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On some special projectively flat (α,β)-metrics

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Abstract. In this paper, we find equations that characterize locally projectively flat Finsler metrics in the form $F=\epsilon\beta+\alpha+\frac{3}{2}\beta\arctan(\beta/\alpha)+\frac{\alpha\beta^2}{2(\alpha^2+\beta^2)}$, where $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta=b_iy^i$ is a 1-form. Then we completely determine the local structure of those with constant flag curvature.

1. Introduction

It is HILBERT's Fourth Problem to characterize the (not-necessarily-reversible) distance functions on an open subset in R^n such that straight lines are geodesics [3]. Regular distance functions with straight geodesics are projectively flat Finsler metrics. It is well-known that every projectively flat metric is of scalar flag curvature, namely, the flag curvature $K(P,y)=K(x,y)$ is independent of the section P containing $y\in T_x\mathcal{U}$ (see [4]). Thus projectively flat metrics become more important and interesting.

In Finsler geometry, (α,β)-metrics are very interesting metrics, which are expressed in the following form,

$$F=\alpha\phi(s),\quad s=\frac{\beta}{\alpha},$$

where $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form with $\|\beta_x\|<b_0$ for $x\in M$, and $\phi=\phi(s)$ is a C^∞ positive function on an open interval $(-b_0,b_0)$ satisfying

$$\phi(0)=1,\quad \phi(s)>0,\quad \phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,$$

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where s and b are arbitrary numbers with $|s| \le b < b_0$. For some functions ϕ, the defined metric $F = \alpha\phi(\beta/\alpha)$ is projectively flat if and only if α is projectively flat and β is parallel with respect to α. For example, the Matsumoto metric defined by $\phi = 1/(1-s)$ and its approximation metrics have this property [1]. For some functions ϕ, the 1-form of projectively flat metric $F = \alpha\phi(\beta/\alpha)$ is not necessarily parallel. The simplest one is Randers metric $F = \alpha + \beta$ defined by $\phi = 1 + s$ (see [7]). Another interesting metric is $F = \frac{(\alpha+\beta)^2}{\alpha}$ defined by $\phi = (1+s)^2$ (see [9]). For the existence of these nontrivial metrics, the Finsler geometry becomes more colorful. In this paper, we consider a special (α, β)-metric

$$F = \epsilon\beta + \alpha + \frac{3}{2}\beta \arctan(\beta/\alpha) + \frac{\alpha\beta^2}{2(\alpha^2+\beta^2)}, \tag{1.1}$$

which was mentioned firstly in [11]. And we find the sufficient and necessary conditions for it to be projectively flat.

Theorem 1.1. *$F = \alpha\big(\epsilon s + 1 + \frac{3}{2}s\arctan(s) + \frac{s^2}{2(1+s^2)}\big)$ where $s = \beta/\alpha$ is locally projectively flat if and only if*

$$b_{i;j} = \frac{1}{2}\tau((1+4b^2)a_{ij} - 3b_ib_j) \tag{1.2}$$

and

$$G^i_\alpha = \theta y^i - \tau\alpha^2 b^i \tag{1.3}$$

where $\tau = \tau(x)$ and $\theta = a_i(x)y^i$. In this case,

$$G^i = \{\theta + \tau\chi\alpha\}y^i,$$

where

$$\chi = \frac{\epsilon(1+s^2)^2 + \frac{3}{2}\arctan(s)(1+s^2)^2 + \frac{3}{2}s(1+s^2) + s}{2(2\epsilon s(1+s^2) + 2(1+s^2) + 3s\arctan(s)(1+s^2) + s^2)} - s, \quad s = \frac{\beta}{\alpha}. \tag{1.4}$$

In [11], Shen studies a larger class of (α, β)-metrics and finds a sufficient condition for the metric in the class to be projectively flat. Then he constructs some special metrics satisfying the sufficient condition. For the family of (α, β)-metrics discussed in this paper, it has been proved in [11] that (1.2) and (1.3) are sufficient conditions for F to be locally projectively. However, the problem whether or not (1.2) and (1.3) are also necessary remains open. In Theorem 1.1 we prove that (1.2) and (1.3) are also necessary. Further, based on the construction in [11], we will give some special solutions to (1.2) and (1.3) in Section 5 below.

