

Finsleroid–Finsler space and geodesic spray coefficients

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Abstract. Starting with particular spray coefficients, we demonstrate how the Landsberg condition can explicitly appear in case of the Finsleroid-type metric function. Calculations are supplementing by a convenient special Maple-program. The general form of the associated geodesic spray coefficients is presented for such metric functions under the condition of constancy of the Finsleroid charge.

1. Introduction

Great parts of many works on the Finsler Geometry (see the books [1]–[3]) deal with the concept of spray. We need the spray coefficients when deriving the geodesic equations, the connection coefficients, the curvature tensors, the geodesic deviation equations, etc. Various ideas of applications of the Finslerian metrics (see [4]–[10]) need also convenient sets of the spray coefficients to succeed in developing handy equations. At the same time, it seems also useful to have the spray coefficients which belong to the Landsberg type (consult [1]–[3, [14]) of Finsler metric function.

Continuing the previous work [10]–[13] dealt with the Finsleroid–Finsler spaces, we below clarify (in a transparent way avoiding any clutter of calculations) how the spray notion may entail the Landsberg-type Finsler space. Any dimension number $N \geq 2$ is embraced by our consideration.

A *spray* \mathbf{G} on an N -dimensional smooth manifold M is a smooth vector field on the slit tangent bundle $TM \setminus 0$ expressed in terms of a standard local coordinate

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system (x^i, y^i) in TM according to the representation

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - G^i(y) \frac{\partial}{\partial y^i}. \quad (1.1)$$

Spray spaces are generalized vector spaces which deep meaning is underlined by the property that a spray \mathbf{G} on M determines a collection of geodesics in M , according to the differential equation

$$\frac{d^2 c^i}{dt^2} + G^i \left(\frac{dc}{dt} \right) = 0 \quad (1.2)$$

for curves $c : (a, b) \rightarrow M$ parametrized by t . The theory of sprays bears close relation on the path spaces. The *Finsler geodesic spray* is the notion which is the adaptation of the general spray notion to the structure of Finsler spaces, by using the Finslerian Christoffel symbols γ^k_{ij} and prescribing the equality

$$G^k_{\{\text{Finsler}\}} = \gamma^k_{ij} y^i y^j. \quad (1.3)$$

On the basis of these coefficients the Finsler connection and curvature can consistently be constructed by following known methods (see [1]–[3]).

Suppose we are given on M a Riemannian metric $S_N = S(x, y)$ and a 1-form $b = b(x, y)$ of the unit Riemannian length. With respect to local coordinates x^i in the Riemannian space $R_N = (M, S_N)$ we have the local representations $b = b_i(x) y^i$, $S = \sqrt{a_{ij}(x) y^i y^j}$, and

$$\|b\|_x := \sqrt{a^{ij}(x) b_i b_j} = 1 \quad (1.4)$$

with the tensor a^{ij} reciprocal to the input a_{ij} . We shall construct from the covariant vector b_i the contravariant vector b^i according to the Riemannian rule $b^i = a^{ij} b_j$.

We also introduce the tensor

$$r_{ij}(x) := a_{ij}(x) - b_i(x) b_j(x), \quad (1.5)$$

obtaining the decomposition

$$S^2 = b^2 + q^2 \quad (1.6)$$

in terms of the scalar

$$q := \sqrt{r_{ij}(x) y^i y^j}. \quad (1.7)$$

In many cases it is convenient to use the variables

$$u_i := a_{ij}y^j, \tag{1.8}$$

$$v^i := y^i - bb^i, \quad v_m := u_m - bb_m = r_{mn}y^n \equiv r_{mn}v^n \equiv a_{mn}v^n. \tag{1.9}$$

Notice that

$$r^i_n := a^{im}r_{mn} = \delta^i_n - b^ib_n = \frac{\partial v^i}{\partial y^n} \tag{1.10}$$

(δ^i_n stands for the Kronecker symbol),

$$v_ib^i = v^ib_i = 0, \quad r_{ij}b^j = r^i_jb^j = b_ir^i_j = 0, \tag{1.11}$$

$$u_iv^i = v_iy^i = q^2, \tag{1.12}$$

and

$$\frac{\partial b}{\partial y^i} = b_i, \quad \frac{\partial q}{\partial y^i} = \frac{v_i}{q}. \tag{1.13}$$

We comply with the notation adopted in [12], [13].

