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An isomorphism theorem for group pairs of finite abelian groups

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Two subgroups A and B of a group G are said to be *equivalent* if there is an automorphism of G that maps A onto B. The equivalence theory of subgroups is a rapidly emerging subject that plays an important role in the structure of abelian groups; see, for example, [H1], [H2] and [HM1]–[HM4]. A solution to the following problem is a basic goal in the equivalence theory.

Problem. Find necessary and/or sufficient conditions for subgroups A and B of an abelian group G to be equivalent.

In this note, we restrict our attention to the case that G is finite. Although well-known classical results determine when two finite abelian groups are isomorphic in terms of numerical invariants (namely, the invariant factors or elementary divisors), there is no known test that can determine by means of numerical invariants when two subgroups are equivalent. Essentially the same problem as that stated above has been studied by E. Szabó [S], but Szabó, as well as K. Búzási [B], treats the problem as the isomorphism of group pairs: given (G, A) and (H, B), where A and B are subgroups of G and H, respectively, when is there an isomorphism $\pi : G \rightarrowtail\!\!\!\!\rightarrow H$ that maps A onto B? Since the isomorphism problem has been solved for finite abelian groups (but not for subgroups) there is no loss of generality in the present context in assuming that $G = H$.

Let A and B be subgroups of the finite abelian group G. In studying the isomorphism of the group pairs (G, A) and (G, B) it is enough to consider the case that G is p-primary, that is, all the elements of G have order a power of a fixed prime p. The reason for this is that the p-primary components of a finite abelian group are fully invariant and appear as direct summands.

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Henceforth, we let A and B denote subgroups of a finite abelian p-group G. We want to know under what conditions is there an automorphism of G that maps A onto B, that is, when are A and B equivalent as subgroups. There are two obvious necessary conditions for A and B to be equivalent, which we call the S and Q Test.

(S) The subgroups are isomorphic, $A \cong B$.

(Q) The quotients are isomorphic, $G/A \cong G/B$.

The S and Q Test is not sufficient except in special circumstances. We remark that the test is clearly sufficient when G is cyclic. However, as is well known, it already fails for a simple example like $G = \langle a\rangle \oplus \langle b\rangle \oplus \langle c\rangle$, where a, b, and c are elements of order p, p^2, and p^3, respectively. Here, we can take $A = \langle a\rangle \oplus \langle pc\rangle$ and $B = \langle pb\rangle \oplus \langle a + pc\rangle$. On the other hand, E. Szabó [S] has found some special cases where the S and Q Test is sufficient.

Realizing the S and Q Test has severe limitations as it applies to the isomorphism of group pairs, we consider the problem from a somewhat different perspective. In order to describe our approach, we first need to establish some notation and terminology. In this connection, first recall that $p^nG = \{p^nx : x \in G\}$ and $G[p^n] = \{x \in G : p^nx = 0\}$. If x is an element of the (additively written) finite abelian group G, we define the *value* of x (usually called the height of x) in G as follows:

$$|x| = n \quad \text{if } x \in p^nG \backslash p^{n+1}G \ \ (\text{and } |x| = \infty \text{ if } x = 0).$$

Dually, we define the *covalue* of x (usually called the exponent of x) in G as follows:

$$\underline{\overline{x}} = n \quad \text{if } x \in G[p^n] \backslash G[p^{n-1}] \ \ (\text{and } \underline{\overline{x}} = 0 \text{ if } x = 0).$$

We shall need to employ not only the covalue of an element but also the covalue of a coset. If H is a subgroup of G and x is an element of G, we define the *covalue of the coset* $x + H$ as follows:

$$\underline{\overline{x + H}} = \min\{\underline{\overline{x + h}} : h \in H\}.$$

If $\varphi : G/A \rightarrowtail\!\!\!\!\!\twoheadrightarrow G/B$ preserves covalues of cosets (in the sense that $\underline{\overline{x + A}} = \underline{\overline{\varphi(x + A)}}$) we often say that φ preserves coset orders; this means that the coset $x+A$ contains an element of minimal order p^n if and only if the coset $\varphi(x + A) = y + B$ contains an elements of the same minimal order p^n.