By Theorem 1.1, we can completely determine the local structure of a projectively flat Finsler metric F in the form (1.1) which is of constant flag curvature.

Theorem 1.2. *Let $F=\alpha\big(\epsilon s+1+\frac{3}{2}s\arctan(s)+\frac{s^2}{2(1+s^2)}\big)$ where $s=\beta/\alpha$. Suppose that F is a locally projectively flat metric with constant flag curvature λ. Then $\lambda=0$. α is a flat metric and β is parallel with respect to α. Thus F is locally Minkowskian.*

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2. (α,β)-metric

In this section, we shall state some properties of (α,β)-metric which is defined above. Let G^i and G^i_α denote the spray coefficients of F and α, respectively, given by

$$G^i=\frac{g^{il}}{4}\big\{[F^2]_{x^ky^l}y^k-[F^2]_{x^k}\big\},\quad G^i_\alpha=\frac{a^{il}}{4}\big\{[\alpha^2]_{x^ky^l}y^k-[\alpha^2]_{x^l}\big\}$$

where $(g_{ij}):=(\frac{1}{2}[F^2]_{y^iy^j})$ and $(a^{ij}):=(a_{ij})^{-1}$. The following formula (2.1) is given in [8] and [10] and a different version is given in [5] and [6].

Lemma 2.1. *The spray coefficients G^i are related to G^i_α by*

$$\begin{aligned}G^i=G^i_\alpha+\alpha Qs^i_0+J\{-2Q\alpha s_0+r_{00}\}\frac{y^i}{\alpha}\\ +H\{-2Q\alpha s_0+r_{00}\}\{b^i-s\frac{y^i}{\alpha}\},\end{aligned}\tag{2.1}$$

where

$$\begin{aligned}Q&:=\frac{\phi'}{\phi-s\phi'},\\ J&:=\frac{\phi'(\phi-s\phi')}{2\phi((\phi-s\phi')+(b^2-s^2)\phi'')},\\ H&:=\frac{\phi''}{2((\phi-s\phi')+(b^2-s^2)\phi'')},\end{aligned}$$

where $s:=\beta/\alpha$ and $b:=\|\beta_x\|_\alpha$.

In [2], G. Hamel proved that a Finsler metric $F=F(x,y)$ on an open subset $\mathcal{U}\subset R^n$ is projectively flat if and only if

$$F_{x^ky^l}y^k-F_{x^l}=0.\tag{2.2}$$

By (2.2), Shen [9] got the following lemma

Lemma 2.2. *An (α,β)-metric $F=\alpha\phi(s)$, where $s=\beta/\alpha$, is projectively flat on an open subset $\mathcal{U}\subset R^n$ if and only if*

$$(a_{ml}\alpha^2-y_my_l)G^m_\alpha+\alpha^3Qs_{l0}+H\alpha(-2\alpha Qs_0+r_{00})(b_l\alpha-sy_l)=0, \tag{2.3}$$

where $s_{ij}=\frac{1}{2}(b_{i;j}-b_{j;i})$, $s_{l0}=s_{li}y^i$, $s_0=s_{l0}b^l$, $r_{ij}=\frac{1}{2}(b_{i;j}+b_{j;i})$, and $r_{00}=r_{ij}y^iy^j$.

3. $F=\alpha\left(\epsilon s+1+\frac{3}{2}s\arctan(s)+\frac{s^2}{2(1+s^2)}\right)$

In this section, we consider the following metric

$$F=\alpha\left(\epsilon s+1+\frac{3}{2}s\arctan(s)+\frac{s^2}{2(1+s^2)}\right)=\alpha\phi(s),\quad s=\beta/\alpha,$$

where ϵ is a constant, $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form on M. Let $b_0=b_0(\epsilon)>0$ be the largest number such that

$$\epsilon s+1+\frac{3}{2}s\arctan(s)+\frac{s^2}{2(1+s^2)}>0,\quad |s|\le b<b_0, \tag{3.1}$$

so that F is a Finsler metric if and only if β satisfies that $b:=\|\beta_x\|_\alpha<b_0$ for any $x\in M$.

$$\begin{aligned}\phi(s)&=\epsilon s+1+\frac{3}{2}s\arctan(s)+\frac{s^2}{2(1+s^2)},\\ \phi'&=\epsilon+\frac{3}{2}\arctan(s)+\frac{3s}{2(1+s^2)}+\frac{s}{(1+s^2)^2},\\ \phi''&=\frac{4}{(1+s^2)^3}.\end{aligned}$$

