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Non-linear commutativity preserving maps on symmetric matrices

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Abstract. We study bijective maps on the set of all symmetric matrices in $M_n(\mathbb{C})$, $n \geq 3$, that preserve commutativity in both directions.

1. Introduction

One of the most active and fertile subjects in matrix theory during the past one hundred years is the linear preserver problem which concerns characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant (see [8], [11]). In the last few decades a lot of results on linear preservers on matrix algebras as well as on more general rings and operator algebras have been obtained. Besides linear preservers also a more general problem of characterizing additive preservers and related problem of characterizing multiplicative preservers on matrix algebras were studied a lot. It is surprising that in some cases we can get nice structural results for preservers without any algebraic assumption like linearity, additivity or multiplicativity. Probably the first fundamental attempt to attack non-linear preserver problems was made by Baribeau and Ransford in [1]. They studied spectrum preserving non-linear maps of matrix algebras under some mild differentiability condition.

Linear preserver problems concerning commutativity are one of the most extensively studied preserver problems both on matrix algebras and on operator algebras (see, for example, [2], [3], [4], [10], and [13] and the references therein).

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The reason is that the assumption of preserving commutativity can be considered as the assumption of preserving zero Lie products. Omitting linearity, the problem can become much more difficult and hence more challenging.

Recently Molnár and Šemrl [9] characterized bijective non-linear maps preserving commutativity in both directions on the set of all self-adjoint complex matrices. In [12] Šemrl considered continuous maps on $M_n(\mathbb{C})$, the algebra of all $n \times n$ complex matrices, $n \geq 3$, that are bijective and preserve commutativity in both directions and are not assumed to be linear. The continuity assumption and the assumption $n \geq 3$ are indispensable in Šemrl's theorem (see [12] for counterexamples). In [5] the problem has been solved also for real matrices. A natural question here is whether the analogues results hold true for the set of all complex symmetric matrices. So, in this paper we will study non-linear commutativity preserving maps on the set of all $n \times n$ complex symmetric matrices which will be denoted by $S_n(\mathbb{C})$.

One of the main results of the paper states that if $\phi : S_n(\mathbb{C}) \to S_n(\mathbb{C})$, $n \geq 3$, is a continuous bijective map preserving commutativity in both directions, then there exist an orthogonal matrix $Q \in M_n(\mathbb{C})$ and for every $A \in S_n(\mathbb{C})$ a complex polynomial p_A such that either $\phi(A) = Qp_A(A)Q^t$, $A \in S_n(\mathbb{C})$, or $\phi(A) = Qp_A(\overline{A})Q^t$, $A \in S_n(\mathbb{C})$. Here, $\overline{A} = \overline{[a_{ij}]} = [\overline{a_{ij}}]$. Note that a complex matrix Q is said to be orthogonal if $QQ^t = I$ (here, Q^t denotes the transpose of the matrix Q). We will also study commutativity preserving maps on $S_n(\mathbb{C})$ without the continuity assumption. Let f be an automorphism of the complex field. For every $A = [a_{ij}] \in S_n(\mathbb{C})$ we will denote $A_f = [f(a_{ij})]$. Then the maps $\phi : S_n(\mathbb{C}) \to S_n(\mathbb{C})$ defined by $\phi(A) = Qp_A(A_f)Q^t$, $A \in S_n(\mathbb{C})$, preserve commutativity. We will prove that every bijective map on $S_3(\mathbb{C})$ that preserves commutativity in both directions is of this nice form. On the other hand we will give an example showing that this is not true for $n > 3$. However, there is a large subset $\mathcal{C} \subset S_n(\mathbb{C})$, $n \geq 3$, which is invariant under every bijective map ϕ on $S_n(\mathbb{C})$ preserving commutativity in both directions and the restriction of ϕ to this subset is of this nice form.

2. Statements of the main results

A map $\phi : S_n(\mathbb{C}) \to S_n(\mathbb{C})$ *preserves commutativity* if $\phi(A)\phi(B) = \phi(B)\phi(A)$ whenever $AB = BA$, $A, B \in S_n(\mathbb{C})$. If ϕ is bijective and both ϕ and ϕ^{-1} preserve commutativity, then we say that ϕ *preserves commutativity in both directions.*

In this paper we will consider maps on $S_n(\mathbb{C})$ that are bijective and preserve commutativity in both directions and are not assumed to be linear.

Let us start by giving some examples of such maps. Let $Q \in M_n(\mathbb{C})$ be an orthogonal matrix. Then, of course, $A \mapsto QAQ^t$ is an example of linear bijective map on $S_n(\mathbb{C})$ preserving commutativity in both directions. Let f be any automorphism of the complex field. Recall that the identity function and the complex conjugation are the only continuous automorphisms of the complex field. On the other hand, there are many noncontinuous automorphisms of $\mathbb{C}$ [7]. The map $A \mapsto A_f$, $A \in S_n(\mathbb{C})$, is a bijective additive map of $S_n(\mathbb{C})$ that preserves commutativity in both directions. But there are also many nonadditive maps $\phi : S_n(\mathbb{C}) \to S_n(\mathbb{C})$ that preserve commutativity. To see this observe that if A and B are two commuting matrices and p and q are any complex polynomials, then $p(A)$ and $q(B)$ commute as well. Therefore, if we associate to each $A \in S_n(\mathbb{C})$ a polynomial p_A, then the map $A \mapsto p_A(A)$ preserves commutativity. Every such map will be called *a locally polynomial map.* This kind of maps are in general neither bijective, nor they preserve commutativity in both directions. However, if such a map ϕ is bijective and if polynomials p_A, $A \in S_n(\mathbb{C})$, are chosen in such a way that for every $A \in S_n(\mathbb{C})$ we can find a polynomial q_A such that $q_A(p_A(A)) = A$ (in other words, if ϕ is bijective and its inverse is again a locally polynomial map), then it preserves commutativity in both directions. Such maps will be called *regular locally polynomial maps.*

Of course, any composition of bijective maps preserving commutativity in both directions is again a bijective map preserving commutativity in both directions. So, at this point it would be tempting to conjecture that every bijective map $\phi : S_n(\mathbb{C}) \to S_n(\mathbb{C})$ preserving commutativity in both directions is of the form $\phi(A) = Qp_A(A_f)Q^t$, $A \in S_n(\mathbb{C})$, where $Q \in M_n(\mathbb{C})$ is any orthogonal matrix, f is any automorphism of the field $\mathbb{C}$, and $A \mapsto p_A(A)$ is a regular locally polynomial map. But it turns out that this conjecture is true just for $n = 3$. On the other hand, for $n > 3$ we define the subset $\mathcal{C} \subset S_n(\mathbb{C})$ of all symmetric matrices with the property that the zeros of their minimal polynomials have multiplicities at most two. Note that the set $\mathcal{C}$ is rather large. In particular, it contains the set of all symmetric matrices with n distinct eigenvalues. We will prove that $\mathcal{C}$ is invariant under every bijective map ϕ on $S_n(\mathbb{C})$ preserving commutativity in both directions. Our first result states that the restriction of ϕ to this subset must be of the nice form described above.

Theorem 2.1. *Let $n \geq 3$ and let $\phi : S_n(\mathbb{C}) \to S_n(\mathbb{C})$ be a bijective map preserving commutativity in both directions. Then there exist an orthogonal* *matrix $Q \in M_n(\mathbb{C})$, an automorphism f of the complex field, and a regular locally polynomial map $A \mapsto p_A(A)$ such that $\phi(A) = Qp_A(A_f)Q^t$ for all $A \in \mathcal{C}$.*

A consequence of this theorem is that every bijective map ϕ on $S_3(\mathbb{C})$ which preserves commutativity in both directions is of the form $\phi(A) = Qp_A(A_f)Q^t$, $A \in S_3(\mathbb{C})$, where $Q \in M_3(\mathbb{C})$ is any orthogonal matrix, f is any automorphism of the complex field, and $A \mapsto p_A(A)$ is a regular locally polynomial map (see Corollary 4.1). However, for $n > 3$ we will give an example showing that outside the set $\mathcal{C}$ bijective maps on $S_n(\mathbb{C})$ preserving commutativity in both directions can have a wild behavior. But under the additional continuity assumption we get a nice result on the whole set of symmetric matrices also for $n > 3$.

