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Hardy spaces and convergence of vector-valued Vilenkin–Fourier series

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Abstract. The atomic decomposition of a vector-valued martingale Hardy space is given. A classical inequality of Marcinkiewicz is generalized for UMD lattice valued (bounded) Vilenkin–Fourier series. It is proved that the Vilenkin–Fourier series of $f \in L_p(X)$ $(1 < p < \infty)$ converges to f in $L_p(X)$ norm if and only if X is a UMD space. Moreover, a lacunary sequence of the UMD lattice valued Vilenkin–Fourier series of $f \in H_1(X)$ converges almost everywhere to f in X norm.

1. Introduction

For trigonometric and Walsh–Fourier series the partial sum operators are bounded on L_p $(1 < p < \infty)$ spaces. An ℓ_r-valued version of this theorem is due to Marcinkiewicz and Zygmund for trigonometric Fourier series (see e.g. Zygmund [28, II. p. 225]), to Sunouchi [19] for Walsh–Fourier series and to Young [27] for Vilenkin–Fourier series.

Ladhawala and Pankratz [9] (see also Weisz [24]) proved that if f is in the dyadic Hardy space H_1 and $(n_k, k \in \mathbb{N})$ is a lacunary sequence of positive integers, then $s_{n_k} f$, the partial sums of the Walsh–Fourier series of f, converges a.e. to f. Moreover, Schipp and Simon [17] verified that if $\Phi(u) = o(\log\log u)$ $(u \to \infty)$ then there exists a function in $H_1\Phi(H_1)$ whose full sequence of partial sums

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diverges everywhere. Especially, if $\Phi(u) = 1$ $(u \geq 1)$ then we get $H_1\Phi(H_1) = H_1$, i.e. it follows the existence of $f \in H_1$ such that $s_n f$ diverges everywhere (see Ladhawala and Pankratz [9]). The analogous results for trigonometric Fourier series can be found in Zygmund [28, II. p. 235] and for Vilenkin–Fourier series in Young [26].

In this paper we extend these results to vector-valued, more exactly to UMD space valued Walsh- and Vilenkin–Fourier series. The UMD (unconditionality property of martingale differences) Banach spaces were introduced by Burkholder [2]. Since that time these spaces itself and their applications to Fourier analysis has been studied very intensively in the literature (e.g. Burkholder [3], [4], Rubio de Francia [15], [16], Tozoni [20], [21], Mishura and Weisz [13], [14], Martinez and Torrea [12] and Girardi and Weis [8]). Hardy spaces of scalar-valued martingales are investigated in the books Long [11] and Weisz [23].

Here we consider Walsh and Vilenkin martingales and give the atomic decomposition of a Banach space valued martingale Hardy space. We generalize the Marcinkiewicz inequality on partial sums for UMD space valued (bounded) Vilenkin–Fourier series. From this it follows that if X is a UMD space then the X valued Vilenkin–Fourier series of $f \in L_p(X)$ $(1 < p < \infty)$ converges to f in norm. The converse is also true: if the Vilenkin–Fourier series converges in $L_p(X)$ norm then X is a UMD space. For Walsh–Fourier series this was proved in Wenzel [25] and Tozoni [20].

It is known that if $f \in L_p(X)$ $(1 < p < \infty)$ and X is UMD then $s_n f \to f$ a.e. in X norm (see Rubio de Francia [15] for trigonometric Fourier series and Weisz [22] for Vilenkin–Fourier series). Finally, we extend this result to Hardy spaces, more exactly we prove that if f is in the Hardy space $H_1(X)$ and $(n_k, k \in \mathbb{N})$ is a lacunary sequence of positive integers, then the partial sums of the Vilenkin–Fourier series $s_{n_k} f$ converge a.e. to f in X norm. In the proofs of these results martingale techniques are used.

2. Vilenkin systems

In this paper we consider the unit interval $[0, 1)$, the σ-algebra $\mathcal{A}$ of the Borel sets and the Lebesgue measure λ. Let $(p_n, n \in \mathbb{N})$ be a sequence of natural numbers with entries at least 2. Introduce the notations $P_0 = 1$ and $P_{n+1} := \prod_{k=0}^{n} p_k$ $(n \in \mathbb{N})$. Every point $x \in [0, 1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{P_{k+1}}, \qquad 0 \leq x_k < p_k,\ x_k \in \mathbb{N}.$$

In case there are two different forms, we choose the one for which $\lim_{k\to\infty} x_k = 0$.