Our main result is that the group pairs (G, A) and (G, B) are isomorphic, for a finite abelian p-group G, if and only if the following condition holds:

(Q^*) There is an order-preserving isomorphism $\varphi : G/A \rightarrowtail\!\!\!\!\!\twoheadrightarrow G/B$ between the quotient groups.

Actually, we can prove a little more, but first we need a technical lemma.

Lemma. *Suppose that A and B are subgroups of the finite abelian p-group G and suppose that $\varphi : G/A \rightarrowtail\!\!\!\!\twoheadrightarrow G/B$ is an isomorphism between the quotients that preserves orders. Furthermore, suppose that $\pi : M \rightarrowtail\!\!\!\!\twoheadrightarrow M'$ is an isomorphism between certain summands M and M' of G, and consider the conditions:*

$$\begin{array}{llll} (1) & A \subseteq M + G\left[p^{n-1}\right], & (1') & B \subseteq M' + G\left[p^{n-1}\right], \\ (2) & A\left[p^{n}\right] \subseteq M + G\left[p^{n-1}\right], & (2') & B\left[p^{n}\right] \subseteq M' + G\left[p^{n-1}\right]. \end{array}$$

If π is compatible with φ in the sense that, for all $x \in M$, $\pi(x) + B = \varphi(x + A)$ and if $p^n(G/M) = 0 = p^n(G/M')$, then (1) $\iff$ (1′) and (2) $\iff$ (2′).

Proof. First, observe that $\varphi((M+G[p^{n-1}]+A)/A) = (M'+G[p^{n-1}]+B)/B$ because $\pi : M \rightarrowtail\!\!\!\!\twoheadrightarrow M'$ is compatible with φ (which yields $\varphi((M + A)/A) = (M' + B)/B$) and because φ preserves coset orders (which yields $\varphi((G\left[p^{n-1}\right] + A)/A) = (G[p^{n-1}] + B)/B$). Therefore, we conclude that φ induces an isomorphism between the quotients,

$$\text{(i)} \quad (G/A)/(M + G\left[p^{n-1}\right] + A)/A \cong (G/B)/(M' + G\left[p^{n-1}\right] + B)/B.$$

Consequently,

$$\begin{aligned} \text{(ii)} \quad & G/\left(M + G\left[p^{n-1}\right]\right)/\left(M + G\left[p^{n-1}\right] + A\right)/\left(M + G\left[p^{n-1}\right]\right) \cong \\ & \cong G/\left(M' + G\left[p^{n-1}\right]\right)/\left(M' + G\left[p^{n-1}\right] + B\right)/\left(M' + G\left[p^{n-1}\right]\right). \end{aligned}$$

Now, $G/\left(M + G\left[p^{n-1}\right]\right)$ and $G/\left(M' + G\left[p^{n-1}\right]\right)$ are annihilated by p becasue $p^n(G/M) = 0 = p^n(G/M')$. Indeed, if $M \oplus D = G = M' \oplus D'$, we have that $D \cong D'$ and

$$\begin{aligned} G/\left(M + G\left[p^{n-1}\right]\right) \cong D/D\left[p^{n-1}\right] \cong & \\ \cong D'/D'\left[p^{n-1}\right] & \cong G/\left(M' + G\left[p^{n-1}\right]\right). \end{aligned}$$