Theorem 2.2. *Let $n \geq 3$ and let $\phi : S_n(\mathbb{C}) \to S_n(\mathbb{C})$ be a continuous bijective map preserving commutativity in both directions. Then there exist an orthogonal matrix $Q \in M_n(\mathbb{C})$ and a regular locally polynomial map $A \mapsto p_A(A)$ such that either $\phi(A) = Qp_A(A)Q^t$ for all $A \in S_n(\mathbb{C})$, or $\phi(A) = Qp_A(\overline{A})Q^t$ for all $A \in S_n(\mathbb{C})$.*

3. Preliminary results

Let $B \in M_k(\mathbb{C})$ be the "backward identity" matrix, $B = E_{1k} + E_{2,k-1} + \ldots + E_{k1}$, and let

$$S_k = \frac{1}{\sqrt{2}}(I + iB) \in S_k(\mathbb{C}). \tag{1}$$

Since $B^2 = I$, we have $S_k\overline{S}_k = I$. Now consider a typical Jordan block $J_k(0)$ with zero main diagonal and $k \geq 2$. It is a simple computation to show that

$$S_k J_k(0) S_k^{-1} = S_k \begin{bmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & 0 \end{bmatrix} \overline{S}_k$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 1 & \ldots & 0 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \ldots & 1 & 0 \end{bmatrix} + \frac{i}{2} \begin{bmatrix} 0 & \ldots & -1 & 0 \\ \vdots & \iddots & 0 & 1 \\ -1 & \iddots & \iddots & \vdots \\ 0 & 1 & \ldots & 0 \end{bmatrix}$$

which is evidently symmetric. Therefore a matrix $S_k J_k(\lambda) S_k^{-1} = S_k J_k(\lambda)\overline{S}_k$ is symmetric as well for any Jordan block $J_k(\lambda)$ with $k \geq 2$.

Every matrix $A \in M_n(\mathbb{C})$ is similar to a Jordan canonical form J. Here, $J = J_{n_1}(\lambda_1, 2) \oplus J_{n_2}(\lambda_2, 2) \oplus \ldots \oplus J_{n_k}(\lambda_k, 2)$ is a direct sum of modified Jordan blocks

$$J_{n_i}(\lambda_i, 2) = \begin{bmatrix} \lambda_i & 2 & \ldots & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 0 & \ldots & 0 & \lambda_i \end{bmatrix} \in M_{n_i}(\mathbb{C}).$$

This observation permits us to drop the coefficient factors of $\frac{1}{2}$ in the above equality. Let $S_{n_i} \in S_{n_i}(\mathbb{C})$ be the matrix of the form (1) for $n_i \geq 2$ and let $S_1 = [1]$. If we set $T = S_{n_1} \oplus S_{n_2} \oplus \ldots \oplus S_{n_k}$, then the preceding argument shows that

$$TJT^{-1} = TJ\overline{T} = (S_{n_1} J_{n_1}(\lambda_1, 2)\overline{S}_{n_1}) \oplus \ldots \oplus (S_{n_k} J_{n_k}(\lambda_k, 2)\overline{S}_{n_k})$$

is a direct sum of symmetric matrices and is therefore symmetric. Thus we have shown that every matrix $A \in M_n(\mathbb{C})$ is similar to a symmetric matrix. Since two symmetric matrices A and B are similar if and only if they are similar via orthogonal similarity, we have the next theorem.

Theorem 3.1. *Every symmetric matrix $A \in S_n(\mathbb{C})$ is similar to a symmetric Jordan canonical form $S = S_{n_1}(\lambda_1) \oplus S_{n_2}(\lambda_2) \oplus \ldots \oplus S_{n_k}(\lambda_k)$, where*

$$S_{n_i}(\lambda_i) = S_{n_i} J_{n_i}(\lambda_i, 2)\overline{S}_{n_i}$$

$$= \lambda_i I + \begin{bmatrix} 0 & 1 & \ldots & 0 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \ldots & 1 & 0 \end{bmatrix} + i \begin{bmatrix} 0 & \ldots & -1 & 0 \\ \vdots & \iddots & 0 & 1 \\ -1 & \iddots & \iddots & \vdots \\ 0 & 1 & \ldots & 0 \end{bmatrix} \in S_{n_i}(\mathbb{C}).$$

Moreover, there exists an orthogonal matrix $Q \in M_n(\mathbb{C})$ such that $A = QSQ^t$.

Note that $S_1(\lambda) = [\lambda]$ and $S_2(\lambda) = \left[\begin{smallmatrix} \lambda - i & 1 \\ 1 & \lambda + i \end{smallmatrix}\right]$. Since the symmetric Jordan canonical form was derived from the Jordan canonical form, its uniqueness is the same as that of the Jordan canonical form.

Let $\mathcal{S}$ be a subset of $S_n(\mathbb{C})$. Recall that its commutant $\mathcal{S}'$ is the space of all matrices from $S_n(\mathbb{C})$ that commute with all matrices from $\mathcal{S}$. When $\mathcal{S} = \{A\}$

we write shortly $A' = \{A\}'$. A matrix A is said to be *nonderogatory* if every eigenvalue of A has geometric multiplicity one.

Let $A = S_n(\lambda)$ for some complex number λ and some positive integer n. Since A is a nonderogatory matrix, a matrix $B \in S_n(\mathbb{C})$ commutes with A if and only if there is a complex polynomial p such that $B = p(A)$ (see [6, p. 135]). Further, a matrix A can be written as $A = S_n J_n(\lambda, 2) S_n^{-1}$, where S_n is a matrix of the form (1) and $J_n(\lambda, 2)$ is a modified Jordan block (see above). Because of this special form an explicit calculation shows that a matrix $B \in S_n(\mathbb{C})$ commutes with A if and only if $B = S_n C S_n^{-1}$, where C must be an upper triangular matrix of Toeplitz type, that is,

$$B = S_n \begin{bmatrix} c_1 & c_2 & \dots & c_n \\ 0 & c_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_2 \\ 0 & \dots & 0 & c_1 \end{bmatrix} S_n^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} 2c_1 & c_2 & \dots & c_n \\ c_2 & 2c_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_2 \\ c_n & \dots & c_2 & 2c_1 \end{bmatrix} + \frac{i}{2} \begin{bmatrix} -c_n & \dots & -c_2 & 0 \\ \vdots & \iddots & 0 & c_2 \\ -c_2 & \iddots & \iddots & \vdots \\ 0 & c_2 & \dots & c_n \end{bmatrix}. \tag{2}$$

Clearly, for $A \in S_n(\mathbb{C})$ we have $A' = S_n(\mathbb{C})$ if and only if A is a scalar matrix. In particular, $B' \subseteq (\lambda I)'$ for every $B \in S_n(\mathbb{C})$ and every complex number λ. We will call a nonscalar matrix $A \in S_n(\mathbb{C})$ *maximal* if every $B \in S_n(\mathbb{C})$ satisfying $A' \subset B'$ and $A' \neq B'$ has to be a scalar matrix. The set of all nonscalar maximal matrices will be denoted by $\mathcal{M}$. Similarly, $A \in S_n(\mathbb{C})$ is *minimal* if there is no $B \in S_n(\mathbb{C})$ satisfying $B' \subset A'$ and $B' \neq A'$.