Under these conditions, it seems attractive to take three scalars $c_1(x)$, $c_2(x)$, $c_3(x)$ and propose to consider on the Riemannian space $R_N = (M, S_N)$ the spray given by the coefficients

$$G^i = c_1(x)\frac{1}{q}y^jy^h(\nabla_jb_h)v^i + c_2(x)y^hb^j(\nabla_jb_h)v^i + c_3(x)qf^i + a^i_{km}y^ky^m, \tag{1.14}$$

which are such that the difference $G^i - a^i_{km}y^ky^m$ involves all the crucial terms linear in ∇_jb_h . Here, the nabla means the covariant derivative in terms of the Riemannian space $R_N = (M, S_N)$; a^i_{nm} stands for the Riemannian Christoffel symbols constructed from the tensor $a_{ij}(x)$; the notation

$$f^i = f^i_ny^n, \quad f^i_n = a^{ik}f_{kn}, \quad f_{mn} = \nabla_m b_n - \nabla_n b_m \equiv \frac{\partial b_n}{\partial x^m} - \frac{\partial b_m}{\partial x^n} \tag{1.15}$$

is used.

In the previous work [10]–[13], the notion of the Finsleroid–Finsler space have been formulated and the necessary and sufficient conditions for the space to be of the Landsberg type have been found. In Section 2 we consider particular G^i which reveal the astonishing property of the nullification (2.11)–(2.12) for contractions. We call them the Landsberg-type spray coefficients, because, under the structural condition (2.13), Finsler metric functions inducing such coefficients must produce the Landsberg-type spaces.

In Section 3 we demonstrate how the use of the generating Finsleroid–Finsler metric functions induces explicitly such coefficients G^i .

In Section 4 the general representation of the geodesic spray coefficients in case of the Finsleroid–Finsler space with $g = \text{const}$ is given. The representation is obviously the kernel from which all the significant spray implications in such spaces are to be grown up.

In Section 5 we present the Maple-program which verifies the Landsberg-type spray coefficients.

The paper ends by Appendix A in which the basic formulas and definitions of the Finsleroid–Finsler space are summarized up.

Key Propositions 1, 2, 3 are motivated and proven.

2. Landsberg-type spray coefficients

Under the condition

$$\nabla_j b_i = k(a_{ij} - b_i b_j) \quad (2.1)$$

with $k = k(x)$, the coefficients (1.14) reduce to

$$G^i = cq(y^i - bb^i) + a^i_{km} y^k y^m \equiv gqk v^i + a^i_{km} y^k y^m, \quad (2.2)$$

where $c = c(x)$ is the scalar that is obtained by

$$c = c_1 k. \quad (2.3)$$

From (2.2) we can readily calculate the entailed coefficients

$$G^i_k := \frac{\partial G^i}{\partial y^k}, \quad G^i_{km} := \frac{\partial G^i_k}{\partial y^m}, \quad G^i_{kmn} := \frac{\partial G^i_{km}}{\partial y^n} \quad (2.4)$$

by applying the rules (1.13), obtaining the representations

$$G^i_k = \frac{c}{q} [(y^i - bb^i)(u_k - bb_k) + q^2(\delta_k^i - b_k b^i)] + 2a^i_{km} y^m, \quad (2.5)$$

$$G^i_{km} = \frac{c}{q} \left[(a_{km} - b_k b_m)(y^i - bb^i) - \frac{1}{q^2} (y^i - bb^i)(u_k - bb_k)(u_m - bb_m) \right. \\ \left. + (u_m - bb_m)(\delta_k^i - b_k b^i) + (u_k - bb_k)(\delta_m^i - b_m b^i) \right] + 2a^i_{km}, \quad (2.6)$$