Thus $G/\left(M + G\left[p^{n-1}\right]\right)$ and $G/\left(M' + G\left[p^{n-1}\right]\right)$ are vector spaces over $\mathbb{Z}/p\mathbb{Z}$ having the same finite dimension $k = \dim\left(D/D\left[p^{n-1}\right]\right) = = \dim\left(D'/D'\left[p^{n-1}\right]\right)$; recall that $p^n D \cong p^n(G/M) = 0 = p^n(G/M') = p^n D'$, so $p\left(D/D\left[p^{n-1}\right]\right) = 0 = p\left(D'/D'\left[p^{n-1}\right]\right)$. Since $G/\left(M+G\left[p^{n-1}\right]\right)$ and $G/\left(M' + G\left[p^{n-1}\right]\right)$ are isomorphic finite dimensional vector spaces, the isomorphism of the quotient spaces in (ii) implies that the subspaces must also be isomorphic,

$$\begin{aligned} \text{(iii)} \quad \left(M + G\left[p^{n-1}\right] + A\right)/\left(M + G\left[p^{n-1}\right]\right) \cong & \\ \cong \left(M' + G\left[p^{n-1}\right] + B\right) & /\left(M' + G\left[p^{n-1}\right]\right). \end{aligned}$$

But (iii) immediately implies that (1) and (1') are equivalent, (1) $\iff$ $(1')$.

In order to verify that $(2) \iff (2')$, we note that $\varphi : (G[p^n]+A)/A \rightarrowtail\!\!\!\!\twoheadrightarrow (G[p^n]+B)/B$, which induces an order-preserving map

$$\varphi_n : G[p^n]/A[p^n] \rightarrowtail\!\!\!\!\twoheadrightarrow G[p^n]/B[p^n].$$

Notice that $M[p^n]$ and $M'[p^n]$ are direct summands of $G[p^n]$ and that $\pi : M[p^n] \rightarrowtail\!\!\!\!\twoheadrightarrow M'[p^n]$ is compatible with φ_n. Therefore, if we replace G by $G[p^n]$, M and M' by $M[p^n]$ and $M'[p^n]$, respectively, and finally A and B by $A[p^n]$ and $B[p^n]$, we retain all the hypotheses of the lemma and conclude from the equivalence of (1) and $(1')$ that

$$A[p^n] \subseteq M[p^n] + G\left[p^{n-1}\right] \iff B[p^n] \subseteq M'[p^n] + G\left[p^{n-1}\right].$$

Obviously, in the preceding it is immaterial whether we use $M[p^n]$ and $M'[p^n]$ or simply M and M', respectively. Therefore, we conclude that $(2) \iff (2')$, which completes the proof of the lemma.

Theorem. *Let A and B be subgroups of a finite abelian p-group G. There is an automorphism π of G that maps A onto B if and only if there exists an isomorphism $\varphi : G/A \rightarrowtail\!\!\!\!\twoheadrightarrow G/B$ between the quotient groups that preserves coset orders (in the sense that $\overline{x+A} = \overline{\varphi(x+A)}$ for all x in G). Moreover, given any isomorphism $\varphi : G/A \rightarrowtail\!\!\!\!\twoheadrightarrow G/B$ that preserves coset orders, there exists an automorphism π of G that not only maps A onto B but also induces φ.*

Proof. An automorphism π of G that maps A onto B clearly yields a coset order-preserving map $\varphi : G/A \rightarrowtail\!\!\!\!\twoheadrightarrow G/B$. Therefore, we shall concentrate entirely on the converse, where $\varphi : G/A \rightarrowtail\!\!\!\!\twoheadrightarrow G/B$ is an isomorphism that preserves coset orders. As we have indicated in the statement of the theorem, the key to the proof that the group pairs (G, A) and (G, B) are isomorphic is to prove more; namely, that there is an automorphism of G that induces φ.