Lemma 3.2. *Let $A \in S_n(\mathbb{C})$ be a nonscalar matrix. Then A is maximal if and only if either A is diagonalizable with exactly two eigenvalues, or $A = \lambda I + N$ for some complex number λ and some square-zero matrix $N \neq 0$.*

Proof. Assume first that A is diagonalizable with exactly two eigenvalues and let $B \in S_n(\mathbb{C})$ be a matrix satisfying $A' \subset B'$ and $A' \neq B'$. Then we may assume that

$$A = \begin{bmatrix} \lambda I & 0 \\ 0 & \mu I \end{bmatrix}$$

with $\lambda \neq \mu$. The commutant of A is the set of all symmetric matrices

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix},$$

where X and Y are any two square symmetric matrices of the appropriate size. It follows from $A' \subset B'$ that every such matrix commutes with B which further yields that

$$B = \begin{bmatrix} \delta I & 0 \\ 0 & \xi I \end{bmatrix}$$

for some complex numbers δ and ξ. In the case $\delta \neq \xi$ we would have $A' = B'$, a contradiction. Therefore B has to be a scalar matrix.

Next, we will prove that also every symmetric matrix A of the form $A = \lambda I + N$, where $\lambda \in \mathbb{C}$ and $N \neq 0$ is a square-zero matrix, is maximal. Replacing A by a similar matrix, if necessary, we may assume that

$$A = \begin{bmatrix} \lambda I + S & 0 \\ 0 & \lambda I \end{bmatrix},$$

where S is the matrix of the form

$$S = \begin{bmatrix} S_2(0) & 0 & \dots & 0 \\ 0 & S_2(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & S_2(0) \end{bmatrix}.$$

The last column and the last row in the matrix A may be absent. For every pair of complex numbers α and β we denote by $S(\alpha, \beta)$ the 2×2 symmetric matrix

$$S(\alpha, \beta) = \begin{bmatrix} \alpha - i\beta & \beta \\ \beta & \alpha + i\beta \end{bmatrix}. \tag{3}$$

The commutant of A is the set of all matrices of the form

$$\begin{bmatrix} X & Y \\ Y^t & Z \end{bmatrix},$$

where

$$X = \begin{bmatrix} S(\alpha_{11}, \beta_{11}) & \dots & S(\alpha_{1k}, \beta_{1k}) \\ \vdots & \ddots & \vdots \\ S(\alpha_{1k}, \beta_{1k}) & \dots & S(\alpha_{kk}, \beta_{kk}) \end{bmatrix} \in S_{2k}(\mathbb{C})$$

(here, $2k$ is the size of the matrix S), Z is an arbitrary symmetric matrix of the appropriate size, and Y is a matrix with the property $SY = 0$. Let $B \in S_n(\mathbb{C})$ be a matrix satisfying $A' \subset B'$ and $A' \neq B'$. Similar argument as above yields that either

$$B = \begin{bmatrix} \mu I + \delta S & 0 \\ 0 & \mu I \end{bmatrix}$$

for some scalars μ and δ with $\delta \neq 0$, or $B = \mu I$. Since $A' \neq B'$ the first possibility cannot occur.

To prove the converse, assume that A is neither diagonalizable with exactly two eigenvalues, nor a scalar plus a nonzero square-zero matrix. Let us start with a case when A has more than two eigenvalues. Then it is similar to a matrix

$$\begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix},$$

where A_1, A_2, and A_3 have pairwise disjoint spectra. We may assume that the matrix A has this block diagonal form. It follows that the commutant of A is contained in the set of all symmetric matrices of the form

$$\begin{bmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{bmatrix},$$

where X, Y, and Z are square symmetric matrices of the appropriate size. But then, obviously, the matrix

$$B = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a nonscalar symmetric matrix whose commutant is larger than the commutant of A.

If A has two eigenvalues and it is not diagonalizable, then there is no loss of generality in assuming that

$$A = \begin{bmatrix} \lambda I + M & 0 \\ 0 & \mu I + N \end{bmatrix},$$

where $\lambda \neq \mu$ and M and N are symmetric nilpotents not both equal to zero. The nonscalar symmetric matrix

$$B = \begin{bmatrix} \lambda I & 0 \\ 0 & \mu I \end{bmatrix}$$

has larger commutant than A, thus showing that A is not maximal also in this case.

The last case we have to treat is that A is of the form $A = \lambda I + N$ for some complex number λ and some symmetric nilpotent N with $N^2 \neq 0$. Using the symmetric Jordan canonical form it is easy to verify that the commutant of A is a proper subset of the commutant of the nonscalar symmetric matrix $\lambda I + N^2$. This completes the proof. $\square$

Lemma 3.3. *Let $A \in S_n(\mathbb{C})$. Then A is minimal if and only if A is nonderogatory.*

Proof. Assume first that A is nonderogatory and let $B \in S_n(\mathbb{C})$ be a matrix satisfying $B' \subseteq A'$. We have to show that $B' = A'$. From $B \in B'$ we conclude that matrices A and B commute. But then $B = p(A)$ for some complex polynomial p since A is nonderogatory. Hence, $A' \subseteq B'$ as desired.

To prove the converse assume that A is in the symmetric Jordan canonical form and that it has more than two blocks corresponding to the same eigenvalue λ. Denote these blocks by $S_{n_1}(\lambda), S_{n_2}(\lambda), \dots, S_{n_k}(\lambda)$. Let B be a symmetric matrix obtained from A by replacing all diagonal entries in $S_{n_1}(\lambda)$ by μ_1, all diagonal entries in $S_{n_2}(\lambda)$ by μ_2, ..., and all diagonal entries in $S_{n_k}(\lambda)$ by μ_k, where $\mu_i \neq \mu_j$ whenever $i \neq j$. Then $B' \subset A'$ and $B' \neq A'$. This completes the proof. $\square$

Our next goal is to characterize nonderogatory symmetric matrices with n different eigenvalues and nonderogatory symmetric matrices with $n-1$ different eigenvalues using commutativity relations. Let A be a nonderogatory symmetric matrix. For two matrices $B, C \in A'$ the commutants B' and C' may be equal or different. We will take all matrices from A', then form the set of their commutants and denote by $\#A$ the cardinality of this set, $\#A = |\{B' : B \in A'\}|$. Note that the quantity $\#A$ does not change if we replace A by a similar symmetric matrix. So, we will assume that A is in the symmetric Jordan canonical form

$$A = \operatorname{diag}(S_{n_1}(\lambda_1), S_{n_2}(\lambda_2), \dots, S_{n_k}(\lambda_k)),$$

where $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigenvalues of A and $n_1 \geq n_2 \geq \dots \geq n_k$.

Assume first that $n_1 \geq 4$ and let $S = S_{n_1} \oplus S_{n_2} \oplus \dots \oplus S_{n_k}$, where $S_{n_1}, S_{n_2}, \dots, \dots, S_{n_k}$ are matrices of the form (1). Then $B_\alpha = S(\alpha E_{1,n_1-1} + \alpha E_{2,n_1} + E_{1,n_1})S^{-1}$ belongs to A' and it is trivial to verify that $B'_\alpha \neq B'_\beta$ whenever $\alpha \neq \beta$. Hence, $\#A = \infty$ in this case. Next, assume that $n_2 \geq 2$. Then $B_\alpha = S(\alpha E_{1,n_1} + E_{n_1+1,n_1+n_2})S^{-1}$ belongs to A' and again $B'_\alpha \neq B'_\beta$ whenever $\alpha \neq \beta$. Hence, $\#A = \infty$ in this case as well.

Thus, we have proved the following result.

Lemma 3.4. *Let $A \in S_n(\mathbb{C})$ be a nonderogatory matrix. If $\#A < \infty$, then A has at most $n-2$ distinct eigenvalues such that at most one of the eigenvalues has algebraic multiplicity larger than one.*

In the next step we will consider only maximal matrices from the commutant A' of a nonderogatory symmetric matrix A. As before we form the set of their commutants and denote by $\#mA$ the cardinality of this set, $\#mA = |\{B' : B \in A' \cap \mathcal{M}\}|$.