The functions

$$r_n(x) := \exp \frac{2\pi \imath x_n}{p_n} \qquad (n \in \mathbb{N})$$

are the *generalized Rademacher functions* where $\imath := \sqrt{-1}$. The product system generated by these functions is called a *Vilenkin system*:

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k}$$

where $n = \sum_{k=0}^{\infty} n_k P_k$, $0 \le n_k < p_k$ and $n_k \in \mathbb{N}$. If $p_n = 2$ for every $n \in \mathbb{N}$ then it is called *Walsh system*. In this paper we suppose that the Vilenkin system is *bounded*, i.e. the sequence (p_n) is bounded. For a detailed investigation of the Walsh- and Vilenkin systems see Schipp, Wade, Simon and Pál [18].

Let $\mathcal{F}_n$ be the σ-algebra generated by $\{r_0, \ldots, r_{n-1}\}$. It is easy to see that

$$\mathcal{F}_n = \sigma\{[kP_n^{-1}, (k+1)P_n^{-1}) : 0 \le k < P_n\}$$

where $\sigma(\mathcal{H})$ denotes the σ-algebra generated by an arbitrary set system $\mathcal{H}$. By a *Vilenkin interval* we mean one of the form $[kP_n^{-1}, (k+1)P_n^{-1})$ for some $k, n \in \mathbb{N}$, $0 \le k < P_n$.

For a Banach space X, the space $L_p(X)$ consists of all *strongly measurable functions* $f : [0,1) \to X$ for which

$$\|f\|_{L_p(X)} := \left(\int_0^1 \|f\|_X^p \, d\lambda \right)^{1/p} \qquad (0 < p \le \infty).$$

If $f \in L_p(X)$ $(p \ge 1)$ then the Bochner integral $\int_0^1 f \, d\lambda$ exists (see Diestel and Uhl [6] and Garcia-Cuerva and Rubio de Francia [7]). The expectation and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E and E_n, respectively. We investigate the class of X-valued (Vilenkin) *martingales* $f = (f_n, \in \mathbb{N})$ with respect to $(\mathcal{F}_n, \in \mathbb{N})$. For a stopping time $\nu : [0,1) \to \mathbb{N} \cup \{\infty\}$ the stopped martingale $(f_n^\nu, \in \mathbb{N})$ is defined by

$$f_n^\nu := \sum_{k=0}^{n} \mathbf{1}_{\{\nu \ge k\}} d_k f,$$

where $d_k f := f_k - f_{k-1}$, $f_{-1} := 0$.

If $f \in L_1(X)$ then $\hat{f}(n) := E(f\overline{w}_n)$ is said to be the nth *Vilenkin–Fourier coefficient* of f $(\in \mathbb{N})$. Denote by $s_n f$ the nth partial sum of the Vilenkin–Fourier series of f, namely,

$$s_n f := \sum_{k=0}^{n-1} \hat{f}(k) w_k.$$

It is easy to see that $(s_{P_n} f, \in \mathbb{N})$ is an X-valued martingale.

We will suppose that X is a *Banach lattice*. As usual, $|\cdot|$ will denote the absolute value in X: $|x| := \sup\{x, -x\}$. For more about Banach lattices see LINDENSTRAUSS and TZAFRIRI [10]. A Banach lattice X is a UMD (unconditionality property for martingale differences) space, if for all $1 < p < \infty$, all X-valued martingale difference sequences $(d_1, d_2, \ldots)$ and all numbers $\epsilon_1, \epsilon_2, \ldots \in \{-1, 1\}$ there exists a positive real number C_p such that

$$\Big\| \sum_{k=1}^{n} \epsilon_k d_k \Big\|_{L_p(X)} \le C_p \Big\| \sum_{k=1}^{n} d_k \Big\|_{L_p(X)} \qquad (\in \mathbb{N}) \tag{1}$$

(see BURKHOLDER [2]). It is enough to assume (1) for some $1 < p < \infty$ and for all X-valued martingale difference sequences $(d_1, d_2, \ldots)$ with respect to $(\mathcal{F}_n)$, because each $\mathcal{F}_n$ is atomic (see RUBIO DE FRANCIA [16] or GIRARDI and WEIS [8]).

The *maximal function* of an X-valued martingale $f = (f_n, \in \mathbb{N})$ is defined by

$$M_n f := \sup_{k \le n} \|f_k\|_X, \qquad Mf := \sup_{k \in \mathbb{N}} \|f_k\|_X.$$

The following theorem can be found in BOURGAIN [1], RUBIO DE FRANCIA [16] and TOZONI [21].