and

$$\begin{aligned}
 G^i_{kmn} = & -\frac{c}{q^3}(u_n - bb_n) \left[(a_{km} - b_k b_m)v^i - \frac{1}{q^2}v^i(u_k - bb_k)(u_m - bb_m) \right. \\
 & \left. + (u_m - bb_m)(\delta_k^i - b_k b^i) + (u_k - bb_k)(\delta_m^i - b_m b^i) \right] \\
 & + \frac{c}{q} \left[(a_{km} - b_k b_m)(\delta_n^i - b_n b^i) + \frac{2}{q^4}(u_n - bb_n)v^i(u_k - bb_k)(u_m - bb_m) \right. \\
 & - \frac{1}{q^2}((\delta_n^i - b_n b^i)(u_k - bb_k)(u_m - bb_m) + v^i(a_{kn} - b_k b_n)(u_m - bb_m) \\
 & + v^i(u_k - bb_k)(a_{mn} - b_m b_n)) \\
 & \left. + (a_{mn} - b_m b_n)(\delta_k^i - b_k b^i) + (a_{kn} - b_k b_n)(\delta_m^i - b_m b^i) \right].
 \end{aligned}$$

The explicitly symmetric form of the latter coefficients reads

$$\begin{aligned}
 G^i_{kmn} = & \frac{3c}{q^5}v^i v_k v_m v_n - \frac{c}{q^3} \left[(\delta_k^i - b_k b^i)v_m v_n + (\delta_m^i - b_m b^i)v_k v_n + (\delta_n^i - b_n b^i)v_k v_m \right. \\
 & \left. + v^i((a_{km} - b_k b_m)v_n + (a_{kn} - b_k b_n)v_m + (a_{mn} - b_m b_n)v_k) \right] \\
 & + \frac{c}{q} \left[(\delta_k^i - b_k b^i)(a_{mn} - b_m b_n) + (\delta_m^i - b_m b^i)(a_{kn} - b_k b_n) \right. \\
 & \left. + (\delta_n^i - b_n b^i)(a_{km} - b_k b_m) \right], \tag{2.7}
 \end{aligned}$$

or

$$G^i_{kmn} = \frac{c}{q}(\eta_k^i \eta_{mn} + \eta_m^i \eta_{kn} + \eta_n^i \eta_{km}), \tag{2.8}$$

where the η -tensors are given by

$$\eta^i_j := r^i_j - \frac{1}{q^2}v^i v_j, \quad \eta_{ij} := r_{ij} - \frac{1}{q^2}v_i v_j \equiv a_{in} \eta^n_j \tag{2.9}$$

(the formulas (1.7)–(1.10) have been used). Because of the nullifications

$$b_i \eta^i_j = u_i \eta^i_j = 0 \tag{2.10}$$

(see the formulas (1.11)–(1.12)) the obtained coefficients (2.8) fulfill the identities

$$b_i G^i_{kmn} = 0 \tag{2.11}$$

and

$$u_i G^i_{kmn} = 0. \tag{2.12}$$

Suppose a Finslerian metric function $F(x, y)$ be obtainable from a function

$\check{F}(S, b)$ of the 1-form b and the Riemannian metric function S , such that

$$F(x, y) = \check{F}(S(x, y), b(x, y)). \tag{2.13}$$

Then it is obvious that covariant vectors $\{y_i\}$ produced by the function F according to the conventional Finsler rule $y_i = \frac{1}{2} \partial F^2(x, y) / \partial y^i$ are linear combinations of b_i and u_i , that is, the equality

$$y_i = p_1 b_i + p_2 u_i \tag{2.14}$$

holds with two scalars p_1, p_2 . Noting the vanishings (2.11) and (2.12), we are justified to claim the following.

Proposition 1. *Suppose a Finsler metric function $F(x, y)$ entail the spray coefficients of the form (2.2). If also the function F is of the structure (2.13), then the function F produces the identity*

$$y_i G^i_{kmn} = 0 \tag{2.15}$$

and, hence, a Landsberg-case Finsler space.

Because of this observation, we introduce the following.

Definition. The coefficients G^i given by the representation (2.2) are called the *Landsberg-type spray coefficients*.