Assume first that A is a diagonal matrix with n different eigenvalues. Then every symmetric matrix $B \in A' \cap \mathcal{M}$ is of the form $B = \alpha P + \beta(I - P)$, where P is a diagonal idempotent, $P \neq 0, I$, and α and β are different complex numbers. Clearly, $B' = P'$. Since two diagonal idempotents P and Q have the same commutant if and only if $P = Q$ or $P = I - Q$ we have

$$\#mA = \frac{1}{2}\left(\binom{n}{1} + \ldots + \binom{n}{n-1}\right) = 2^{n-1} - 1.$$

Now, let A be a nonderogatory symmetric matrix with $n-1$ different eigenvalues. Then its symmetric Jordan canonical form has one, say the first cell of the size 2×2, while all the others are 1×1 trivial cells. Hence, $B \in A' \cap \mathcal{M}$ if and only if $B = \alpha I + \beta \left[\begin{smallmatrix} S_2(0) & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with $\beta \neq 0$ or B is a diagonal matrix with exactly two eigenvalues and the first two diagonal entries must be equal. Since all the matrices $B \in A' \cap \mathcal{M}$ that are of the form scalar plus square-zero matrix have the same commutant, we have

$$\#mA = 1 + \frac{1}{2}\left(\binom{n-1}{1} + \ldots + \binom{n-1}{n-2}\right) = 2^{n-2}.$$

Similarly, if A has $n-2$ different eigenvalues one of them being of algebraic multiplicity 3, then $\#mA = 2^{n-3}$.

Hence, we have the following statement.

Lemma 3.5. *Let $A \in S_n(\mathbb{C})$, $n \geq 3$, be a nonderogatory matrix and $\#A < \infty$. Then*

(i) *A has n different eigenvalues if and only if $\#mA = 2^{n-1} - 1$,*

(ii) *A has $n-1$ different eigenvalues if and only if $\#mA = 2^{n-2}$,*

(iii) *A has $n-2$ different eigenvalues one of them being of algebraic multiplicity 3 if and only if $\#mA = 2^{n-3}$.*

In the proof of Theorem 2.1 we will also use the next simple lemma. Before setting this lemma we introduce some notation. Recall that every symmetric idempotent P of rank one can be written as $P = xx^t$, where x is a $n \times 1$ matrix satisfying $x^t x = 1$. Similarly, every symmetric nilpotent N of rank one can be written as $N = xx^t$, where x is a $n \times 1$ matrix satisfying $x^t x = 0$. The space of all $n \times 1$ matrices will be identified with $\mathbb{C}^n$. For a nonzero $x \in \mathbb{C}^n$ we denote by $[x]$ the one-dimensional space spanned by x. As usual, $P\mathbb{C}^n = \{[x] : x \in \mathbb{C}^n \setminus 0\}$.

Lemma 3.6. *Let $n \geq 3$ and $x, y, z \in \mathbb{C}^n$. The following two statements are equivalent:*

(i) $[z] \subset [x] + [y]$,

(ii) $w^t z = 0$ *for every vector* $w \in \mathbb{C}^n$ *satisfying* $w^t x = 0$ *and* $w^t y = 0$.

4. Commutativity preserving maps

The goal of this section is to prove Theorem 2.1 and Corollary 4.1. So, we will start with the study of commutativity preserving maps on $S_n(\mathbb{C})$, $n \geq 3$, without assuming the continuity of such maps.

Proof of Theorem 2.1. Let us assume that $n \geq 3$ and let $\phi : S_n(\mathbb{C}) \to S_n(\mathbb{C})$ be a bijective map preserving commutativity in both directions. Then, obviously, for every subset $\mathcal{S} \subseteq S_n(\mathbb{C})$ we have $\phi(\mathcal{S}') = \phi(\mathcal{S})'$. If $A \in S_n(\mathbb{C})$ has n different eigenvalues, then A is a diagonalizable nonderogatory matrix. Assume that A is already in a diagonal form and that $B \in A'$. Then B is diagonal and the commutant B' is completely determined if we know which of the diagonal entries of B are equal. Thus, $\#A < \infty$ and $\#mA = 2^{n-1} - 1$. Therefore $\phi(A)$ is also a nonderogatory symmetric matrix with $\#\phi(A) < \infty$ and $\#m\phi(A) = 2^{n-1} - 1$. It follows from Lemma 3.5 that ϕ maps the set of all symmetric matrices with n different eigenvalues onto itself. Further, a symmetric matrix A is diagonalizable if and only if it commutes with some symmetric matrix with n different eigenvalues. Thus, $\mathcal{D}$, the set of all diagonalizable symmetric matrices is mapped by ϕ onto itself. Denote by $\mathcal{D}_k \subset S_n(\mathbb{C})$, $k = 1, 2, \ldots, n$, the set of all diagonalizable symmetric matrices with exactly k eigenvalues. We have $A \in \mathcal{D}_1$ if and only if $A = \lambda I$ for some $\lambda \in \mathbb{C}$ and this is equivalent to $A' = S_n(\mathbb{C})$. Thus, $\mathcal{D}_1$ is mapped onto itself. The same is true for $\mathcal{D}_2 = \mathcal{M} \cap \mathcal{D}$. Observe that for $A \in \mathcal{D}$ the following two statements are equivalent:

(i) $A \in \mathcal{D}_3$,

(ii) $A \notin \mathcal{D}_1 \cup \mathcal{D}_2$ and every symmetric matrix $B \in \mathcal{D}$ satisfying $B \in A'$, $A' \subset B'$, and $A' \neq B'$ belongs to $\mathcal{D}_1 \cup \mathcal{D}_2$.

It follows easily that $\phi(\mathcal{D}_3) = \mathcal{D}_3$. Repeating this procedure we get $\phi(\mathcal{D}_k) = \mathcal{D}_k$, $k = 1, 2, \ldots, n$.

Let $A \in S_n(\mathbb{C})$ be a nonderogatory symmetric matrix with $n-1$ different eigenvalues. Assume that A is already in the symmetric Jordan canonical form with the first cell of the size 2×2 and let $B \in A'$. Then

$$B = \begin{bmatrix} S(\alpha, \beta) & 0 \\ 0 & X \end{bmatrix},$$

where X is a diagonal matrix of the appropriate size and $S(\alpha, \beta)$ is a matrix of the form (3) for some complex numbers α and β. Thus, $\# A < \infty$ and $\# mA = 2^{n-2}$ (see above). This yields that $\phi(A)$ is also a nonderogatory symmetric matrix with $\#\phi(A) < \infty$ and $\# m\phi(A) = 2^{n-2}$. It follows from Lemma 3.5 that ϕ maps the set of all nonderogatory symmetric matrices with $n-1$ different eigenvalues onto itself. Further, let $A \in S_n(\mathbb{C})$ be a matrix of the form $A = \lambda I + N$ for some complex number λ and a symmetric nilpotent N of rank one. Then A commutes with some nonderogatory symmetric matrix with $n-1$ different eigenvalues. Hence, $\phi(A)$ also commutes with some nonderogatory symmetric matrix with $n-1$ different eigenvalues. Therefore, since A is maximal and it is not diagonalizable, $\phi(A) = \mu I + M$ for some complex number μ and a symmetric nilpotent M of rank one. This yields that $\mathcal{N} \subset S_n(\mathbb{C})$, the set of all matrices that can be written as $\lambda I + N$, where λ is any complex number and N is any symmetric nilpotent of rank one, is mapped by ϕ onto itself.

We denote by $\mathcal{P} \subset \mathcal{D}_2$ the set of all matrices of the form $\lambda P + \mu(I-P)$, where λ and μ are different complex numbers and P is a symmetric idempotent of rank one. Note that $\mathcal{P}$ is the set of all diagonalizable symmetric matrices with exactly two eigenvalues one of them having the eigenspace of dimension one. In our next step we will prove that ϕ maps the set $\mathcal{P}$ onto itself. We will verify that for $A \in \mathcal{D}_2$ the following two statements are equivalent:

(i) $A \in \mathcal{P}$,

(ii) for every $B \in A' \cap \mathcal{D}_2$ we have $\{A, B\}'' \subseteq \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$.

