setminus \{0\}$.

In what follows $\lambda$, $\mu$, $\nu$ denote fixed numbers from $(0, 1)$.

We start with a number of useful lemmas. The first one is a particular case of some results of Zs. Páles [14] (cf. Corollary 6 and Example 2).

**Lemma 1.** *Let $J \subset \mathbb{R}$ be an open interval and $f : J \to \mathbb{R}$ be a strictly increasing function such that*

$$J \ni s \mapsto f(s) - f(\mu s + (1-\mu)t)$$

*strictly increases for every $t \in J$. Then for every $s_0 \in J$ there exist numbers $\delta \in (0, \infty)$ and $K, L \in (0, \infty)$ such that $(s_0 - \delta, s_0 + \delta) \subset J$ and*

$$K \leq \frac{f(s) - f(t)}{s - t} \leq L$$

*for every $s, t \in (s_0 - \delta, s_0 + \delta)$, $s \neq t$.*

In the case $J = \mathbb{R}$ the next lemma follows directly from [1; Sec. 2.2.6, Theorem 1], (cf. also [11; Chapter XII, Sec. 10, Lemma 1]). Nevertheless, for the convenience of the reader we give an immediate argument.

**Lemma 2.** *Let $J \subset \mathbb{R}$ be an interval and let $\vartheta \in \mathbb{R} \setminus \{0, 1\}$. If $f : J \to \mathbb{R}$ satisfies*

$$f(\mu s + (1-\mu)t) = \vartheta f(s) + (1-\vartheta) f(t) \tag{3}$$

*for all $s, t \in J$, then there exist an additive function $a : \mathbb{R} \to \mathbb{R}$ and a real $b \in \mathbb{R}$ such that*

$$f(s) = a(s) + b, \qquad s \in J.$$

Proof. Applying the Daróczy–Páles identity (see [3])

$$\mu\left((1-\mu)\frac{s+t}{2}+\mu s\right)+(1-\mu)\left(\mu\frac{s+t}{2}+(1-\mu)t\right)=\mu s+(1-\mu)t$$

and (3) we obtain

$$\vartheta f(s)+(1-\vartheta)f(t)=\vartheta^2 f(s)+2\vartheta(1-\vartheta)f\left(\frac{s+t}{2}\right)+(1-\vartheta)^2 f(t)$$

for every $s,t\in J$ and, consequently,

$$\frac{f(s)+f(t)}{2}=f\left(\frac{s+t}{2}\right),\qquad s,t\in J,$$

whence the assertion follows (cf., for instance, [11; Chapter XIII, Sec. 2, Theorem 1], also [1; Sec. 2.1.4]). □

*Remark 2.* We can also argue in a different way. Namely, it follows from the general extension theorem of Zs. Páles [13; Theorem 5] that any solution of (3) admits an extension from $J$ to $\mathbb{R}$ satisfying (3) for all $s,t\in\mathbb{R}$. Using the Daróczy–Páles identity we see that this extension satisfies Jensen's equation on $\mathbb{R}$ and, consequently, is affine as stated in Lemma 2.

Making use of some results of A. Járai from the monograph [9] and reasoning similarly as Z. Daróczy and Zs. Páles in [5] we prove the following

**Lemma 3.** *Let $J\subset\mathbb{R}$ be an open interval and $f,g:J\to(0,\infty)$ satisfy*

$$\begin{aligned}f(\mu s+(1-\mu)t)&[(1-\nu)g(t)-(1-\mu)g(s)]\\&=\mu(1-\nu)f(s)g(t)-(1-\mu)\nu f(t)g(s)\end{aligned}\tag{4}$$

*for all $s,t\in J$. If $f$ is Lebesgue measurable and $g$ is of the first Baire class then $f$ and $g$ are continuous on a nonempty subinterval of $J$.*

Proof. If $f$ is constant on an interval contained in $J$ then, by (4), we infer that $g$ is also constant on the same subinterval.

Now assume that $g$ is constant on an interval $J_0\subset J$ . Then equation (4) can be rewritten in the form

$$f(\mu s+(1-\mu)t)(\mu-\nu)=\mu(1-\nu)f(s)-(1-\mu)\nu f(t),\qquad s,t\in J_0.\tag{5}$$

If $\mu = \nu$ then, by (5), $f$ is constant on $J_0$. If $\mu \neq \nu$ then (5) is equivalent to the condition

$$f(\mu s + (1-\mu)t) = \frac{\mu(1-\nu)}{\mu-\nu} f(s) - \frac{(1-\mu)\nu}{\mu-\nu} f(t), \qquad s,t \in J_0.$$

Let $\vartheta := \frac{\mu(1-\nu)}{\mu-\nu}$. Then

$$f(\mu s + (1-\mu)t) = \vartheta f(s) + (1-\vartheta) f(t), \qquad s,t \in J_0.$$

Applying Lemma 2 we obtain the existence of an additive function $a : \mathbb{R} \to \mathbb{R}$ and a number $b \in \mathbb{R}$ such that

$$f(s) = a(s) + b, \qquad s \in J_0.$$

Since $f$ is positive $a$ is bounded below, whence there exists a $c \in \mathbb{R}$ such that

$$f(s) = cs + b, \qquad s \in J_0,$$

and, consequently, $f$ is continuous on $J_0$. (In fact, inserting the form of $f$ into (5) one can observe that $f$ is constant on $J_0$.)