In the two-dimensional case,

$$N = 2, \tag{2.16}$$

such a unit 1-form $e = e_j(x) y^j$ exists that

$$a_{ij} = e_i e_j + b_i b_j, \tag{2.17}$$

whence the definitions (1.5), (1.7), and (1.9) reduce to

$$r_{ij} = e_i e_j, \tag{2.18}$$

$$q = |e|, \tag{2.19}$$

$$v^i = e e^i, \tag{2.20}$$

the η -tensors (2.9) vanish

$$\eta_k^i = \eta_{kn} = 0, \tag{2.21}$$

and the implication

$$(N = 2) \rightarrow G^i_{kmn} = 0 \tag{2.22}$$

is applicable to (2.8) *independently of the value of the scalar c*. Therefore, in the dimension $N = 2$ the coefficients (2.2), (2.5), and (2.6) reduce to

$$G^i = ce|e|e^i + a^i_{km}y^ky^m, \tag{2.23}$$

$$G^i_k = 2c|e|e^ie_k + 2a^i_{km}y^m, \tag{2.24}$$

and

$$G^i_{km} = \frac{2ce}{|e|}e^ie_ke_m + 2a^i_{km}. \tag{2.25}$$

The latter coefficients are *independent* of vectors y , thereby corresponding to the Berwald case.

The formula (2.2) obtained for the spray coefficients, as well as the very condition (2.1), is applicable *in any* dimension $N \geq 2$. The right-hand side of the formula involves q which is a square root of a quadratic form of rank $N - 1$, so that in the dimensions $N \geq 3$ the coefficients can not be quadratic in vectors y (unless the Riemannian case occurs), – this note may be regarded as the reason proper why the Landsberg case treated does not degenerate to the Berwald case at $N \geq 3$. In the dimension $N = 2$, however, the quadratic form mentioned is a square of the 1-form e introduced above, hence the square root is extracted up (see (2.19)), leaving us with the expression (2.23) quadratic in vectors y , that is with the Berwald case. In the Finsleroid–Finsler space of the dimension $N = 2$ the associated main scalar I proves to be $I = I(x) = |g(x)|$ (cf. p. 26 in [12]).

3. Use of generating metric functions in the Finsleroid–Finsler case

Let us inquire into whether the Finsleroid–Finsler metric function K with $g = \text{const}$ may fulfil the conditions which underlined Proposition 1.

Accordingly, we use a constant g ranging over $-2 < g < 2$, together with the notation

$$h = \sqrt{1 - \frac{1}{4}g^2}, \quad G = g/h. \tag{3.1}$$

The respective Finsleroid–Finsler metric function K does belong to the class (2.13).

Put $w = q/b$ whenever $b \neq 0$ and rewrite the function K in the form

$$K = bV(w), \tag{3.2}$$

where the *generating metric function* $V(w)$ is smooth of the class C^∞ on all the region

$$I_w = (-\infty, \infty). \tag{3.3}$$

In terms of the quadratic form

$$Q(w) = 1 + gw + w^2 \quad (3.4)$$

we have

$$V(w) = \sqrt{Q(w)} e^{\frac{1}{2}G\Phi(w)}, \quad (3.5)$$

where

$$\Phi(w) = \frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \left(\frac{w + \frac{g}{2}}{h} \right), \quad \text{if } b \geq 0, \quad (3.6)$$

and

$$\Phi(w) = -\frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \left(\frac{w + \frac{g}{2}}{h} \right), \quad \text{if } 0 \geq b, \quad (3.7)$$

We obtain

$$V' = wV/Q, \quad V'' = V/Q^2, \quad (3.8)$$

$$(V^2/Q)' = -gV^2/Q^2, \quad (V^2/Q^2)' = -2(g+w)V^2/Q^3, \quad \Phi' = -h/Q \quad (3.9)$$

and also

$$\begin{aligned} \frac{1}{2}(V^2)' &= wV^2/Q, & \frac{1}{2}(V^2)'' &= (Q - gw)V^2/Q^2, \\ \frac{1}{4}(V^2)''' &= -gV^2/Q^3, \end{aligned} \quad (3.10)$$

where the prime (') denotes the differentiation with respect to w .

If, alternatively, we use the variable $s = b/S$ and consider the function K to read

$$K = S\phi(s), \quad (3.11)$$

we obtain the *generating metric function* $\phi(s)$ which is smooth of the class C^∞ on the interval

$$I_s = (-1, 1), \quad (3.12)$$

with

$$\phi(s) = \sqrt{1 + gs\sqrt{1 - s^2}} e^{\frac{1}{2}G\Phi(s)}, \quad (3.13)$$

where

$$\Phi(s) = \frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \left(\frac{\sqrt{1-s^2} + \frac{g}{2}s}{hs} \right), \quad \text{if } 1 > s \geq 0, \quad (3.14)$$

and

$$\Phi(s) = -\frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \left(\frac{\sqrt{1-s^2} + \frac{g}{2}s}{hs} \right), \quad \text{if } 0 \geq s > -1, \quad (3.15)$$