Assume for a moment that we have already proved this. Then, because ϕ preserves the first commutants, it has to preserve also the second commutants and since it preserves $\mathcal{D}_k$, $k = 1, 2, 3$, we have necessarily $\phi(\mathcal{P}) = \mathcal{P}$, as desired. So, assume that $A = \lambda P + \mu(I-P) \in \mathcal{P}$ and $B \in A' \cap \mathcal{D}_2$. A symmetric matrix B

commutes with A if and only if it commutes with P. Therefore there is no loss of generality in assuming that already A is a symmetric idempotent of rank one, and after applying an orthogonal similarity, if necessary, we may assume that $A = E_{11}$. Moreover, two diagonalizable matrices commute if and only if they are simultaneously diagonalizable. Therefore there is no loss of generality in assuming that $B = \delta(E_{11} + \ldots + E_{kk}) + \xi(E_{k+1,k+1} + \ldots + E_{nn})$, where $1 \le k \le n-1$ and $\delta \ne \xi$. If $k = 1$, then $\{A, B\}'' = \operatorname{span}\{E_{11}, I - E_{11}\} \subseteq \mathcal{D}_1 \cup \mathcal{D}_2$. Here, $\operatorname{span}\{E_{11}, I - E_{11}\}$ denotes the linear span of the set $\{E_{11}, I - E_{11}\}$. Similarly, if $2 \le k \le n-1$, then $\{A, B\}'' = \operatorname{span}\{E_{11}, E_{22} + \ldots + E_{kk}, I - (E_{11} + \ldots + E_{kk})\} \subseteq \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$. To prove the other direction assume that $A \in \mathcal{D}_2 \setminus \mathcal{P}$. As before there is no loss of generality in assuming that $A = E_{11} + \ldots + E_{kk}$ for some k, $2 \le k \le n-2$. Take $B = E_{11} + E_{k+1,k+1}$ and observe that then $\{A, B\}'' = \operatorname{span}\{E_{11}, E_{22} + \ldots + E_{kk}, E_{k+1,k+1}, I - (E_{11} + \ldots + E_{k+1,k+1})\}$ contains matrices with four different eigenvalues.

To each $A \in \mathcal{P}$ we associate the unique symmetric idempotent $P \in S_n(\mathbb{C})$ of rank one satisfying $A = \lambda P + \mu(I - P)$, $\lambda, \mu \in \mathbb{C}$. If $A, B \in \mathcal{P}$ and P and Q are the corresponding symmetric idempotents of rank one, then $P = Q$ if and only if $A' = B'$. Moreover, $PQ = QP = 0$ if and only if A and B commute and $A' \ne B'$. Similarly, to each $A \in \mathcal{N}$ we associate the unique symmetric nilpotent $N \in S_n(\mathbb{C})$ of rank one satisfying $A = \lambda I + N$, $\lambda \in \mathbb{C}$. If $A, B \in \mathcal{N}$ and N and M are the corresponding symmetric nilpotents of rank one, then $N = \mu M$ for some complex number μ if and only if $A' = B'$. Moreover, $NM = MN = 0$ if and only if A and B commute. Further, let $A = \lambda P + \mu(I - P) \in \mathcal{P}$ and let $B = \xi I + N \in \mathcal{N}$. Then $PN = NP = 0$ if and only if A and B commute. Thus, ϕ induces a bijective map φ defined on the projective space $P\mathbb{C}^n$ which has the property that $[z] \subset [x] + [y]$ if and only if $\varphi([z]) \subset \varphi([x]) + \varphi([y])$ (see Lemma 3.6). Hence, by the fundamental theorem of projective geometry there exist a bijective linear map $Q : \mathbb{C}^n \to \mathbb{C}^n$ and an automorphism $f : \mathbb{C} \to \mathbb{C}$ such that $\varphi([x]) = [Qx_f]$ for every nonzero vector x. Here,

$$x_f = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_f = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}.$$

Since $x^t y = 0$ if and only if $(Qx_f)^t Q y_f = 0$ for every pair $x, y \in \mathbb{C}^n$ the map Q can be chosen in such a way that $QQ^t = I$. Replacing ϕ by $A \mapsto Q^t \phi(A_{f^{-1}}) Q$ we may assume without loss of generality that for every symmetric idempotent P of rank one the set of all matrices of the form $\lambda P + \mu(I - P)$, $\lambda, \mu \in \mathbb{C}$, $\lambda \ne \mu$, is mapped bijectively onto itself and that for every symmetric nilpotent N of rank one the set of all matrices of the form $\lambda I + \mu N$, $\lambda, \mu \in \mathbb{C}$, is mapped bijectively onto itself. In other words, for every $A \in \mathcal{P} \cup \mathcal{N} \cup \mathbb{C}I$ there exist complex polynomials p_A and q_A such that $\phi(A) = p_A(A)$ and $A = q_A(p_A(A))$. Hence, after composing ϕ by an appropriate regular locally polynomial map (this map acts like the identity outside $\mathcal{P} \cup \mathcal{N} \cup \mathbb{C}I$), we may assume that $\phi(A) = A$ for every $A \in \mathcal{P} \cup \mathcal{N} \cup \mathbb{C}I$.

In the next step we will prove that after composing ϕ by yet another regular locally polynomial map we may assume that $\phi(A) = A$ for every diagonalizable symmetric matrix A. As before, we need to show that for every diagonalizable $A \in \mathcal{D}$ there are polynomials p_A and q_A such that $\phi(A) = p_A(A)$ and $A = q_A(p_A(A))$. In fact, it is enough to prove this only for diagonal matrices. Indeed, assume that we have proved the existence of such polynomials for diagonal matrices and let $A \in \mathcal{D}$ be any diagonalizable symmetric matrix. Then there exists an orthogonal matrix $Q \in M_n(\mathbb{C})$ such that $QAQ^t = D$ is diagonal. The map $\psi : S_n(\mathbb{C}) \to S_n(\mathbb{C})$ defined by $\psi(X) = Q\phi(Q^t X Q)Q^t$ is a bijective map preserving commutativity in both directions with the additional property that $\psi(A) = A$ for every $A \in \mathcal{P} \cup \mathcal{N} \cup \mathbb{C}I$. Thus, by our assumption, $\psi(D)$ and D have the same commutant or equivalently $\phi(A)$ and A have the same commutant which is the same as the existence of polynomials p_A and q_A such that $\phi(A) = p_A(A)$ and $A = q_A(p_A(A))$. Hence, let D be a diagonal matrix. It is easy to see that $D' = \operatorname{span}(I_n(\mathbb{C}) \cap D')$, where $I_n(\mathbb{C}) \subset S_n(\mathbb{C})$ denotes the subset of all symmetric idempotents of rank one. Since ϕ acts like the identity on $I_n(\mathbb{C})$ we have $\phi(D)' = D'$, as desired. Thus, from now on we will assume that $\phi(A) = A$ for every diagonalizable matrix $A \in \mathcal{D}$.

To complete the proof we have to show that for every $A \in \mathcal{C}$ there exist polynomials p_A and q_A such that $\phi(A) = p_A(A)$ and $A = q_A(p_A(A))$. Note that this is equivalent to the requirement that $\phi(A) = p_A(A)$, $A \in \mathcal{C}$, for some complex polynomial p_A with the property $A' = p_A(A)'$.