Now assume that neither $f$, nor $g$ is constant on a subinterval of $J$ and let

$$C(g) := \{v \in J : g \text{ is continuous at } v\}.$$

Since $g$ is of the first Baire class $C(g)$ is a dense $G_\delta$ subset of $J$. We will show that there exist $s_0, t_0 \in C(g)$, $s_0 \neq t_0$, such that

$$(1-\mu)g(s_0) \neq (1-\nu)g(t_0). \tag{6}$$

Suppose that

$$(1-\mu)g(s) = (1-\nu)g(t)$$

for all different $s,t \in C(g)$. If $v \in C(g)$ then $g$ is constant on $C(g) \setminus \{v\}$. The set $C(g)$ being uncountable, contains two different points. Consequently, $g$ is constant on $C(g)$, i.e. there exists a $k \in \mathbb{R}$ such that

$$g(t) = k \quad \text{for every} \quad t \in C(g). \tag{7}$$

Therefore $\mu = \nu$ and equation (4) can be rewritten in the form

$$f(\mu s + (1-\mu)t)(g(t) - g(s)) = \mu(f(s)g(t) - f(t)g(s)). \tag{8}$$

Thus, by (7),

$$\mu k(f(s) - f(t)) = 0, \qquad s, t \in C(g),$$

whence $f$ is constant on $C(g)$, i.e. there exists an $l \in \mathbb{R}$ such that $f(t) = l$ for every $t \in C(g)$.

If there existed an $s_0 \in J$ such that $\mu s_0 + (1-\mu)t \in J \setminus C(g)$ for every $t \in C(g)$ then $C(g)$ would be homeomorphic with a subset of $J \setminus C(g)$. This, however, is impossible since $C(g)$ is a dense $G_\delta$ subset of $J$ and, consequently, $J \setminus C(g)$ is of the first Baire category. Therefore for every $s \in J$ there exists a $t \in C(g)$ such that $\mu s + (1-\mu)t \in C(g)$. Now if $s \in J$ and $t \in C(g)$ are such that $\mu s + (1-\mu)t \in C(g)$ then, by (8), we have

$$l[k - g(s)] = \mu[kf(s) - lg(s)].$$

Hence

$$f(s) = \frac{kl - l(1-\mu)g(s)}{k\mu}, \qquad s \in J.$$

Using again (8) we obtain

$$\begin{aligned} &\frac{kl - l(1-\mu)g(\mu s + (1-\mu)t)}{k\mu}[g(t) - g(s)] \\ &\qquad = \mu\left(\frac{kl - l(1-\mu)g(s)}{k\mu}g(t) - \frac{kl - l(1-\mu)g(t)}{k\mu}g(s)\right), \quad s, t \in J, \end{aligned}$$

and, consequently,

$$[g(t) - g(s)][k - g(\mu s + (1-\mu)t)] = 0, \quad s, t \in J. \tag{9}$$

Since $g$ is not constant on $J$ there exists a $v_0 \in J$ such that $m := g(v_0) \neq k$. Take arbitrary $v \in J$ and $\varepsilon > 0$ with $(v - \varepsilon, v + \varepsilon) \subset J$. Since $g$ is not constant on intervals there exists an $s \in (v - \varepsilon, v + \varepsilon)$ such that

$$g(\mu s + (1-\mu)v_0) \neq k.$$

By (9) we have $g(s) = g(v_0) = m$. Therefore in every neighbourhood of $v$ there exist an $s$ with $g(s) = m$ and, since $C(g)$ is dense in $J$, a point $u$ such that $g(u) = k \neq m$. Thus $g$ is not continuous at $v$ and, consequently, $C(g) = \emptyset$ which is impossible. This proves the existence of different $s_0, t_0 \in C(g)$ satisfying (6).