Thus

$$\phi-s\phi'=\frac{1}{(1+s^2)^2}.$$

From now on, we always assume that ϵ satisfies (3.1). By Lemma 2.1, the spray coefficients G^i of F are given by (2.1) with

$$\begin{aligned}
Q &:= \frac{\phi'}{\phi - s\phi'} = \left(\epsilon + \frac{3}{2}\arctan(s) + \frac{3s}{2(1+s^2)} + \frac{s}{(1+s^2)^2}\right)(1+s^2)^2,\\
J &:= \frac{\phi'(\phi - s\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}\\
&= \frac{\left(\epsilon + \frac{3}{2}\arctan(s) + \frac{3s}{2(1+s^2)} + \frac{s}{(1+s^2)^2}\right)(1+s^2)}{2\left(\epsilon s + 1 + \frac{3}{2}s\arctan(s) + \frac{s^2}{2(1+s^2)}\right)(1+4b^2-3s^2)},\\
H &:= \frac{\phi''}{2((\phi - s\phi') + (b^2 - s^2)\phi'')} = \frac{2}{1+4b^2-3s^2}.
\end{aligned}$$

Thus (2.3) goes to

$$\begin{aligned}
&(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha\\
&\quad + \alpha^3\left(\epsilon + \frac{3}{2}\arctan(s) + \frac{3s}{2(1+s^2)} + \frac{s}{(1+s^2)^2}\right)(1+s^2)^2 s_{l0}\\
&\quad + \frac{2}{1+4b^2-3s^2}\alpha(-2\alpha\left(\epsilon + \frac{3}{2}\arctan(s) + \frac{3s}{2(1+s^2)} + \frac{s}{(1+s^2)^2}\right)\\
&\quad \times (1+s^2)^2 s_0 + r_{00})(b_l\alpha - s y_l) = 0.
\end{aligned} \tag{3.2}$$

Proof of Theorem 1.1. First, we rewrite (3.2) as

$$\begin{aligned}
&\alpha((1+4b^2)\alpha^2 - 3\beta^2)(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + ((1+4b^2)\alpha^2 - 3\beta^2)\\
&\quad\times\left(\epsilon(\alpha^2+\beta^2)^2 + \frac{3}{2}(\alpha^2+\beta^2)^2\arctan(s) + \frac{3}{2}\alpha\beta(\alpha^2+\beta^2) + \alpha^3\beta\right)s_{l0}\\
&\qquad - 4\left(\epsilon(\alpha^2+\beta^2)^2 + \frac{3}{2}(\alpha^2+\beta^2)^2\arctan(s) + \frac{3}{2}\alpha\beta(\alpha^2+\beta^2) + \alpha^3\beta\right)\\
&\quad\times(b_l\alpha^2 - \beta y_l)s_0 + 2\alpha^3(b_l\alpha^2 - \beta y_l)r_{00} = 0.
\end{aligned} \tag{3.3}$$

The coefficients of $\arctan(s)$ must be zero, because the other terms are algebraic functions of y^i. We obtain

$$\frac{3}{2}(\alpha^2+\beta^2)^2((1+4b^2)\alpha^2 - 3\beta^2)s_{l0} = 6(\alpha^2+\beta^2)^2(b_l\alpha^2 - \beta y_l)s_0. \tag{3.4}$$

Contracting (3.4) with b^l yields

$$(\alpha^2+\beta^2)^3 s_0 = 0.$$

By assumption, for any $y \neq 0$

$$\alpha^2 + \beta^2 \neq 0.$$

Thus

$$s_0 = 0.$$

Then it follows from (3.4) that

$$s_{l0} = 0. \tag{3.5}$$

Thus β is closed.