Theorem 1. *If X is a UMD lattice and $f \in L_p(X)$ then*

$$\rho \lambda(Mf > \rho) \le C \|f\|_{L_1(X)}, \qquad (\rho > 0)$$

and

$$\|f\|_{L_p(X)} \sim \|Mf\|_{L_p[0,1)} \sim \Big\| \Big(\sum_{n=0}^{\infty} |d_n f|^2 \Big)^{1/2} \Big\|_{L_p(X)} \sim \Big\| \Big(\sum_{n=0}^{\infty} E_{n-1} |d_n f|^2 \Big)^{1/2} \Big\|_{L_p(X)}$$

for all $1 < p < \infty$, where $\sim$ denotes the equivalence of the norms.

Note that the sequence $(\mathcal{F}_n)$ is regular. In this paper the positive constants C_p depend only on p and may denote different constants in different contexts.

3. Hardy spaces and atomic decomposition

The Hardy space $H_p(X)$ $(1 \le p \le \infty)$ consists of all X-valued martingales f for which

$$\|f\|_{H_p(X)} := \|Mf\|_{L_p(\mathbb{R})} < \infty.$$

By Theorem 1, if X is UMD then $H_p(X) \sim L_p(X)$ for all $1 < p < \infty$. Moreover, if $(f_n) \in H_p(X)$ for some $1 \le p < \infty$ then there exists $f \in L_p(X)$ such that $f = \lim_{n\to\infty} f_n$ in $L_p(X)$ norm and $f_n = E_n f$ (see e.g. Diestel and Uhl [6]).

The atomic decomposition is a useful characterization of Hardy spaces (for scalar valued martingales see e.g. Weisz [23]). Let us introduce first the concept of atoms. A function a is an *atom* if there exists a Vilenkin interval I such that

$$\int_I a\, d\lambda = 0, \qquad \|a\|_{L_\infty(X)} \le \lambda(I)^{-1}, \qquad \{a \ne 0\} \subset I.$$

Though the proof of the next atomic decomposition is similar to the scalar valued case, for the sake of completeness we present a short proof.

Theorem 2. *Assume that X is a UMD lattice. Then $f \in H_1(X)$ if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} \mu_k a^k = f \qquad \text{a.e. in } X \text{ norm and} \qquad \sum_{k=0}^{\infty} |\mu_k| < \infty. \tag{2}$$

Moreover,

$$\|f\|_{H_1(X)} \sim \inf \sum_{k=0}^{\infty} |\mu_k|$$

where the infimum is taken over all decompositions of f of the form (2).

Proof. Assume that $f \in H_1(X)$. Define the stopping time ν_k by

$$\nu_k(x) := \inf\{n \in \mathbb{N} : E_n \mathbf{1}_{\{M_{n+1}f > 2^k\}}(x) \ge 1/d\}, \qquad (k \in \mathbb{Z}),$$

where $d = \sup_n p_n$. From this it follows that $\nu_k \le \nu_{k+1}$, $(k \in \mathbb{Z})$,

$$\{Mf > 2^k\} \subset \{\nu_k < \infty\}, \qquad \lambda(\nu_k < \infty) \le d\lambda(Mf > 2^k) \tag{3}$$

and $M_{\nu_k} f \le 2^k$ for all $k \in \mathbb{Z}$, where $M_{\nu_k} f := M_n f$ if $\nu_k = n$. It is easy to see that

$$f = \sum_{k\in\mathbb{Z}} (f_{\nu_{k+1}} - f_{\nu_k}) \qquad \text{a.e. in } X \text{ norm.}$$

Indeed, $\lambda(\nu_k < \infty) \to 0$ by (3) and so $f_{\nu_{k+1}} \to f$ a.e. in X norm as $k \to \infty$ and $\|f_{\nu_k}\|_X \le 2^k \to 0$ as $k \to -\infty$. We decompose $\{\nu_k = l\} = \cup_n I_{k,n}^l$, where $I_{k,n}^l \in \mathcal{F}_l$ are Vilenkin intervals. If we define

$$\mu_{k,n}^l := 3 \cdot 2^k \lambda(I_{k,n}^l), \qquad a_{k,n}^l := (\mu_{k,n}^l)^{-1} \mathbf{1}_{I_{k,n}^l} (f_{\nu_{k+1}} - f_{\nu_k})$$

then

$$f = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{N}} \sum_n \mu_{k,n}^l a_{k,n}^l \qquad \text{a.e. in } X \text{ norm.} \tag{4}$$