According to (6) there exist open intervals $U$, $V$ containing $s_0$, $t_0$, respectively, and such that for every $s \in U$ and $t \in V$ we have $(1-\mu)g(s) \neq (1-\nu)g(t)$. Making use of (4) we obtain

$$f(\mu s + (1-\mu)t) = \frac{\mu(1-\nu)f(s)g(t) - \nu(1-\mu)f(t)g(s)}{(1-\nu)g(t) - (1-\mu)g(s)}, \qquad s \in U,\ t \in V.$$

Now we are going to apply [9; Theorem 8.6] by A. JÁRAI. To this aim put $n = 4$, $T := J$, $Z = Z_1 = \ldots = Z_4 = Y := \mathbb{R}$, $X_1 = X_3 = A_1 = A_3 := U$ and $X_2 = X_4 = A_2 = A_4 := V$. Fix an $\eta > 0$ with $(t_0 - \eta, t_0 + \eta) \subset V$ and define

$$D := \Big\{(v, y) \subset J \times U : |v - (\mu s_0 + (1-\mu)t_0)| < \frac{\eta}{2}(1-\mu) \\ \text{and } |y - s_0| < \left(\frac{1}{\mu} - 1\right)\frac{\eta}{2}\Big\}$$

and

$$W := \left\{(v, y, z_1, z_2, z_3, z_4) \in D \times \mathbb{R}^4 : (1-\nu)z_4 \neq (1-\mu)z_3\right\}.$$

Put also $f := f$, $f_1 := f|_U$, $f_2 := f|_V$, $f_3 := g|_U$, $f_4 := g|_V$ and define $g_1, g_3 : D \to U$, $g_2, g_4 : D \to V$ by

$$g_1(v, y) = g_3(v, y) = y, \qquad g_2(v, y) = g_4(v, y) = \frac{v - \mu y}{1-\mu},$$

and $h : W \to \mathbb{R}$ by

$$h(v, y, z_1, z_2, z_3, z_4) = \frac{\mu(1-\nu)z_1 z_4 - \nu(1-\mu)z_2 z_3}{(1-\nu)z_4 - (1-\mu)z_3}.$$

Put $K := [s_0 - \delta, s_0 + \delta]$, where $0 < \delta < \left(\frac{1}{\mu} - 1\right)\eta/2$ and $[s_0 - \delta, s_0 + \delta] \subset U$. Making use of [9; Theorem 8.6] applied to the Lebesgue measure we infer that $f$ is continuous on the interval

$$J_f := \Big\{v \in J : |v - (\mu s_0 + (1-\mu)t_0)| < (1-\mu)\frac{\eta}{2}\Big\}.$$

Fix an $s^* \in J_f$. Since $f$ is not constant on intervals there is a $t^* \in J_f$ such that $f(\mu s^* + (1-\mu)t^*) \neq \mu f(s^*)$. By the continuity of $f$ at $t^*$ we have $f(\mu s^* + (1-\mu)t) \neq \mu f(s^*)$ for $t$'s from an interval $J_g \subset J_f$. Then, by (4),

$$g(t) = \frac{1-\mu}{1-\nu}\,\frac{f(\mu s^* + (1-\mu)t) - \nu f(t)}{f(\mu s^* + (1-\mu)t) - \mu f(s^*)}\,g(s^*), \qquad t \in J_g,$$

and, consequently, $g$ is continuous on $J_g$. □

The next result is fundamental in determining the form of $f$ and $g$ and, consequently, $\varphi$ and $\psi$. To prove it we use another theorem of Járai as well as a result proved jointly with J. MATKOWSKI in [10].

**Lemma 4.** *Let $J \subset \mathbb{R}$ be an open interval. If continuous $f, g : J \to \mathbb{R}$ satisfy equation* (4) *then there exist a $c \in (0, \infty)$ such that*

$$f(s)^{\mu} g(s)^{1-\nu} = c, \qquad s \in J.$$

Proof. We consider two cases. In the first one assume that $\mu \neq \nu$. Then equation (4) can be rewritten in the form

$$(1-\mu)g(s)[f(\mu s + (1-\mu)t) - \nu f(t)] = (1-\nu)g(t)[f(\mu s + (1-\mu)t) - \mu f(s)]$$

for every $s, t \in J$. Interchanging $s$ by $t$ here we obtain

$$(1-\mu)g(t)[f(\mu t + (1-\mu)s) - \nu f(s)] = (1-\nu)g(s)[f(\mu t + (1-\mu)s) - \mu f(t)]$$

for every $s, t \in J$. Multiplying these equalities by sides we have

$$\begin{aligned}&(1-\mu)^2 g(s)g(t)[f(\mu s + (1-\mu)t) - \nu f(t)][f(\mu t + (1-\mu)s) - \nu f(s)]\\ &\quad = (1-\nu)^2 g(t)g(s)[f(\mu s + (1-\mu)t) - \mu f(s)][f(\mu t + (1-\mu)s) - \mu f(t)]\end{aligned}$$

for every $s, t \in J$, whence dividing it by positive $g(s)$, $g(t)$ we get

$$\begin{aligned}&(1-\mu)^2[f(\mu s + (1-\mu)t) - \nu f(t)][f(\mu t + (1-\mu)s) - \nu f(s)]\\ &\qquad = (1-\nu)^2[f(\mu s + (1-\mu)t) - \mu f(s)][f(\mu t + (1-\mu)s) - \mu f(t)] \end{aligned} \tag{10}$$

for every $s, t \in J$. Put

$$k(s,t) := \nu(1-\mu)^2[f(\mu s + (1-\mu)t) - \nu f(t)] - \mu(1-\nu)^2[f(\mu t + (1-\mu)s) - \mu f(t)]$$

for every $s, t \in J$. Fix an $s_0 \in J$. Then

$$\begin{aligned}k(s_0, s_0) &= \nu(1-\mu)^2[f(s_0) - \nu f(s_0)] - \mu(1-\nu)^2[f(s_0) - \mu f(s_0)]\\ &= (1-\mu)(1-\nu)(\nu-\mu)f(s_0).\end{aligned}$$