Suppose that we have already constructed an isomorphism $\pi : M \rightarrowtail\!\!\!\!\twoheadrightarrow M'$ that satisfies

$$(*) \qquad \pi(x) + B = \varphi(x + A)$$

for all $x \in M$, where M and M' are summands of G. In this connection, two quick observations should be made. We can certainly do this for $M = 0 = M'$, and we will have finished the proof if we can accomplish this for $M = G = M'$. Thus, assume that M and M' are proper summands of G, and let $n \geq 1$ be the smallest positive integer for which $p^n(G/M) = 0$. The strategy is to extend π to a mapping from $M \oplus \langle x \rangle$ to $M' \oplus \langle y \rangle$, where $M \oplus \langle x \rangle$ and $M' \oplus \langle y \rangle$, are themselves direct summands of G with $x, y \neq 0$ and where π continues to satisfy $(*)$ for the larger summands. In order to find the appropriate x and y, we distinguish three cases.

Case 1: condition (2) of the lemma does not hold. Since (1) $\Longrightarrow$ (2), condition (1) also fails. The lemma asserts that (1′) and (2′) must also fail. Hence, in Case 1, none of the conditions (1), (1′), (2), nor (2′) is valid. Let $M \oplus D = G = M' \oplus D'$. Choose $a \in A\left[p^n\right] \setminus \left(M + G\left[p^{n-1}\right]\right)$ and $b \in B\left[p^n\right] \setminus \left(M' + G\left[p^{n-1}\right]\right)$, and let $a = m + d$, $b = m' + d'$, where $m \in M$, $m' \in M'$, $d \in D$ and $d' \in D'$. Clearly, $\overline{\underline{d}} = n = \overline{\underline{d'}}$ because $a \notin M + G\left[p^{n-1}\right]$ and $b \notin M' + G\left[p^{n-1}\right]$. Hence, $\langle d\rangle$ and $\langle d'\rangle$ are summands of D and D', respectively, due to the fact that $p^n D = 0 = p^n D'$; see, for example, [F], p.77. Letting $D = \langle d\rangle \oplus C$ and $D' = \langle d'\rangle \oplus C'$, we notice that since $p^n m = 0 = p^n m'$ it follows immediately that

$$M \oplus \langle a\rangle \oplus C = M \oplus \langle d\rangle \oplus C = G = M' \oplus \langle d'\rangle \oplus C' = M' \oplus \langle b\rangle \oplus C'.$$

Therefore, if we let $x = a$ and $y = b$, the following conditions hold.

(a) $x \in A$.
(b) $y \in B$.
(c) $G = M \oplus \langle x\rangle \oplus C$ for some subgroup C of G.
(d) $G = M' \oplus \langle y\rangle \oplus C'$ for some subgroup C' of G.
(e) $\overline{\underline{x}} = n = \overline{\underline{y}}$, that is, the order of x and y is p^n.

Clearly, in the case at hand, we can simply map x onto y and obtain the desired extension of π that continues to satisfy condition $(*)$, because $x + A$ and $y + B$ are zero and $\varphi(x + A) = y + B$.

Case 2: condition (1) of the lemma fails, but (2) is valid. As before, let $M \oplus D = G = M' \oplus D'$ and choose $a \in A \setminus \left(M + G\left[p^{n-1}\right]\right)$. Write $a = m - x$ where $m \in M$ and x is an element of D order p^n that satisfies condition (c) of Case 1, $G = M \oplus \langle x\rangle \oplus C$. Observe that $\overline{\underline{m + A}} = \overline{\underline{x + A}} = n$, for $\overline{\underline{x + A}} < n$ implies that there exists $a_0 \in A$ for which $p^{n-1}\left(x - a_0\right) = 0$. But this leads to $a + a_0 = m - \left(x - a_0\right) \in M + G\left[p^{n-1}\right]$. Since $a_0 \in A\left[p^n\right]$ and since condition (2) holds, this would imply that $a \in M + G\left[p^{n-1}\right]$ contrary to the choice of a. Therefore, $\overline{\underline{m + A}} = n = \overline{\underline{\pi(m) + B}}$ because π is compatible with φ, by virture of condition $(*)$, and because φ preserves coset orders. This means that there exist $b \in B$ and $y \in G$ such that $b = \pi(m) - y$, where $p^n y = 0$. Indeed, we know that $\overline{\underline{y + B}} = \overline{\underline{\pi(m) + B}} = n$, so y has order exactly p^n but we want to show even more; we claim that $y \notin M' + G\left[p^{n-1}\right]$.