Let Q be an arbitrary $n \times n$ orthogonal matrix, k and h nonnegative integers with $2k + h = n$, and $\lambda_1, \lambda_2, \ldots, \lambda_k, \mu_1, \mu_2, \ldots, \mu_h$ complex numbers. We have to prove that

$$A = Q \ \operatorname{diag}(S_2(\lambda_1), S_2(\lambda_2), \ldots, S_2(\lambda_k), \mu_1, \mu_2, \ldots, \mu_h) \ Q^t$$

is mapped into

$$\phi(A) = Q \ \operatorname{diag}(S(\alpha_1, \beta_1), S(\alpha_2, \beta_2), \ldots, S(\alpha_k, \beta_k), \delta_1, \delta_2, \ldots, \delta_h) \ Q^t,$$

where $S(\alpha_i, \beta_i)$, $i = 1, 2, \ldots k$, are matrices of the form (3), $\lambda_i = \lambda_j$ if and only if $\alpha_i = \alpha_j$ and $\beta_i = \beta_j$, $\mu_i = \mu_j$ if and only if $\delta_i = \delta_j$, and $\lambda_i = \mu_j$ if and only if $\alpha_i = \delta_j$.

Because A commutes with idempotents

$$\begin{gathered} Q\ \operatorname{diag}(I,\ldots,0,0,\ldots,0)\ Q^t, \\ \vdots \\ Q\ \operatorname{diag}(0,\ldots,I,0,\ldots,0)\ Q^t, \\ Q\ \operatorname{diag}(0,\ldots,0,1,\ldots,0)\ Q^t, \\ \vdots \\ Q\ \operatorname{diag}(0,\ldots,0,0,\ldots,1)\ Q^t, \end{gathered}$$

the matrix $\phi(A)$ commutes with these idempotents as well, and therefore,

$$\phi(A) = Q\ \operatorname{diag}(A_1, A_2, \ldots, A_k, \delta_1, \delta_2, \ldots, \delta_h)\ Q^t,$$

where $A_1, A_2, \ldots, A_k$ are 2×2 symmetric matrices and $\delta_1, \delta_2, \ldots, \delta_h \in \mathbb{C}$. The matrix A commutes with

$$Q\ \operatorname{diag}(S_2(0), \ldots, 0, 0, \ldots, 0)\ Q^t \in \mathcal{N}$$

and, of course, the same must be true for $\phi(A)$. Thus, $A_1 = S(\alpha_1, \beta_1)$ for some complex numbers α_1 and β_1. If $\beta_1 = 0$, then $\phi(A)$ commutes with the idempotent $QE_{11}Q^t$ and then the same must be true for A. This contradiction shows that $\beta_1 \neq 0$. In the same way we show that all the matrices A_i, $i = 2, 3, \ldots, k$, have a similar form. So, we have proved that

$$\phi(A) = Q\ \operatorname{diag}(S(\alpha_1, \beta_1), S(\alpha_2, \beta_2), \ldots, S(\alpha_k, \beta_k), \delta_1, \delta_2, \ldots, \delta_h)\ Q^t,$$

for some complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, $\delta_1, \delta_2, \ldots, \delta_h$ and some nonzero complex numbers $\beta_1, \beta_2, \ldots, \beta_k$.

Assume that two of the λ's, say λ_1 and λ_2, are equal. Then A commutes with the matrix

$$D = Q\ \left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \oplus 0 \right)\ Q^t \in \mathcal{D}.$$

Here, I stands for the 2×2 identity matrix and the last 0 denotes the $(n-4)\times(n-4)$ zero matrix. It follows that $\phi(A)$ commutes with D which further yields $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. The same argument shows that $S(\alpha_1, \beta_1) = S(\alpha_2, \beta_2)$ implies $\lambda_1 = \lambda_2$. Hence, $\lambda_i = \lambda_j$ if and only if $\alpha_i = \alpha_j$ and $\beta_i = \beta_j$. Similarly, if two of the μ's, say μ_1 and μ_2, are equal, then A commutes with the matrix

$Q(E_{2k+1,2k+2}+E_{2k+2,2k+1})Q^t \in \mathcal{D}$. Thus, the matrix $\phi(A)$ has to commute with $Q(E_{2k+1,2k+2}+E_{2k+2,2k+1})Q^t$ as well which implies $\delta_1 = \delta_2$. In the same way we show that $\delta_1 = \delta_2$ implies $\mu_1 = \mu_2$. Hence, $\mu_i = \mu_j$ if and only if $\delta_i = \delta_j$.

It remains to prove that $\lambda_i = \mu_j$ if and only if $\alpha_i = \delta_j$. The only case we have to consider is that $i = j = 1$ and $k = 1$ since the same simple idea works in the general case as well. So, assume that

$$A = Q\ \operatorname{diag}(S_2(\lambda_1), \mu_1, \mu_2, \dots, \mu_h)\ Q^t$$

with $\lambda_1 = \mu_1$. We already know that

$$\phi(A) = Q\ \operatorname{diag}(S(\alpha_1, \beta_1), \delta_1, \delta_2, \dots, \delta_h)\ Q^t,$$

where $\beta_1 \neq 0$. We want to prove that $\alpha_1 = \delta_1$. Clearly, the matrix A commutes with the symmetric matrix

$$B = Q\ \left(\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 1 \\ -i & 1 & 0 \end{bmatrix} \oplus 0 \right)\ Q^t,$$

where the last 0 denotes the $(n-3) \times (n-3)$ zero matrix. As above we can prove that $\phi(B) = Q\ (C \oplus \xi I)\ Q^t$. Here, I stands for the $(n-3) \times (n-3)$ identity matrix, $\xi \in \mathbb{C}$, and $C = [c_{ij}] \in S_3(\mathbb{C})$. Suppose that $c_{13} = 0$ and $c_{23} = 0$. Then the matrix $\phi(B)$ commutes with the diagonalizable symmetric matrix

$$Q\ \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus 0 \right)\ Q^t \in \mathcal{D}$$

and the same must be true for the matrix B, a contradiction. Therefore, either $c_{13} \neq 0$, or $c_{23} \neq 0$. Since B commutes with the matrix A, $\phi(B)$ commutes with $\phi(A)$. This yields $\alpha_1 = \delta_1$, as desired. Similarly, $\alpha_1 = \delta_1$ implies $\lambda_1 = \mu_1$. This completes the proof. $\square$

Corollary 4.1. *Let $\phi : S_3(\mathbb{C}) \to S_3(\mathbb{C})$ be a bijective map preserving commutativity in both directions. Then there exist an orthogonal matrix $Q \in M_3(\mathbb{C})$, an automorphism f of the complex field, and a regular locally polynomial map $A \mapsto p_A(A)$ such that $\phi(A) = Qp_A(A_f)Q^t$ for all $A \in S_3(\mathbb{C})$.*

Proof. Without loss of generality we can assume that $\phi(A) = A$ for every $A \in \mathcal{C} \subset S_3(\mathbb{C})$ (see the proof of Theorem 2.1). Now, let Q be an arbitrary 3×3 orthogonal matrix, $\lambda \in \mathbb{C}$, and let $A = QS_3(\lambda)Q^t$. All we have to do is to show that $\phi(A) = p_A(A)$ for some complex polynomial p_A with the property $A' = p_A(A)'$.

A matrix A commutes with a diagonalizable symmetric matrix

$$Q\left(S_3\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}S_3^{-1}\right)Q^t$$

(here, $S_3 \in S_3(\mathbb{C})$ is the matrix of the form (1)) and therefore $\phi(A)$ has to commute with this matrix as well. This yields that

$$\phi(A) = Q\left(S_3\begin{bmatrix}\alpha&\gamma&\delta\\0&\beta&\gamma\\0&0&\alpha\end{bmatrix}S_3^{-1}\right)Q^t,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. On the other hand $\phi(A)$ is a symmetric matrix with one eigenvalue of geometric multiplicity one since $\phi(\mathcal{C}) = \mathcal{C}$. Hence, $\alpha = \beta$ and $\gamma \neq 0$. This completes the proof. $\square$

5. Continuous commutativity preserving maps

The goal of this section is to prove Theorem 2.2.