Since $f(s_0) > 0$, $\mu \neq 1$, $\nu \neq 1$, $\mu \neq \nu$ we have $k(s_0, s_0) \neq 0$. Thus there exists an $\varepsilon > 0$ such that $k(s,t) \neq 0$ for all $s, t \in (s_0 - \varepsilon, s_0 + \varepsilon)$. Let $J_0 := (s_0 - \varepsilon, s_0 + \varepsilon)$. By (10)

$$\begin{aligned}f(s) = &\frac{(1-\mu)^2 f(\mu t + (1-\mu)s)[f(\mu s + (1-\mu)t) - \nu f(t)]}{k(s,t)}\\ &- \frac{(1-\nu)^2 f(st + (1-\mu)t)[f(\mu t + (1-\mu)s) - \mu f(t)]}{k(s,t)}\end{aligned}$$

for every $s, t \in J_0$.

Put $s = k = 1$, $n = 3$, $Z := \mathbb{R}$, $T := J_0$, $Y := \mathbb{R}$, $D := {J_0}^2$, $C := [s_0 - \vartheta_0\varepsilon, s_0 + \vartheta_0\varepsilon]$ with $\vartheta_0 := \max\{\mu, 1-\mu\}$,

$$W := D \times \{(w_1, w_2, w_3) \in \mathbb{R}^3 : \nu(1-\mu)^2[\nu w_1 - w_2] \\ \neq \mu(1-\nu)^2[\mu w_1 - w_3]\}.$$

Define $f := f|_{J_0}$, $g_1, g_2, g_3 : D \to \mathbb{R}$, by

$$g_1(s,t) = t, \quad g_2(s,t) = \mu s + (1-\mu)t, \quad g_3(s,t) = \mu t + (1-\mu)s, \tag{11}$$

and $h : W \to \mathbb{R}$ by

$$h(s,t,w_1,w_2,w_3) := \frac{(1-\mu)^2 w_3[w_2 - \nu w_1] - (1-\nu)^2 w_2[w_3 - \mu w_1]}{\nu(1-\mu)^2[w_2 - \nu w_1] - \mu(1-\nu)^2[w_3 - \mu w_1]}. \tag{12}$$

Then, according to [9; Theorem 11.6] by A. Járai, $f$ is locally Lipschitzian on $J_0$. On account of [8; Theorem 3.1.9] $f$ is almost everywhere (with respect to the Lebesgue measure) differentiable on $J_0$.

Now let $s = k = 1$, $n = 3$, $Z_1 = Z_2 = Z_3 = Z := \mathbb{R}$, $Y = T = X_1 = X_2 = X_3 := J_0$, $D := {J_0}^2$, $D \times \mathbb{R}^3$ and take $r_1 = r_2 = r_3 = 1$. Define $f = f_1 = f_2 = f_3 := f|_{J_0}$, $g_1, g_2, g_3 : D \to \mathbb{R}$ by (12) and $h : D \times \mathbb{R}^3 \to \mathbb{R}$ by (13). According to [9; Theorem 14.2] $f$ is continuously differentiable on $J_0$. Since $s_0$ was chosen arbitrarily in $J$ we have come to the differentiability of $f$ in $J$. Due to [10; Lemma 3] there exists a $c \in (0,\infty)$ such that $f(s)^\mu g(s)^{1-\nu} = c$ for every $s \in J$.

If $\mu = \nu$ then equation (4) can be rewritten in the form

$$f(\mu s + (1-\mu)t)[g(t) - g(s)] = \mu[f(s)g(t) - f(t)g(s)], \qquad s, t \in J.$$

Now it is enough to use [7; Theorem 2]. $\square$

Now we will prove an extension lemma.