Since $\nu_{k+1} \ge \nu_k = l$ on $I_{k,n}^l$, by the martingale property

$$\int_{I_{k,n}^l} (f_{\nu_{k+1}} - f_{\nu_k})\, d\lambda = \int_{I_{k,n}^l} (f_{\nu_{k+1}} - f_l)\, d\lambda = 0.$$

This and

$$\|a_{k,n}^l\|_X \le |\mu_{k,n}^l|^{-1} (\|f_{\nu_{k+1}}\|_X + \|f_{\nu_k}\|_X) \le \lambda(I_{k,n}^l)^{-1}$$

imply that $a_{k,n}^l$ are atoms. By (3),

$$\sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{N}} \sum_n |\mu_k| = 3 \sum_{k\in\mathbb{Z}} 2^k \lambda(\nu_k < \infty) \le 3d \sum_{k\in\mathbb{Z}} 2^k \lambda(Mf > 2^k) \le CE(Mf).$$

Since $E(\|a\|_X) \le 1$, the sum

$$\sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{N}} \sum_n |\mu_{k,n}^l| \|a_{k,n}^l\|_X$$

is convergent a.e. Thus the sum in (4) can be rearranged to get (2).

Conversely, suppose that f has a decomposition of the form (2). Since the sum in (2) converges in $L_1(X)$ norm, we have

$$E_n f = \sum_{k=0}^{\infty} \mu_k E_n a^k$$

and so we conclude

$$E(|Mf|) \le \sum_{k=0}^{\infty} |\mu_k| E(\sup_n \|E_n a^k\|_X) \le \sum_{k=0}^{\infty} |\mu_k| \int_{I_k} \sup_{n\ge n_k} E_n \|a^k\|_X \, d\lambda \le \sum_{k=0}^{\infty} |\mu_k|,$$

where the Vilenkin interval $I_k \in \mathcal{F}_{n_k}$ is the support of a_k. □

Note that the same proof works if we suppose only that X is a Banach space having the Radon–Nikodym property.

4. Marcinkiewicz inequality

Now we generalize the classical Marcinkiewicz inequality, mentioned in the Introduction, for UMD valued functions.

Theorem 3. *Assume that X is a UMD lattice and n_k is an arbitrary natural number for each $k \in \mathbb{N}$. If $(f^k, k \in \mathbb{N}) \in L_p(\ell_r(X))$ for some $1 < p, r < \infty$ then*

$$\bigg\| \Big(\sum_{k=0}^{\infty} \|s_{n_k} f^k\|_X^r \Big)^{1/r} \bigg\|_p \le C_{p,r} \bigg\| \Big(\sum_{k=0}^{\infty} \|f^k\|_X^r \Big)^{1/r} \bigg\|_p . \tag{5}$$

If $(f^k, k \in \mathbb{N}) \in L_1(\ell_r(X))$ for some $1 < r < \infty$ then

$$\rho\lambda\bigg(\Big(\sum_{k=0}^{\infty} \|s_{n_k} f^k\|_X^r \Big)^{1/r} > \rho \bigg) \le C_r \bigg\| \Big(\sum_{k=0}^{\infty} \|f^k\|_X^r \Big)^{1/r} \bigg\|_1 , \qquad (\rho > 0). \tag{6}$$

Proof. It is known that

$$\overline{w}_n s_n f = \sum_{j=0}^{\infty} \overline{w}_n T_j^n \Big(w_n \big(E_{j+1}(f\overline{w}_n) - E_j(f\overline{w}_n) \big) \Big) =: \sum_{j=0}^{\infty} d_j^n, \tag{7}$$

where the operator T_j^n is linear,

$$|T_j^n f|^2 \le C E_j |f|^2, \qquad (j, n \in \mathbb{N}) \tag{8}$$

and $(d_j^n, j \in \mathbb{N})$ is a martingale difference sequence with respect to $(\mathcal{F}_{j+1})$ (see Weisz [23]). Note that (7) is a finite sum. Since ℓ_r $(1 < r < \infty)$ is a UMD lattice, so is $\ell_r(X)$ (see Rubio de Francia [16]). Then we may apply Theorem 1 and (8) to obtain