Now equation (3.3) is reduced to the following

$$((1 + 4b^2)\alpha^2 - 3\beta^2)(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + 2\alpha^2(b_l\alpha^2 - \beta y_l)r_{00} = 0. \tag{3.6}$$

Contracting (3.6) with b^l, we get

$$((1 + 4b^2)\alpha^2 - 3\beta^2)(a_m\alpha^2 - y_m\beta)G^m_\alpha = -2\alpha^2(b^2\alpha^2 - \beta^2)r_{00}.$$

Note that the polynomial $(1 + 4b^2)\alpha^2 - 3\beta^2$ is not divisible by α^2 and $b^2\alpha^2 - \beta^2$. Thus $(a_m\alpha^2 - y_m\beta)G^m_\alpha$ is divisible by $\alpha^2(b^2\alpha^2 - \beta^2)$. Therefore, there is a scalar function $\tau = \tau(x)$ such that

$$r_{00} = \frac{\tau}{2}[(1 + 4b^2)\alpha^2 - 3\beta^2]. \tag{3.7}$$

By (3.5) and (3.7), the formula (2.1) for G^i can be simplified to

$$G^i = G^i_\alpha + \tau\chi\alpha y^i + \tau\alpha^2 b^i, \tag{3.8}$$

where χ is given in (1.4). We know that F is projectively flat if and only if

$$G^i = P y^i.$$

By (3.8), this is equivalent to the following

$$G^i_\alpha = \theta y^i - \tau\alpha^2 b^i,$$

where $\theta = a_i y^i$ is a 1-form. In this case,

$$G^i = \{\theta + \tau\chi\alpha\} y^i. \qquad \Box$$

4. Flag curvature

In this section, we shall study the following metric with constant flag curvature $\mathbf{K}=\lambda$,

$$F=\alpha\left(\epsilon s+1+\frac{3}{2}s\arctan(s)+\frac{s^2}{2(1+s^2)}\right)=\alpha\phi(s),\quad s=\beta/\alpha.$$

We assume that F is locally projectively flat. It is known that if the spray coefficients of F are in the form $G^i=Py^i$, then F is of scalar curvature with flag curvature

$$\mathbf{K}=\frac{P^2-P_{x^k}y^k}{F^2}.$$

Then

$$\begin{aligned}\mathbf{K}&=\frac{[\theta+\tau\chi\alpha]^2-[\theta+\tau\chi\alpha]_{x^k}y^k}{F^2}\\&=\frac{(\theta+\tau\chi\alpha)^2-\theta_{x^k}y^k-\tau_{x^k}y^k\chi\alpha-\tau\chi'(s)s_{x^k}y^k\alpha-\tau\chi\alpha_{x^k}y^k}{F^2}.\end{aligned}$$

Observe that

$$\alpha_{x^k}y^k=\frac{2}{\alpha}G^m_\alpha y_m=\frac{2}{\alpha}\{\theta y^m-\tau\alpha^2b^m\}y_m=2(\theta-\tau\beta)\alpha,$$

$$\begin{aligned}s_{x^k}y^k&=\frac{r_{00}}{\alpha}+\frac{2}{\alpha^2}\{b_m\alpha-sy_m\}G^m_\alpha\\&=\frac{\tau}{2}\{(1+4b^2)-3s^2\}\alpha+\frac{2}{\alpha^2}\{b_m\alpha-sy_m\}\{\theta y^m-\tau\alpha^2b^m\}\\&=\frac{\tau}{2}\{(1+4b^2)-3s^2\}\alpha-2\tau(b^2-s^2)\alpha=\frac{\tau}{2}(1+s^2)\alpha.\end{aligned}$$

We obtain

$$\mathbf{K}=\frac{\theta^2-\theta_{x^k}y^k+\tau^2\chi^2\alpha^2-\tau_{x^k}y^k\chi\alpha-\frac{\tau^2}{2}\chi'(1+s^2)\alpha^2+2s\tau^2\chi\alpha^2}{F^2}.\tag{4.1}$$

Lemma 4.1. *Suppose that* $F=\alpha\big(\epsilon s+1+\frac{3}{2}s\arctan(s)+\frac{s^2}{2(1+s^2)}\big)$, $s=\beta/\alpha$ *is locally projectively flat with constant flag curvature* $\mathbf{K}=\lambda=$ *constant, then* $\lambda=0$.