**Lemma 5.** *If $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotonic functions satisfying (2) and there exists a $p \in \mathbb{R}$ such that $\varphi \sim \chi_p$ and $\psi \sim \chi_{-p}$ on an open subinterval of $I$ then $\varphi \sim \chi_p$ and $\psi \sim \chi_{-p}$ on $I$.*

Proof. Assume that $\varphi \sim \chi_p$ and $\psi \sim \chi_{-p}$ on an open subinterval $I_0 \subset I$. Replacing the satisfying (2) $\varphi$ and $\psi$ by $a\varphi + b$ and $c\psi + d$ with some $a, b, c, d \in \mathbb{R}$, $a, c \neq 0$, we may additionally assume that

$$\varphi(x) = \chi_p(x) \quad \text{and} \quad \psi(x) = \chi_{-p}(x), \qquad x \in I_0, \tag{13}$$

and $I_0$ is the maximal interval with this property. Let $I_0 = (a, b)$ and suppose that $I_0 \neq I$. Then $a \in I$ or $b \in I$. Consider the case when $a \in I$. Choose a $b^* \in (a, b)$. By the continuity and strictly monotonicity of $\varphi$ and $\psi$ there exists a positive $\delta$ such that $(a - \delta, a] \subset I$, $(b^* - \delta, b^*] \subset I_0$ and

$$\mu\varphi(x) + (1-\mu)\varphi(y) \in \varphi(I_0) \quad \text{and} \quad \nu\psi(x) + (1-\nu)\psi(y) \in \varphi(I_0) \tag{14}$$

for all $x \in (a - \delta, a]$ and $y \in (b^* - \delta, b^*]$.

Assume that $p \neq 0$. Then, inserting the form of $\varphi$ and $\psi$ into (2), we infer that

$$(e^{-px})^{\lambda}(\mu e^{px} + (1-\mu)e^{py})^{\lambda} = (e^{py})^{1-\lambda}(\nu e^{-px} + (1-\nu)e^{-py})^{1-\lambda}$$

for all $x, y \in I_0$, whence, by putting $z := e^{p(y-x)}$, we obtain

$$((1-\mu)z + \mu)^{\lambda} = (\nu z + (1-\nu))^{1-\lambda}$$

for $z$'s from an interval of positive reals. This implies $\lambda = 1 - \lambda$, i.e. $\lambda = 1/2$ and, consequently, $1 - \mu = \nu$. Moreover,

$$\varphi^{-1}(v) = \frac{1}{p}\log v, \qquad v \in \varphi(I_0), \tag{15}$$

and

$$\psi^{-1}(v) = -\frac{1}{p}\log v, \qquad v \in \psi(I_0). \tag{16}$$

Making use of (14)–(17) and (2) we have

$$\frac{1}{p}\log\frac{(1-\nu)\varphi(x) + \nu e^{py}}{\nu\psi(x) + (1-\nu)e^{-py}} = x + y$$

for all $x \in (a - \delta, a]$, $y \in (b^* - \delta, b^*]$. Hence

$$\nu e^{py}(1 - e^{px}\psi(x)) = (1-\nu)(e^{px} - \varphi(x)), \qquad x \in (a-\delta, a],\ y \in (b^* - \delta, b^*]$$

and, consequently, $\psi(x) = e^{-px}$ and $\varphi(x) = e^{px}$ for every $x \in (a - \delta, a]$ which contradicts the maximality of $I_0$.

Now assume that $p = 0$. Then $\psi(x) = \varphi(x) = x$ for all $x \in I_0$. It follows from (2) that

$$\lambda(\mu\varphi(x) + (1-\mu)y) + (1-\lambda)(\nu\psi(x) + (1-\nu)y) = \lambda x + (1-\lambda)y \tag{17}$$

for every $x \in (a-\delta, a]$ and $y \in (b^*-\delta, b^*]$ whence

$$\lambda\mu\varphi(x) + (1-\lambda)\nu\psi(x) = \lambda x + y(\nu(1-\lambda) - \lambda(1-\mu))$$

for every $x \in (a-\delta, a]$ and $y \in (b^*-\delta, b^*]$ and, consequently, $\lambda = \frac{\nu}{1-\mu+\nu}$. By (18) we obtain

$$\mu\varphi(x) + (1-\mu)\psi(x) = x, \qquad x \in (a-\delta, a].$$

On the other hand, taking in (2) $y \in (a-\delta, a]$ and $x \in (b^*-\delta, b^*]$, we get

$$\nu\varphi(y) + (1-\mu)\psi(y) = y, \qquad y \in (a-\delta, a].$$

Then, for every $x \in (a-\delta, a]$, we have

$$\mu\varphi(x) + (1-\mu)\psi(x) = x \quad \text{and} \quad \nu\varphi(x) + (1-\nu)\psi(x) = x,$$

and, consequently,

$$(\nu-\mu)(\varphi(x) - \psi(x)) = 0, \qquad x \in (a-\delta, a].$$

If $\mu = \nu$ then it is enough to use [2; Theorem 3] by Z. Daróczy and G. Hajdu. Otherwise $\varphi(x) = \psi(x) = x$ for every $x \in (a-\delta, a]$ which contradicts the maximality of $I_0$. $\square$

Proof of Theorem 1. Let $J := \varphi(I)$. Without loss of generality we may assume that $\varphi$, $\psi$ are strictly increasing.