Assume, by way of contradiction, that $y \in M' + G\left[p^{n-1}\right]$. Then we can write y as $y = \pi\left(m_1\right) + z$, where $m_1 \in M$ and $z \in G\left[p^{n-1}\right]$. Actually, $m_1 \in M\left[p^n\right]$ since $p^n y = 0 = p^n z$. Since $\varphi\left(x - m_1 + A\right) = \varphi\left(m - m_1 + A\right) = \pi(m) - \pi\left(m_1\right) + B = y - \pi\left(m_1\right) + B = z + B$ and since $p^{n-1} = 0$, there exists an element $a_1 \in A$ such that $x - m_1 + a_1 \in G\left[p^{n-1}\right]$. Recall that x and m_1 are both in $G\left[p^n\right]$, so $a_1 \in A\left[p^n\right]$. Since condition

(2) is satisfied, we conclude that $a_1 \in M + G\left[p^{n-1}\right]$. But this implies that $x \in M + G\left[p^{n-1}\right]$. This, however, is impossible because $a = m - x \notin M + G\left[p^{n-1}\right]$. Now that we have shown that $y \notin M' + G\left[p^{n-1}\right]$ and that y has order p^n, we can assume that y satisfies condition (d) of Case 1, $G = M' \oplus \langle y \rangle \oplus C'$. In fact, if $y = m' + d'$, where $m' \in M'$ and $d' \in D'$ we know that $D' = \langle d' \rangle \oplus C'$ and that $G = M' \oplus \langle y \rangle \oplus C'$ since $p^n m' = 0$. Since $\varphi(x + A) = \varphi(m + A) = \pi(m) + B = y + B$ and $M \oplus \langle x \rangle \oplus C = G = M' \oplus \langle y \rangle \oplus C'$, we can obtain the desired extension of π that continues to satisfy $(*)$ by mapping x onto y.

We remark that it would be easy to extend π in all cases if we did not have to be concerned with condition $(*)$.

Case 3: conditions (1) and (2), as well as (1′) and (2′), are satisfied. Choose an element $x \in D$ of order p^n, where $G = M \oplus D$, so that condition (c) of Case 1 is satisfied. Since $x \notin M + G\left[p^{n-1}\right]$ and since $A \subseteq M + G\left[p^{n-1}\right]$, it quickly follows that $\overline{\underline{x + A}} = \overline{\underline{x}} = n$. Indeed, $x - a \in G\left[p^{n-1}\right]$ with $a \in A$ immediately implies, under the special hypothesis of Case 3, that $x \in M + G[p^{n-1}]$. Since φ preserves coset orders, we know we can choose an element y in G of order p^n so that $\varphi(x + A) = y + B$. In fact, we can easily demonstrate that $y \notin M' + G\left[p^{n-1}\right]$. If y were in $M' + G\left[p^{n-1}\right]$, then certainly we would have $x \in M + G\left[p^{n-1}\right] + A$. But, in Case 3, $A \subseteq M + G\left[p^{n-1}\right]$. Hence, we would have $x \in M + G\left[p^{n-1}\right]$, but this is impossible since x has order p^n and satisfies $G = M \oplus \langle x \rangle \oplus C$. Now, since $y \notin M' + G\left[p^{n-1}\right]$ we can write, as in Case 2, $G = M' \oplus \langle y \rangle \oplus C'$, and we can once again obtain the desired extension of π by mapping x onto y.

We have demonstrated in all cases that we can extend π so that it continues to satisfy condition $(*)$, which enables us to construct an automorphism of G that induces φ. Since any such automorphism necessarily takes A onto B, the proof is finished.