The limits at $s = 0$ from the left and from the right are the same value

$$\phi(0) = e^{\frac{1}{2}G \arctan(G/2)}.$$

Evaluating derivatives yields merely

$$\phi' = \frac{g\sqrt{1-s^2} e^{\frac{1}{2}G\Phi(s)}}{\sqrt{1+gs\sqrt{1-s^2}}}, \quad \phi'' = -\frac{gs e^{\frac{1}{2}G\Phi(s)}}{\sqrt{1-s^2}(\sqrt{1+gs\sqrt{1-s^2}})^3}, \quad (3.16)$$

where the prime ($'$) denotes the differentiation with respect to the variable s . Singularities appear when $|s| \rightarrow 1$.

Using each of the two generating metric functions, $V(w)$ or $\phi(s)$, it is easy to observe that with the symmetry assumption

$$\nabla_j b_i = \nabla_i b_j \quad (3.17)$$

the Finsler spray coefficients $G^i = \gamma^i_{nm} y^n y^m$ (see (1.3)) prove to be of the explicit form

$$G^i = \frac{gy^m y^n \nabla_n b_m}{\sqrt{S^2 - b^2}} (y^i - bb^i) + a^i_{nm} y^n y^m. \quad (3.18)$$

If we plug here the condition (2.1), we obtain the spray coefficients

$$G^i = gk\sqrt{S^2 - b^2} (y^i - bb^i) + a^i_{nm} y^n y^m \quad (3.19)$$

which are tantamount to the Landsberg-type spray coefficients (2.2).

This way we have arrived at the following.

Proposition 2. *Under the assumption (2.1), the Finsleroid–Finsler metric function K with $g = \text{const}$ induces the Landsberg-type spray coefficients. The entailed coefficients G^i_{kmn} are of the simple form that is given by (2.8), with*

$$c = kg. \quad (3.20)$$

The Berwald case corresponds to $k = 0$ in dimensions $N \geq 3$, and holds uniquely in dimension $N = 2$.

4. General form of Finsleroid–Finsler geodesic spray coefficients

Straightforward calculations of the Finsleroid–Finsler Christoffel symbols γ_{ikj} (on the basis of the representation (A.15) in Appendix below) results in the following representation:

$$\begin{aligned}
2\gamma_{ikj} = & \frac{\partial g_{kj}}{\partial g} \frac{\partial g}{\partial x^i} + \frac{\partial g_{ik}}{\partial g} \frac{\partial g}{\partial x^j} - \frac{\partial g_{ij}}{\partial g} \frac{\partial g}{\partial x^k} \\
& + \frac{2gb^2}{Bq} (c_i g_{kj} + c_j g_{ik} - c_k g_{ij}) + \frac{g}{B} \left(q - \frac{b^2}{q} \right) (c_i a_{kj} + c_j a_{ik} - c_k a_{ij}) \frac{K^2}{B} \\
& - \frac{g \frac{K^2}{B}}{B} \left[c_i \left[\left(2gb + \frac{S^2}{q} \right) b_k b_j + \frac{1}{q^2} \frac{S^2}{q} (b^2 b_k b_j - bb_k u_j - bb_j u_k + u_k u_j) \right] \right. \\
& + \left. \left(q + \frac{b^2}{q} \right) (b_{k,i} (bb_j - u_j) + b_{j,i} (bb_k - u_k)) - gq^2 (b_{k,i} b_j + b_{j,i} b_k) \right] \\
& - \frac{g \frac{K^2}{B}}{B} \left[c_j \left[\left(2gb + \frac{S^2}{q} \right) b_k b_i + \frac{1}{q^2} \frac{S^2}{q} (b^2 b_k b_i - bb_k u_i - bb_i u_k + u_k u_i) \right] \right. \\
& + \left. \left(q + \frac{b^2}{q} \right) (b_{k,j} (bb_i - u_i) + b_{i,j} (bb_k - u_k)) - gq^2 (b_{k,j} b_i + b_{i,j} b_k) \right] \\
& + \frac{g \frac{K^2}{B}}{B} \left[c_k \left[\left(2gb + \frac{S^2}{q} \right) b_i b_j + \frac{1}{q^2} \frac{S^2}{q} (b^2 b_i b_j - bb_i u_j - bb_j u_i + u_i u_j) \right] \right. \\
& + \left. \left(q + \frac{b^2}{q} \right) (b_{i,k} (bb_j - u_j) + b_{j,k} (bb_i - u_i)) - gq^2 (b_{i,k} b_j + b_{j,k} b_i) \right] + \Delta. \quad (4.1)
\end{aligned}$$