Proof of Theorem 2.2. Let $n \geq 3$ and let $\phi : S_n(\mathbb{C}) \to S_n(\mathbb{C})$ be a continuous bijective map preserving commutativity in both directions. Then we may assume without loss of generality that $\phi(A) = A$ for every $A \in \mathcal{C} \subset S_n(\mathbb{C})$ (see the proof of Theorem 2.1).

Let $\alpha \in \mathbb{C}$. Denote $S_\alpha = S_n(0) + \alpha(iE_{11} + E_{1n} + E_{n1} - iE_{nn})$. We observe first that for every $\alpha \neq 0$ the matrix S_α is diagonalizable and consequently $\phi(S_\alpha) = S_\alpha$. Note also that $S_n(0) = \lim_{\alpha\to 0} S_\alpha$ and since ϕ is continuous we have $\phi(S_n(0)) = \lim_{\alpha\to 0}\phi(S_\alpha) = \lim_{\alpha\to 0} S_\alpha = S_n(0)$. In a similar way we prove that for every orthogonal matrix $Q \in M_n(\mathbb{C})$ the matrix $Q\ \mathrm{diag}(0, S_m(0), 0)\ Q^t$, where 0 stands for the zero matrices of the appropriate size (possibly different size and one of them possibly absent), is mapped by ϕ into itself.

Using exactly the same ideas as in the proof of Theorem 2.1 we conclude that every matrix

$$A = Q\ \mathrm{diag}(S_{n_1}(\lambda_1), S_{n_2}(\lambda_2), \ldots, S_{n_k}(\lambda_k))\ Q^t$$

(here, $Q \in M_n(\mathbb{C})$ is an orthogonal matrix) is mapped into

$$\phi(A) = Q\ \mathrm{diag}(A_1, A_2, \ldots, A_k)\ Q^t,$$

where the A_i's are symmetric matrices of the appropriate size. Since A commutes with

$$Q\ \operatorname{diag}(S_{n_1}(0), 0, \ldots, 0)\ Q^t,$$
$$Q\ \operatorname{diag}(0, S_{n_2}(0), \ldots, 0)\ Q^t,$$
$$\vdots$$
$$Q\ \operatorname{diag}(0, 0, \ldots, S_{n_k}(0))\ Q^t,$$

$\phi(A)$ commutes with these matrices as well, and consequently,

$$A_i = S_{n_i} \begin{bmatrix} a_1^{(i)} & a_2^{(i)} & \ldots & a_{n_i}^{(i)} \\ 0 & a_1^{(i)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2^{(i)} \\ 0 & \ldots & 0 & a_1^{(i)} \end{bmatrix} S_{n_i}^{-1},$$

where $S_{n_i} \in S_{n_i}(\mathbb{C})$ is the matrix of the form (1). Suppose that $a_2^{(1)} = 0$. Then there exists a symmetric matrix $B_1 \in A_1'$ such that $B_1 \notin S_{n_1}(0)'$. Thus, the matrix $B = Q\ \operatorname{diag}(B_1, 0, \ldots, 0)\ Q^t$ commutes with $\phi(A)$ and does not commute with $Q\ \operatorname{diag}(S_{n_1}(0), 0, \ldots, 0)\ Q^t$. Note that $B = \phi(C)$, where $C = Q\ \operatorname{diag}(C_1, \xi I, \ldots, \xi I)\ Q^t$ with $C_1 \in S_{n_i}(\mathbb{C})$ and $\xi \in \mathbb{C}$ (see above). Since ϕ preserves commutativity in both directions C commutes with A and does not commute with the matrix $Q\ \operatorname{diag}(S_{n_1}(0), 0, \ldots, 0)\ Q^t$. This yields that there exists a symmetric matrix $C_1 \in S_{n_1}(\lambda_1)'$ such that $C_1 \notin S_{n_1}(0)'$, a contradiction. In the same way we prove that $a_2^{(i)} \neq 0$ for $i = 2, 3, \ldots, k$.

Assume that two of the λ's, say λ_1 and λ_2, are equal. Then A commutes with some symmetric matrix $Q\ \operatorname{diag}(B, 0, \ldots, 0)\ Q^t$, where

$$B = \begin{bmatrix} 0 & C \\ C^t & 0 \end{bmatrix} \in S_{n_1+n_2}(\mathbb{C}).$$

Here, the first 0 denotes the $n_1 \times n_1$ zero matrix, the second 0 denotes the $n_2 \times n_2$ zero matrix, and C is some nonzero complex matrix of the appropriate size. This implies that $\phi(A)$ commutes with a symmetric matrix

$$Q\ \operatorname{diag}(X, \xi I, \ldots, \xi I)\ Q^t,$$

where $\xi \in \mathbb{C}$ and

$$X = \begin{bmatrix} Y & Z \\ Z^t & W \end{bmatrix} \in S_{n_1+n_2}(\mathbb{C}).$$

Here, $Y \in S_{n_1}(\mathbb{C})$, $W \in S_{n_2}(\mathbb{C})$, and Z stands for some nonzero complex matrix of the appropriate size (see the proof of Theorem 2.1). Hence, the eigenvalue of A_1 coincides with the eigenvalue of A_2. The same argument shows that if A_1 and A_2 have the same eigenvalue, then $\lambda_1 = \lambda_2$. So we proved that the cells $S_{n_i}(\lambda_i)$ and $S_{n_j}(\lambda_j)$ correspond to the same eigenvalue of A if and only if the eigenvalue of A_i coincides with the eigenvalue of A_j. Thus, we have

$$\begin{aligned}\phi(A) &= \phi(Q\ \operatorname{diag}(S_{n_1}(\lambda_1), S_{n_2}(\lambda_2), \ldots, S_{n_k}(\lambda_k))\ Q^t) \\ &= Q\ \operatorname{diag}(p_1(S_{n_1}(\lambda_1)), p_2(S_{n_2}(\lambda_2)), \ldots, p_k(S_{n_k}(\lambda_k)))\ Q^t\end{aligned}$$

for some complex polynomials $p_1, p_2, \ldots, p_k$. Moreover, $p_i(\lambda_i) = p_j(\lambda_j)$ if and only if $\lambda_i = \lambda_j$. In the next step we will show that we can take $p_i = p_j$ whenever $\lambda_i = \lambda_j$. In other words, we want to prove that $a_1^{(i)} = a_1^{(j)}$, $a_2^{(i)} = a_2^{(j)}$, $\ldots$, $a_t^{(i)} = a_t^{(i)}$ whenever $\lambda_i = \lambda_j$. Here, $t = \min\{n_i, n_j\}$. If $\lambda_i = \lambda_j$, we already know that $a_1^{(i)} = a_1^{(j)}$. Thus we have to consider only the cases when $n_i, n_j \geq 2$.

Let $A \in S_n(\mathbb{C})$ be a matrix with the largest cell in the symmetric Jordan canonical form of the size $m \times m$. Of course, $1 \leq m \leq n$. If $m \leq 2$, we already know that $\phi(A) = A$. Suppose that $m = 3$ and that two of the λ's, say λ_1 and λ_2, are equal. Without loss of generality we can assume that $n_1 \geq n_2$. If $n_1 = n_2$, then A commutes with the diagonalizable symmetric matrix

$$B = Q\ (\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \oplus 0)\ Q^t.$$

Here, I stands for the $n_1 \times n_1$ identity matrix and the last 0 denotes the $(n - 2n_1) \times (n - 2n_1)$ zero matrix. It follows that $\phi(A)$ commutes with B which further yields $p_1 = p_2$. It remains to consider the case when $n_1 = 3$ and $n_2 = 2$. Let $S = S_3 \oplus S_2$ and let

$$B = S \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} S^{-1}.$$

Then the matrix A commutes with the symmetric matrix $C = Q\ (B \oplus 0)\ Q^t$. Here, the last 0 denotes the $(n-5) \times (n-5)$ zero matrix. Since $\phi(C) = C$ (the matrix B is similar to the matrix $S_5(0)$), $\phi(A)$ commutes with C as well. Thus, $p_1 = p_2$. Hence, after composing ϕ by an appropriate regular locally polynomial map we may assume that $\phi(A) = A$ for all symmetric matrices $A \in S_n(\mathbb{C})$ with the property that all the cells in the symmetric Jordan canonical form of A are of the size 1×1, 2×2 or 3×3.