At first we will show that $\varphi, \varphi^{-1}, \psi, \psi^{-1}$ are locally Lipschitzian and their derivatives do not vanish wherever they exist. Putting $s = \varphi(x)$ and $t = \varphi(y)$ in (2) we get

$$\begin{aligned}(1-\lambda)\psi^{-1}\left(\nu\psi(\varphi^{-1}(s)) + (1-\nu)\psi\left(\varphi^{-1}(t)\right)\right)\\ = \lambda\varphi^{-1}(s) + (1-\lambda)\varphi^{-1}(t) - \lambda\varphi^{-1}(\mu s + (1-\mu)t)\end{aligned}$$

for every $s, t \in J$. Since the left-hand side increases as a function of $s$ so does the right-hand side. Hence

$$J \ni s \mapsto \varphi^{-1}(s) - \varphi^{-1}(\mu s + (1-\mu)t)$$

strictly increases for every $t \in J$. For all $v_0 \in J$, by Lemma 1, we can find $\delta \in (0, \infty)$ and $K, L \in (0, \infty)$ such that $(v_0 - \delta, v_0 + \delta) \subset J$ and

$$K \le \frac{\varphi^{-1}(u) - \varphi^{-1}(v)}{u - v} \le L, \qquad u, v \in (v_0 - \delta, v_0 + \delta).$$

Then also for every $x_0 \in I$ there exist $\delta > 0$ and $K, L > 0$ such that

$$\frac{1}{L} \le \frac{\varphi(x)-\varphi(y)}{x-y} \le \frac{1}{K}, \qquad x, y \in (x_0-\delta, x_0+\delta).$$

Interchanging $\varphi$ and $\psi$ here we obtain the analogous conditions for $\psi$ and $\psi^{-1}$. In particular, it follows that if the function $\varphi$ [function $\psi$] is differentiable at a point $x_0 \in I$ then $\varphi'(x_0) \ne 0$ [$\psi'(x_0) \ne 0$] and if the function $\varphi^{-1}$ [function $\psi^{-1}$] is differentiable at $v_0 \in \varphi(I)$ [$v_0 \in \psi(I)$] then $(\varphi^{-1})'(v_0) \ne 0$ [$(\psi^{-1})'(v_0) \ne 0$].

Now we will show that $\varphi$, $\psi$ are differentiable on some nonempty open subinterval $I_1 \subset I$. For every $v \in J$ put

$$J(v) := \frac{1}{1-\mu}(J-v) \cap \frac{1}{\mu}(v-J);$$

observe that $J(v)$ is an open interval containing 0. Let

$$\begin{aligned} G := \{v \in J : \lambda\varphi^{-1}(v+(1-\mu)u) + (1-\lambda)\varphi^{-1}(v-\mu u) \ne \varphi^{-1}(v) \\ \text{for a } u \in J(v)\}. \end{aligned}$$

Now we consider two cases. In the first one assume that $G = \emptyset$. Then

$$\lambda\varphi^{-1}(v+(1-\mu)u) + (1-\lambda)\varphi^{-1}(v-\mu u) = \varphi^{-1}(v), \qquad v \in J,\ u \in J(v),$$

whence, putting $s := v + (1-\mu)u$ and $t := v - \mu u$, we get

$$\lambda\varphi^{-1}(s) + (1-\lambda)\varphi^{-1}(t) = \varphi^{-1}(\mu s + (1-\mu)t), \qquad s, t \in J.$$

On account of Lemma 2 there exist an additive function $a : \mathbb{R} \to \mathbb{R}$ and a $b \in \mathbb{R}$ such that

$$\varphi^{-1}(s) = a(s) + b, \qquad s \in J.$$

Since $\varphi^{-1}$ is monotonic then a is linear and, consequently, $\varphi^{-1}$ is differentiable on $J$.

In the second case assume that $G \ne \emptyset$. According to the continuity of $\varphi$ the set $G$ is open so it contains a nonempty open interval $J_1$. Putting $x := \varphi^{-1}(v+(1-\mu)u)$ and $y := \varphi^{-1}(v-\mu u)$ in (2) we get

$$\begin{aligned} \lambda\varphi^{-1}(v) = \lambda\varphi^{-1}(v+(1-\mu)u) + (1-\lambda)\varphi^{-1}(v-\mu u) \\ - (1-\lambda)\psi^{-1}(\nu h(v+(1-\mu)u) + (1-\nu)h(v-\mu u)) \end{aligned} \tag{18}$$

for every $v \in J$ and $u \in J(v)$, where $h := \psi \circ \varphi^{-1}$. Fix a $v_0 \in J_1$ and define functions $g_i : J(v_0) \to \mathbb{R}$, $i \in \{1, 2, 3, 4\}$, by

$$g_1(u) = \varphi^{-1}(v_0 + (1-\mu)u), \qquad g_2(u) = \varphi^{-1}(v_0 - \mu u),$$
$$g_3(u) = h(v_0 + (1-\mu)u), \qquad g_4(u) = h(v_0 - \mu u).$$

Let

$$N_{g_i} := \{u \in J(v_0) : g_i \text{ is not differentiable at } u\}, \quad i = 1, \dots, 4.$$

By the monotonicity of $g_i$ the sets $N_{g_i}$ are of measure 0 for $i = 1, \dots, 4$ and, consequently, the measure of $N := \bigcup_{i=1}^{4} N_{g_i}$ is 0.