Here, $S^2 = b^2 + q^2$, $u_i = a_{ij} y^j$, $b_{j,k} = \partial b_j / \partial x^k$, $c_i = y^k b_{k,i}$, and Δ symbolizes the summary of the terms which involve partial derivatives of the input Riemannian metric tensor a_{ij} with respect to the coordinate variables x^k . By contracting we find

$$\begin{aligned}
\gamma_{ikj} y^i y^j = & \frac{\partial y_k}{\partial g} \frac{\partial g}{\partial x^i} y^i - \frac{1}{2} \frac{\partial K^2}{\partial g} \frac{\partial g}{\partial x^k} \\
& + \frac{gS^2}{Bq} (c_j y^j) y_k - gq \frac{K^2}{B} c_k + gq \frac{K^2}{B} b_{ki} y^i - \frac{g \frac{K^2}{B} b}{q} (c_i y^i) b_k + \Delta y \quad (4.2)
\end{aligned}$$

and

$$\gamma_{ikj} y^i y^j y^k = \frac{1}{2} \frac{\partial K^2}{\partial g} \frac{\partial g}{\partial x^k} y^k + gq (c_i y^i) \frac{K^2}{B} + y \Delta. \quad (4.3)$$

Using this result, we obtain after due calculations (applying the representation (A.16) of Appendix A below) the following.

Proposition 3. *In the Finsleroid–Finsler space under the only condition that the Finsleroid charge is a constant, $g = \text{const}$, the induced spray coefficients $G^i = \gamma^i_{nm} y^n y^m$ can explicitly be written in the form (1.14) with*

$$c_1 = g, \quad c_2 = g^2, \quad c_3 = -g, \quad (4.4)$$

so that

$$G^i = g \left(\frac{1}{q} y^j y^h \nabla_j b_h + g y^h b^j \nabla_j b_h \right) v^i - g q f^i + a^i_{km} y^k y^m. \quad (4.5)$$

We have here $f^i = 0$ if the symmetry (3.17) is assumed (see (1.15)). If the condition (2.1) is plugged in (4.5), the spray coefficients (2.2) appear with c given by (3.20).

It is remarkable to note that the Finsleroid–Finsler metric function K does not enter the right-hand side of (4.5). The presence of the constant g in the right-hand side of (4.5) is the only trace of the function K in the spray coefficients G^i obtained.

5. Maple-verification

Below we check the vanishing $\dot{A}_{jkl} = 0$ by the resource of the Maple10, using the formulas

$$G^i = c q (y^i - b b^i) + a^i_{km} y^k y^m, \quad \dot{A}_{jkl} = -\frac{1}{4} y_i \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} \quad (5.1)$$

with $q = \sqrt{r_{ij} y^i y^j}$ and $b = b_i y^i$; c is independent of y . In the program, G^i will be denoted by `gammas[i]`, \dot{A}_{jkl} by `dotA[j,k,l]`, and b by `bs`.

```
> restart:
> N:=2:q:=sqrt(add(add(r[i,j]*y[i]*y[j], j=1..N), i=1..N)):
  bs:=add(b[i]*y[i], i=1..N):
  for i from 1 to N do
    c*q*(y[i]-bs*b[i])+add(add(a[i,j,k]*y[j]*y[k], j=1..N), k=1..N):
    gammas[i]:=eval(%, {seq(seq(r[i,j]=r[j,i], i=1..j), j=1..N)});
  end do:
Apply  $y_i = p_1 b_i + p_2 r_{ij} y^j$  (equation (2.14)) with arbitrary  $p_1$  and  $p_2$ .
> for j from 1 to N do for k from 1 to N do for l from 1 to N do
  dotA[j,k,l]:=-1/4*factor(add((add(r[i,a]*y[a]*p2, a=1..N)+b[i]*p1)
  *diff(diff(diff(gammas[i], y[l]), y[k]), y[j]), i=1..N));
end do: end do: end do:
```