$$\begin{aligned}
\bigg\| \Big(\sum_{k=0}^{\infty} \|s_{n_k} f^k\|_X^r \Big)^{1/r} \bigg\|_p &= \bigg\| \Big(\sum_{k=0}^{\infty} \Big\| \sum_{j=0}^{\infty} d_j^{n_k} \Big\|_X^r \Big)^{1/r} \bigg\|_p \\
&\le C_{p,r} \bigg\| \Big(\sum_{k=0}^{\infty} \Big\| \Big(\sum_{j=0}^{\infty} |d_j^{n_k}|^2 \Big)^{1/2} \Big\|_X^r \Big)^{1/r} \bigg\|_p \\
&\le C_{p,r} \bigg\| \Big(\sum_{k=0}^{\infty} \Big\| \Big(\sum_{j=0}^{\infty} E_j |E_{j+1}(f^k \overline{w}_{n_k}) - E_j(f^k \overline{w}_{n_k})|^2 \Big)^{1/2} \Big\|_X^r \Big)^{1/r} \bigg\|_p \\
&\le C_{p,r} \bigg\| \Big(\sum_{k=0}^{\infty} \Big\| \Big(\sum_{j=0}^{\infty} |E_{j+1}(f^k \overline{w}_{n_k}) - E_j(f^k \overline{w}_{n_k})|^2 \Big)^{1/2} \Big\|_X^r \Big)^{1/r} \bigg\|_p
\end{aligned}$$

$$\leq C_{p,r}\left\|\left(\sum_{k=0}^{\infty}\left\|f^k\overline{w}_{n_k}\right\|_X^r\right)^{1/r}\right\|_p, \tag{9}$$

which proves (5). Note that in the last step we have used that

$$(E_{j+1}(f^k\overline{w}_{n_k})-E_j(f^k\overline{w}_{n_k}), j\in\mathbb{N})$$

is a martingale difference sequence.

To prove (6) let us define the stopping time

$$\nu(x):=\inf\{n\in\mathbb{N}:E_n\mathbf{1}_{\{(\sum_{k=0}^{\infty}\|E_{n+1}(f^k\overline{w}_{n_k})\|_X^r)^{1/r}>\rho\}}(x)\geq 1/d\}.$$

Then

$$\lambda(\nu<\infty)\leq d\lambda\left(\sup_{\in\mathbb{N}}\left(\sum_{k=0}^{\infty}\|E_n(f^k\overline{w}_{n_k})\|_X^r\right)^{1/r}>\rho\right) \tag{10}$$

and

$$\sup_{n\leq\nu}\left(\sum_{k=0}^{\infty}\|E_n(f^k\overline{w}_{n_k})\|_X^r\right)^{1/r}\leq\rho. \tag{11}$$

Obviously,

$$\begin{aligned}\left(\sum_{k=0}^{\infty}\|s_{n_k}f^k\|_X^r\right)^{1/r} &= \left(\sum_{k=0}^{\infty}\left\|\sum_{j=0}^{\infty}d_j^{n_k}\mathbf{1}_{\{\nu\geq j+1\}}\right\|_X^r\right)^{1/r}\\ &\quad+\left(\sum_{k=0}^{\infty}\left\|\sum_{j=0}^{\infty}d_j^{n_k}\mathbf{1}_{\{\nu<j+1\}}\right\|_X^r\right)^{1/r}=(A)+(B).\end{aligned}$$

Using the definition of stopped martingales and (11) we get similarly to (9) that

$$\begin{aligned}\lambda((A)>\rho) &\leq \frac{1}{\rho^2}E((A)^2)\\ &\leq \frac{C_r}{\rho^2}E\left(\sum_{k=0}^{\infty}\left\|\sum_{j=0}^{\infty}(E_{j+1}(f^k\overline{w}_{n_k})-E_j(f^k\overline{w}_{n_k}))\mathbf{1}_{\{\nu\geq j+1\}}\right\|_X^r\right)^{2/r}\\ &\leq \frac{C_r}{\rho^2}E\left(\sum_{k=0}^{\infty}\|E_\nu(f^k\overline{w}_{n_k})\|_X^r\right)^{2/r}\leq\frac{C_r}{\rho}E\left(\sum_{k=0}^{\infty}\|E_\nu(f^k\overline{w}_{n_k})\|_X^r\right)^{1/r}\\ &\leq \frac{C_r}{\rho}E\left(\sum_{k=0}^{\infty}\|f^k\|_X^r\right)^{1/r}.\end{aligned} \tag{12}$$