Corollary 1. *Suppose that A and B are subgroups of the finite abelian p-group G. The following conditions are equivalent.*

- (S^*) *There is an isomorphism between the subgroups which preserves values (heights).*
- (Q^*) *There is an isomorphism between the quotient groups which preserves coset covalues (orders).*

Proof. If the group pairs (G, A) and (G, B) are isomorphic, that is, if there is an automorphism π of G that takes A onto B, then both (S^*) and (Q^*) must hold (by virtue of isomorphisms induced by π).

If (S^*) holds, let $\pi : A \rightarrowtail\!\!\!\!\!\twoheadrightarrow B$ be a value-preserving map. It is well known (the proof essentially goes back at least to Zippin [Z]) that π can be extended to an automorphism of G, from whence (Q^*) follows.

Conversely, if (Q^*) holds, our theorem demonstrates that there is an automorphism of G that maps A onto B, so (Q^*) also implies (S^*).

Corollary 2. *Let $G = \oplus\langle g_i\rangle$ be a finite abelian p-group written as a direct sum of cyclic groups $\langle g_i\rangle$ and let A be an arbitrary subgroup of G. There exists stacked bases for G and A, in the sense that $G = \oplus\langle x_i\rangle$ and $A = \oplus\langle n_i x_i\rangle$ for suitable nonnegative integers n_i if and only if there exist an order-preserving isomorphism between G/A and G/B, where $B = \oplus\langle n_i g_i\rangle$.*

Proof. One direction is trivial. If we have stacked bases x_i and $n_i x_i$ for G and A, then we can arrange the x_i's so that x_i and g_i have the same order. If $B = \oplus\langle n_i g_i\rangle$, clearly the automorphism of G that maps x_i onto g_i maps A onto B and yields an order-preserving isomorohism from G/A onto G/B.

Conversely, if $B = \oplus\langle n_i g_i\rangle$ and $\varphi : G/A \rightarrowtail\!\!\!\!\rightarrow G/B$ is an order-preserving isomorphism, then our theorem states there is an automorphism π of G that induces φ. Letting $x_i = \pi^{-1}(g_i)$, we obtain stacked bases, $\{x_i\}$ and $\{n_i x_i\}$, for G and A.

We conclude with the following problem.

Problem. Find other necessary and sufficient conditions for a group pair (G, A) to have stacked bases when G is a finite abelian p-group.

References

[B] K. Búzási, Invariants of pairs of finite abelian groups, *Publ. Math. Debrecen* **28** (3-4) (1981), 317–326, (Russian).
[F] L. Fuchs, Infinite Abelian Groups, vol. I, *Academic Press*, *New York*, 1970.
[H1] P. Hill, On the structure of abelian p-groups, *Trans. Amer. Math. Soc.* **288** (1985), 505–525.
[H2] P. Hill, Equivalence theorems, *Rocky Mountain Jour. of Mathematics* **23** (1993), 203–221.
[HM1] P. Hill and C. Megibben, On the theory and classification of abelian p-groups, *Math. Zeit.* **190** (1985), 17–38.
[HM2] P. Hill and C. Megibben, The local equivalence theorem, Contemporary Mathematics, *Amer. Math. Soc.* **87** (1989), 201–220.
[HM3] P. Hill and C. Megibben, Generalization of the stacked bases theorems, *Trans. Amer. Math. Soc.* **312** (1989), 377–402.
[HM4] P. Hill and C. Megibben, Mixed groups, *Trans. Amer. Math. Soc.* **334** (1992), 121–142.
[S] E. Szabó, Über isomorphe Paare endlicher abelscher Gruppen (On isomorphic pairs of finite abelian groups), *Publ. Math. Debrecen* **35** (1–2) (1988), 93–101.
[Z] L. Zippin, Countable torsion group, *Ann. Math.* **36** (1935), 86–99.

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