Plug the symmetry $r_{ij} = r_{ji}$ and simplify the arisen quantities \dot{A}_{jkl} by the use of the constrains $b_i b^i = 1$ and $r_{ij} b^j = 0$.

```
> for a1 from 1 to N do for a2 from 1 to N do for a3 from 1 to N do
  dotA[a1,a2,a3]:factor(eval(%,{seq(seq(r[i,j]=r[j,i],i=1..j),j=1..N)})):
  algsubs(add(b[i]^2,i=1..N)=1,%):factor(%);
  for j from 1 to N do
    algsubs(eval(add(r[i,j]*b[i],i=1..N)=0,
      {seq(seq(r[i,j]=r[j,i],i=1..j),j=1..N)}),%);end do:
  simplify(%);print(%);end do:end do:end do:
```

The result of the simplification is just the succession of zeros:

0
0
0
0
0
0
0
0
0
0

This result supports Proposition 1 of Section 2.

The calculation times at the dimensions $N = 2$ and $N = 3$ are short.

Appendix A: Involved Finsleroid–Finsler representations

We introduce on the manifold M a scalar $g = g(x)$ subject to ranging

$$-2 < g(x) < 2, \quad (\text{A.1})$$

and apply the convenient notation

$$h = \sqrt{1 - \frac{1}{4}g^2}, \quad G = g/h. \quad (\text{A.2})$$

The *characteristic quadratic form*

$$B(x, y) := b^2 + gqb + q^2 \equiv \frac{1}{2} [(b + g_+q)^2 + (b + g_-q)^2] > 0 \quad (\text{A.3})$$

where $g_+ = \frac{1}{2}g + h$ and $g_- = \frac{1}{2}g - h$, is of the negative discriminant

$$D_{\{B\}} = -4h^2 < 0 \tag{A.4}$$

and, therefore, is positively definite.

Definition. The scalar function $K(x, y)$ given by the formulas

$$K(x, y) = \sqrt{B(x, y)} J(x, y) \tag{A.5}$$

and

$$J(x, y) = e^{\frac{1}{2}G\Phi(x, y)}, \tag{A.6}$$

where

$$\Phi(x, y) = \frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \left(\frac{L(x, y)}{hb} \right), \quad \text{if } b \geq 0, \tag{A.7}$$

and

$$\Phi(x, y) = -\frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \left(\frac{L(x, y)}{hb} \right), \quad \text{if } b \leq 0, \tag{A.8}$$

with

$$L(x, y) = q + \frac{g}{2}b, \tag{A.9}$$

is called the *Finsleroid–Finsler metric function*.

The positive (not absolute) homogeneity holds fine: $K(x, \lambda y) = \lambda K(x, y)$ for all $\lambda > 0$.

In the limit $g \rightarrow 0$, the definition degenerates to the input Riemannian metric function:

$$K|_{g=0} = S. \tag{A.10}$$

Definition. The arisen space

$$FF_g^{PD} := \{R_N; b(x, y); g(x); K(x, y)\} \tag{A.11}$$

is called the *Finsleroid–Finsler space*.

Definition. The space R_N entering the above definition is called the *associated Riemannian space*.

Definition. Within each tangent space T_xM , the Finsleroid-metric function $K(x, y)$ produces the *Finsleroid*

$$F_g^{PD}_{\{x\}} := \{y \in F_g^{PD}_{\{x\}} : y \in T_xM, K(x, y) \leq 1\}. \tag{A.12}$$