According to (19) the following equalities are equivalent:

$$\lambda\varphi^{-1}(v + (1-\mu)u) + (1-\lambda)\varphi^{-1}(v - \mu u) = \varphi^{-1}(v), \qquad v \in J,\ u \in J(v),$$

and

$$\nu h(v + (1-\mu)u) + (1-\nu)h(v - \mu u) = h(v), \qquad v \in J,\ u \in J(v).$$

Therefore the function $h_{v_0} : J(v_0) \to \mathbb{R}$, given by

$$h_{v_0}(u) = \nu h(v_0 + (1-\mu)u) + (1-\nu)h(v_0 - \mu u),$$

takes a different value from $h(v_0)$. Since $h_{v_0}(0) = h(v_0)$ then it is not constant.

Let $K := h_{v_0}(J(v_0))$ and $C := \{s \in K : \psi^{-1} \text{ is not differentiable at } s\}$. Then $K$ is a nonempty interval and $C$ is of measure 0, whence $K \setminus C$ has a positive measure. Let $D := {h_{v_0}}^{-1}(K \setminus C) \subset J(v_0)$. Then $h_{v_0}(D) = K \setminus C$. If $D$ were of measure 0 then, since $h$ is locally Lipschitzian, $h_{v_0}(D)$ would be of measure 0. Therefore $D$ has a positive measure and so is $D \setminus N$; in particular, it is nonempty, i.e. there exists a $u_0 \in D \setminus N$. Then $g_i$, $i \in \{1, \dots, 4\}$, are differentiable at $u_0$ and $\psi^{-1}$ is differentiable at $h_{v_0}(u_0)$. Consequently, $\varphi^{-1}$ is differentiable at $v_0 + (1-\mu)u_0$ and $v_0 - \mu u_0$, so, on account of (19), also at $v_0$. Thus we have proved that $\varphi^{-1}$ is differentiable in $J_1$.

Since the derivative of $\varphi^{-1}$ does not vanish, $\varphi$ is differentiable on a subinterval of $\varphi^{-1}(J_1)$. Now, considering equation (2) on this interval and interchanging the role of $\varphi$ and $\psi$, we infer that $\varphi$ and $\psi$ are differentiable on an interval $I_1 \subset \varphi^{-1}(J_1)$.

Define functions $f, g : \varphi(I_1) \to (0, \infty)$ by

$$f(s) = \varphi'(\varphi^{-1}(s)), \qquad g(s) = \psi'(\varphi^{-1}(s)).$$

We will show that there exist a nonempty open interval $J_0 \subset \varphi(I_1)$ and a number $c \in (0, \infty)$ such that $f, g$ are continuous on $J_0$ and

$$f(s)^{\mu} g(s)^{1-\nu} = c, \qquad s \in J_0.$$

Differentiating both sides of equality (2) with respect to $x$ we get

$$\frac{\lambda\mu\varphi'(x)}{\varphi'(\varphi^{-1}(\mu\varphi(x)+(1-\mu)\varphi(y)))}+\frac{(1-\lambda)\nu\psi'(x)}{\psi'(\psi^{-1}(\nu\psi(x)+(1-\nu)\psi(y)))}=\lambda \tag{19}$$

for all $x, y \in I_1$. Putting $y = x$ in (20) we have

$$\lambda = \frac{\nu}{1-\mu+\nu}.$$

Differentiating equality (2) with respect to $y$ we have

$$\frac{\lambda(1-\mu)\varphi'(y)}{\varphi'(\varphi^{-1}(\mu\varphi(x)+(1-\mu)\varphi(y)))}+\frac{(1-\lambda)(1-\nu)\psi'(y)}{\psi'(\psi^{-1}(\nu\psi(x)+(1-\nu)\psi(y)))}=1-\lambda \tag{20}$$

for all $x, y \in I_1$. Multiplying equality (19) by $(1-\nu)\psi'(y)$ and (21) by $-\nu\psi'(x)$, adding the obtained equalities by sides and then using the fact that $\lambda = \frac{\nu}{1-\mu+\nu}$ we have

$$\frac{\mu(1-\nu)\varphi'(x)\psi'(y)-\nu(1-\mu)\varphi'(y)\psi'(x)}{\varphi'(\varphi^{-1}(\mu\varphi(x)+(1-\mu)\varphi(y)))}=(1-\nu)\psi'(y)-(1-\mu)\psi'(x)$$