Proof. First by (4.1), multiplied the equation $\mathbf{K}=\lambda$ by $\alpha^8F^4(1+s^2)^4$, then the coefficients of $\arctan^4(s)$ must be zero because it is a transcendental function of y^i. We get

$$\lambda\left(\frac{3}{2}\right)^4\beta^4(\alpha^2+\beta^2)^4=0.$$

Thus $\lambda=0$. □

By Lemma 4.1, we have the following

Proposition 4.2. *Let* $F = \alpha\phi(s) = \alpha\big(\epsilon s + 1 + \frac{3}{2}s\arctan(s) + \frac{s^2}{2(1+s^2)}\big)$ *where* $s = \beta/\alpha$. *Suppose that* F *is a locally projectively flat metric with zero flag curvature. Then* α *is a flat metric and* β *is parallel with respect to* α. *Thus* F *is locally Minkowskian.*

Proof. Under the assumption that $\mathbf{K} = 0$, we obtain

$$\theta^2 - \theta_{x^k}y^k + \tau^2\chi^2\alpha^2 - \tau_{x^k}y^k\chi\alpha - \frac{\tau^2}{2}\chi'(1+s^2)\alpha^2 + 2s\tau^2\chi\alpha^2 = 0. \tag{4.2}$$

Multiplied (4.2) by $4\phi^2$, then the coefficients of $\arctan^2(s)$ must be zero (note: $\arctan^2(s)$ is a transcendental function of y^i). We obtain

$$9s^2(\theta^2 - \theta_{x^k}y^k) + \frac{3}{4}\tau^2\alpha^2(1-3s^2)^2 + \frac{9}{4}(2s\tau^2\alpha^2 - \tau_{x^k}y^k\alpha)s(1-3s^2)s + \frac{9}{2}\tau^2\alpha^2(1+s^2)\left[\frac{1}{4}(1+s^2)^2(1+3s^2) + s^2\right] = 0. \tag{4.3}$$

$\alpha^6 \times$ (4.3) yields

$$9\beta^2\alpha^4(\theta^2 - \theta_{x^k}y^k) + \frac{3}{4}\tau^2\alpha^4(\alpha^2 - 3\beta^2)^2 + \frac{9}{4}\alpha^4(2\beta\tau^2 - \tau_{x^k}y^k)(\alpha^2 - 3\beta^2)\beta + \frac{9}{2}\tau^2(\alpha^2+\beta^2)\left[\frac{1}{4}(\alpha^2+\beta^2)^2(\alpha^2+3\beta^2) + \alpha^4\beta^2\right] = 0.$$

Note that the polynomial $\frac{9}{2}\tau^2(\alpha^2+\beta^2)[\frac{1}{4}(\alpha^2+\beta^2)^2(\alpha^2+3\beta^2) + \alpha^4\beta^2]$ is not divisible by β. Thus $\tau = 0$. Therefore

$$b_{i;j} = 0, \quad G^i = G^i_\alpha = \theta y^i.$$

By assumption, F has zero flag curvature, thus α has zero sectional curvature. Thus α is locally isometric to the Euclidean metric. □

5. Special solutions

In the last section, we have shown that (1.2) and (1.3) are necessary and sufficient conditions for F to be locally projectively flat. Now we give some special solutions to (1.2) and (1.3), based on the construction of examples in [11].

Example 5.1 ([11]). Define

$$\alpha := e^{\rho(h)}\alpha_\mu, \quad \beta := C_2 e^{\frac{5}{4}\rho(h)}dh,$$

where

$$\alpha_\mu = \frac{\sqrt{|y|^2+\mu(|x|^2|y|^2-\langle x,y\rangle^2)}}{1+\mu|x|^2},$$

$$h := \frac{1}{\sqrt{1+\mu|x|^2}}\left\{C_1+\langle a,x\rangle+\frac{\eta|x|^2}{1+\sqrt{1+\eta|x|^2}}\right\},$$

$\rho=\rho(t)$ be given by

$$\rho(t)=\ln\left[-2(C_2)^2\left(C_3+\mu t-\frac{1}{2}\mu t^2\right)\right]^{-2},$$

η and C_i are constants ($C_2>0$) and $a\in R^n$ is a constant vector.

Then we can simply check that α and β satisfy (1.2) and (1.3) with

$$\tau=\frac{\rho'(h)}{2C_2 e^{\frac{5}{4}\rho(h)}}.$$

Thus the Finsler metric $F=\epsilon\beta+\alpha+\frac{3}{2}\beta\arctan(\beta/\alpha)+\frac{\alpha\beta^2}{2(\alpha^2+\beta^2)}$ is projectively flat.

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