We calculate from the function K the covariant tangent vector $\hat{y} = \{y_i\}$ and the Finslerian metric tensor $\{g_{ij}\}$, by making use of the conventional Finslerian rules

$$y_i := \frac{1}{2} \frac{\partial K^2}{\partial y^i}, \quad g_{ij} := \frac{1}{2} \frac{\partial^2 K^2}{\partial y^i \partial y^j} = \frac{\partial y_i}{\partial y^j}, \quad (\text{A.13})$$

obtaining

$$y_i = (a_{ij}y^j + gqb_i) \frac{K^2}{B} \quad (\text{A.14})$$

and

$$g_{ij} = \left[a_{ij} + \frac{g}{B} \left((gq^2 - \frac{bS^2}{q})b_i b_j - \frac{b}{q}u_i u_j + \frac{S^2}{q}(b_i u_j + b_j u_i) \right) \right] \frac{K^2}{B}, \quad (\text{A.15})$$

where the notation (1.8) has been used. The reciprocal components $(g^{ij}) = (g_{ij})^{-1}$ read

$$g^{ij} = \left[a^{ij} + \frac{g}{q}(bb^i b^j - b^i y^j - b^j y^i) + \frac{g}{Bq}(b + gq)y^i y^j \right] \frac{B}{K^2}, \quad (\text{A.16})$$

In terms of the variables (1.9) we obtain the representations

$$y_i = \left(v_i + (b + gq)b_i \right) \frac{K^2}{B}, \quad (\text{A.17})$$

$$g_{ij} = \left[a_{ij} + \frac{g}{B} \left(q(b + gq)b_i b_j + q(b_i v_j + b_j v_i) - b \frac{v_i v_j}{q} \right) \right] \frac{K^2}{B}, \quad (\text{A.18})$$

and

$$g^{ij} = \left[a^{ij} + \frac{g}{B} \left(-bqb^i b^j - q(b^i v^j + b^j v^i) + (b + gq) \frac{v^i v^j}{q} \right) \right] \frac{B}{K^2} \quad (\text{A.19})$$

which are alternative to (A.14)–(A.16).

The determinant of the metric tensor is the smooth and positive function as follows:

$$\det(g_{ij}) = \left(\frac{K^2}{B} \right)^N \det(a_{ij}) > 0. \quad (\text{A.20})$$

For the component of the contracted Cartan tensor we find

$$A_i = \frac{NK}{2} g \frac{1}{q} \left(b_i - \frac{b}{K^2} y_i \right) = \frac{NK}{2} g \frac{1}{qB} (q^2 b_i - b v_i). \quad (\text{A.21})$$

Since

$$\frac{v^i v^j}{q} \rightarrow 0 \quad \text{when} \quad v^i \rightarrow 0 \tag{A.22}$$

(notice the definition (1.7) of q) the components y_i , g_{ij} , and g^{ij} , as given by (A.17), (A.18), and (A.19), are smooth on all the slit tangent bundle $TM \setminus 0$. However, the components (A.21) are singular at $v^i = 0$. Therefore, on $TM \setminus 0$ the Finsleroid–Finsler space is smooth of the class C^2 and not of the class C^3 , and at the same time the space is smooth of the class C^∞ on $TM \setminus \{0, -b, b\}$.

We use the Riemannian Christoffel symbols

$$a^k{}_{ij} := \frac{1}{2} a^{kn} (\partial_j a_{ni} + \partial_i a_{nj} - \partial_n a_{ji}) \tag{A.23}$$

($\partial_j = \partial/\partial x^j$) given rise to by the associated Riemannian metric S_N , and also the Finslerian Christoffel symbols

$$\gamma^k{}_{ij} := g^{kn} \gamma_{inj} \tag{A.24}$$

with

$$\gamma_{inj} := \frac{1}{2} (\partial_j g_{ni} + \partial_i g_{nj} - \partial_n g_{ji}). \tag{A.25}$$

In terms of the tensors

$$H_{mn} = \eta_{mn} \frac{K^2}{B}, \quad H_k{}^i = \eta_k{}^i \tag{A.26}$$

(cf. equations (A.11) and (A.16) in [12]) the coefficients given by (2.8) take on the form

$$G^i{}_{kmn} = \frac{c}{q} (H_k{}^i H_{mn} + H_m{}^i H_{kn} + H_n{}^i H_{km}) \frac{B}{K^2}, \tag{A.27}$$

and if we lower here the first index, we obtain the *totally symmetric coefficients*

$$G_{ikmn} = \frac{c}{q} (H_{ik} H_{mn} + H_{im} H_{kn} + H_{in} H_{km}). \tag{A.28}$$

The vanishings

$$y^i G_{ikmn} = 0, \quad b^i G_{ikmn} = 0, \quad A^i G_{ikmn} = 0 \tag{A.29}$$

hold.

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