Now, suppose that $m = 4$ and that $\lambda_1 = \lambda_2$ $(n_1 \geq n_2)$. In the same way as above we can prove that $p_1 = p_2$ if $n_1 = n_2$ or $n_1 = 3$, $n_2 = 2$. So, it remains to consider the cases when $n_1 = 4$, $n_2 = 2$ or $n_1 = 4$, $n_2 = 3$. Suppose that $n_1 = 4$ and $n_2 = 2$. Let $S = S_4 \oplus S_2$ and let

$$B = S \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} S^{-1}.$$

Then the matrix A commutes with the symmetric matrix $C = Q\ (B \oplus 0)\ Q^t$. Here, the last 0 denotes the $(n-6) \times (n-6)$ zero matrix. With the symmetric Jordan canonical form it is easy to see that $\phi(C) = C$. Thus, $\phi(A)$ commutes with C as well which further yields $p_1 = p_2$. The proof is the same in the second case $(n_1 = 4,\ n_2 = 3)$. The only difference is that

$$B = S \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} S^{-1},$$

where $S = S_4 \oplus S_3$. Hence, after composing ϕ by an appropriate regular locally polynomial map we may assume that $\phi(A) = A$ for all symmetric matrices $A \in S_n(\mathbb{C})$ with the property that all the cells in the symmetric Jordan canonical form of A are of the size 1×1, 2×2, 3×3 or 4×4. We continue in the same way for $m = 5, \ldots, n-1$. For $m = n$ we already know that $\phi(A) = A$.

So we proved that there exist an orthogonal matrix $Q \in M_n(\mathbb{C})$, an automorphism $f : \mathbb{C} \to \mathbb{C}$, and a regular locally polynomial map $A \mapsto p_A(A)$ such that $\phi(A) = Qp_A(A_f)Q^t$ for all $A \in S_n(\mathbb{C})$.

To complete the proof we have to show that either $f(\lambda) = \lambda$ for all $\lambda \in \mathbb{C}$, or $f(\lambda) = \overline{\lambda}$ for all $\lambda \in \mathbb{C}$. Replacing ϕ by $A \mapsto Q^t\phi(A)Q$ we may assume without loss of generality that $\phi(A) = p_A(A_f)$ for all $A \in S_n(\mathbb{C})$. In particular, we have

$\phi(E_{11}) = \lambda E_{11} + \mu I$ for some scalars λ, μ with $\lambda \neq 0$. Moreover, $\phi((1+x)E_{11} + \sqrt{-x-x^2}(E_{12}+E_{21}) - xE_{22}) = \lambda(x)((1+f(x))E_{11} + f(\sqrt{-x-x^2})(E_{12}+E_{21}) - f(x)E_{22}) + \mu(x)I$ for some functions $\lambda, \mu : \mathbb{C} \to \mathbb{C}$. By the continuity assumption we have $\phi(E_{11}) = \lim_{x\to 0} \phi((1+x)E_{11} + \sqrt{-x-x^2}(E_{12}+E_{21}) - xE_{22})$, and consequently, $\lim_{x\to 0} \mu(x) = \mu$, which further yields $\lim_{x\to 0} \lambda(x)f(x) = 0$ since $\lim_{x\to 0}(\mu(x) - \lambda(x)f(x)) = \mu$. We also have $\lim_{x\to 0}(\lambda(x)(1+f(x)) + \mu(x)) = \lambda + \mu$ and therefore $\lim_{x\to 0} \lambda(x) = \lambda \neq 0$. All these yields that $\lim_{x\to 0} \lambda(x)f(x) = \lambda \lim_{x\to 0} f(x) = 0$. Thus, f is an automorphism of the complex field that is continuous at zero. Therefore, we have either $f(\lambda) = \lambda$, $\lambda \in \mathbb{C}$, or $f(\lambda) = \overline{\lambda}$, $\lambda \in \mathbb{C}$. This completes the proof of Theorem 2.2. $\square$

The next example will show that there exist bijective maps on the whole set $S_n(\mathbb{C})$, $n > 3$, preserving commutativity in both directions which are not of the nice form given in Theorem 2.1. Nevertheless it should be mentioned that the main idea of this example is taken from [12].

Example 5.1. Let $n > 3$ and let $\mathcal{S} \subset S_n(\mathbb{C})$ be the set of all symmetric matrices of the form $\lambda I + N$, where λ is any complex number and N is a symmetric nilpotent of maximal nilindex (i.e., $N^n = 0$ and $N^{n-1} \neq 0$). For $A, B \in \mathcal{S}$ we will write $A \sim B$ if $A' = B'$. Clearly, $A \sim B$ if and only if there exist polynomials p and q with complex coefficients such that $A = p(B)$ and $B = q(A)$. In other words, $A \sim B$ if and only if $AB = BA$. Now suppose that $A = \lambda I + N \in \mathcal{S}$, $B = \mu I + M \in \mathcal{S}$, and that N is already in the symmetric Jordan canonical form (i.e., $N = S_n(0)$). Then $A \sim B$ if and only if $N \sim M$ and this is true if and only if M is of the form (3) with $c_1 = 0$ and $c_2 \neq 0$.

Further, for $A = \lambda I + N \in \mathcal{S}$ and $B = \mu I + M \in \mathcal{S}$ we will write $A \approx B$ if $A' \setminus \mathcal{S} = B' \setminus \mathcal{S}$. Note that $A' \setminus \mathcal{S} = \operatorname{span}\{I, N^2, \ldots, N^{n-1}\}$ and $B' \setminus \mathcal{S} = \operatorname{span}\{I, M^2, \ldots, M^{n-1}\}$. Of course, $A \approx B$ if and only if $N \approx M$. Now suppose that $N = S_n(0)$. Then it is easy to verify that every symmetric matrix $M = T + \delta S_n E_{2,n-1} S_n^{-1}$, where δ is any complex number and T is a symmetric matrix of the form (3) with $c_1 = 0$ and $c_2 \neq 0$, satisfies $N \approx M$. On the other hand, if a symmetric nilpotent M of maximal nilindex commutes with $N^2, N^3, \ldots, N^{n-1}$, then M has to be of the form $M = T + \delta S_n E_{2,n-1} S_n^{-1}$, where $\delta \in \mathbb{C}$ and T is as above (see [12, Lemma 3.5]). This yields that $N \approx M$ if and only if M is of the form described above. Clearly, $A \sim B$ yields $A \approx B$. Therefore the relation $\approx$ induces an equivalence relation on $\mathcal{S}/_\sim = \{[A] : A \in \mathcal{S}\}$, the set of all equivalence classes with respect to the relation $\sim$.

Let $\phi : S_n(\mathbb{C}) \to S_n(\mathbb{C})$ be any bijective map such that $\phi(A) = A$ for all $A \notin \mathcal{S}$, $A \sim B$ if and only if $\phi(A) \sim \phi(B)$ for every pair $A, B \in \mathcal{S}$, and $\phi(A) \approx A$

for all $A \in \mathcal{S}$. In other words, ϕ acts like the identity outside $\mathcal{S}$ and it maps every equivalence class $[A] \in \mathcal{S}/_\sim$ bijectively onto the equivalence class $[\phi(A)]$ with the property $\phi(A) \approx A$. Moreover, the correspondence between equivalence classes $[A]$ and $[\phi(A)]$ induced by the map ϕ is a bijection of the set $\mathcal{S}/_\sim$ onto itself. Such a map obviously preserves commutativity on the whole set $S_n(\mathbb{C})$ but does not need to be of the nice form given in Theorem 2.1.

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