for all $x, y \in I_1$, whence, setting here $x = \varphi^{-1}(s)$ and $y = \psi^{-1}(t)$, we obtain

$$\begin{aligned} f(\mu s+(1-\mu)t)&[(1-\nu)g(t)-(1-\mu)g(s)] \\ &= \mu(1-\nu)f(s)g(t)-\nu(1-\mu)f(t)g(s) \end{aligned} \tag{21}$$

for every $s, t \in \varphi(I_1)$. Since $\varphi$ is locally Lipschitzian and $\varphi'$ is measurable $\varphi' \circ \varphi^{-1}$ is Lebesgue measurable. Moreover, $\psi'$ is of the first Baire class and $\varphi^{-1}$ is continuous whence $\psi' \circ \varphi^{-1}$ is of the first Baire class. Therefore, due to Lemma 3, we infer that $f, g$ are continuous in an open interval $J_0 \subset \varphi(I_1)$. According to Lemma 4 there exists a $c > 0$ such that

$$f(s)^{\mu}g(s)^{1-\nu}=c, \qquad s \in J_0. \tag{22}$$

Now we will show that there exist an open interval $I_0 \subset I$ and a $p \in \mathbb{R}$ such that $\varphi \sim \chi_p$ and $\psi \sim \chi_{-p}$ on $I_0$, and if $p \neq 0$ then $\lambda = 1/2$ and $\mu + \nu = 1$. Using (23) we can rewrite (22) as

$$\begin{aligned} f(\mu s+(1-\mu)t)&\left[(1-\nu)f(t)^{\frac{-\mu}{1-\nu}}-(1-\mu)f(s)^{\frac{-\mu}{1-\nu}}\right] \\ &= \mu(1-\nu)f(s)f(t)^{\frac{-\mu}{1-\nu}}-\nu(1-\mu)f(t)f(s)^{\frac{-\mu}{1-\nu}}, \quad s, t \in J_0. \end{aligned}$$

By virtue of [10; Lemma 4] we infer that either $f$ is constant, or $\mu+\nu=1$ and there exists $p\in\mathbb{R}\setminus\{0\}$ and $b\in\mathbb{R}$ such that

$$f(s)=p(s-b),\qquad s\in J_0.$$

If $f$ is constant then, by (22), $g$ is constant and, consequently, $\varphi$ and $\psi$ are affine. In the second case, since $\mu+\nu=1$ we get $\lambda=1/2$ and

$$\varphi'(x)=p(\varphi(x)-b),\qquad x\in I_0,$$

where $I_0=\varphi^{-1}(J_0)$. Thus

$$\varphi(x)=ae^{px}+b,\qquad x\in I_0,$$

for an $a\in\mathbb{R}\setminus\{0\}$. Similarly, we infer that

$$\psi(x)=ce^{-px}+d,\qquad x\in I_0,$$

with some $c\in\mathbb{R}\setminus\{0\}$ and $d\in\mathbb{R}$. Hence $\varphi(x)\sim\chi_p(x)$ and $\psi(x)\sim\chi_{-p}(x)$, $x\in I_0$. According to Lemma 5 either $\varphi\sim\chi_0$ and $\psi\sim\chi_0$ on $I$, or $\lambda=1/2$, $\mu+\nu=1$ and there exists a $p\in\mathbb{R}\setminus\{0\}$ such that $\varphi\sim\chi_p$ and $\psi\sim\chi_{-p}$ on $I$.

The converse implication can be easily verified. □

The next result is an immediate consequence of Theorem 1.

**Theorem 2.** *Let $I\subset\mathbb{R}$ be an open interval. Continuous and strictly monotonic functions $\alpha,\beta,\gamma:I\to\mathbb{R}$ and numbers $\lambda,\mu,\nu\in(0,1)$ satisfy* (1) *if and only if the following two conditions are fulfilled:*

(i) $\lambda=\dfrac{\nu}{1-\mu+\nu}$,

(ii) *there exist $a,c\in\mathbb{R}\setminus\{0\}$ and $b,d\in\mathbb{R}$ such that*

$$\beta(x)=a\alpha(x)+b\quad\text{and}\quad\gamma(x)=c\alpha(x)+d,\quad x\in I,$$

*or $\lambda=1/2$ and*

$$\beta(x)=ae^{p\alpha(x)}+b\quad\text{and}\quad\gamma(x)=ce^{-p\alpha(x)}+d,\quad x\in I,$$

*with some $p\in\mathbb{R}\setminus\{0\}$.*

*Remark 3.* The statement (ii) of Theorem 2 can be reformulated in the following way:

(ii') $$\beta\sim\alpha\quad\text{and}\quad\gamma\sim\alpha\text{ on }I$$

or $\lambda=1/2$ and

$$\beta\sim\chi_p\circ\alpha\quad\text{and}\quad\gamma\sim\chi_{-p}\circ\alpha\text{ on }I$$

with some $p\in\mathbb{R}\setminus\{0\}$.

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