It is easy to see that $(B)=0$ if $\nu=\infty$, and so $\{(B)>\rho\}\subset\{\nu<\infty\}$. Since $\big(\sum_{k=0}^{\infty}\|E_n(f^k\overline{w}_{n_k})\|_X^r\big)^{1/r}$ is a non-negative submartingale, we obtain

$$\lambda(\nu<\infty)\le d\lambda\bigg(\sup_{\in\mathbb{N}}\bigg(\sum_{k=0}^{\infty}\|E_n(f^k\overline{w}_{n_k})\|_X^r\bigg)^{1/r}>\rho\bigg)\le\frac{d}{\rho}E\bigg(\sum_{k=0}^{\infty}\|f^k\|_X^r\bigg)^{1/r}.$$

This together with (12) implies (6). □

If we apply Theorem 3 for one k, only, then we get

Corollary 1. *If X is a UMD lattice and $f\in L_p(X)$ for some $1<p<\infty$ then*

$$\|s_nf\|_{L_p(X)}\le C_p\|f\|_{L_p(X)}\qquad(\in\mathbb{N})\tag{13}$$

and $s_nf\to f$ in $L_p(X)$ norm as $n\to\infty$.

The converse of this result easily follows from the proof of Theorem 3:

Theorem 4. *Assume that X is a Banach lattice. Inequality* (13) *holds for some (or equivalently for all) $1<p<\infty$ if and only if X is UMD.*

Proof. One can show that

$$T_j^n(w_n(E_{j+1}(f\overline{w}_n)-E_j(f\overline{w}_n)))=n_j(w_n(E_{j+1}(f\overline{w}_n)-E_j(f\overline{w}_n)))$$

if $n_j=0$ or 1, where $n=\sum_{j=0}^{\infty}n_jP_j$, $0\le n_j<p_j$ (see Weisz [23]). Consider only such numbers n for which $n_j=0$ or 1 for each j. Then

$$\overline{w}_ns_nf=\sum_{j=0}^{\infty}n_j\big(E_{j+1}(f\overline{w}_n)-E_j(f\overline{w}_n)\big).$$

Inequality (13) implies

$$\bigg\|\sum_{j=0}^{\infty}n_j\big(E_{j+1}(f\overline{w}_n)-E_j(f\overline{w}_n)\big)\bigg\|_{L_p(X)}\le C_p\|f\|_{L_p(X)}.$$

Writing fw_n instead of f we obtain

$$\bigg\|\sum_{j=0}^{\infty}n_j\big(E_{j+1}f-E_jf\big)\bigg\|_{L_p(X)}\le C_p\|f\|_{L_p(X)}$$

and this implies that X is UMD (see (1)). □

For other versions of this theorem see also Wenzel [25], Tozoni [20] and Clément at al. [5].

5. Almost everywhere convergence

It is known (see Weisz [22]) that $s_n f \to f$ a.e. in X norm as $n \to \infty$, whenever $f \in L_p(X)$ for some $1 < p < \infty$. However, this does not hold for $L_1(X)$ or $H_1(X)$ even if $X = \mathbb{R}$ (see Ladhawala and Pankratz [9] or Schipp and Simon [17]). We say that an increasing sequence $(n_k, k \in \mathbb{N})$ of positive integers is *lacunary* if $n_{k+1}/n_k > \alpha > 1$ for all $k \in \mathbb{N}$. Now we are ready to prove our main result.

Theorem 5. *Assume that X is a UMD lattice and $(n_k, k \in \mathbb{N})$ is a lacunary sequence of positive integers. If $f \in H_1(X)$ then*

$$\rho\lambda\big(\sup_k \|s_{n_k} f\|_X > \rho\big) \le C\|f\|_{H_1(X)}, \qquad (\rho > 0).$$

Proof. It is well known that every lacunary sequence $(n_k, k \in \mathbb{N})$ can be split into a finite number of lacunary subsequences $(n_k^j, k \in \mathbb{N})$ with $n_{k+1}^j \ge d n_k^j$ $(k \in \mathbb{N})$. Thus we may assume that $P_k \le n_k < P_{k+1}$. Then $s_{n_k} f = s_{P_k} f + s_{n_k}(d_k f)$, where $d_k f := s_{P_{k+1}} f - s_{P_k} f$. Since $(s_{P_k} f)$ is a martingale, Theorem 1 implies

$$\rho\lambda\big(\sup_k \|s_{P_k} f\|_X > \rho\big) \le C\|f\|_{L_1(X)} \le C\|f\|_{H_1(X)} \qquad (\rho > 0). \tag{14}$$

On the other hand, by Theorem 3,

$$\begin{aligned}\rho\lambda\big(\sup_k \|s_{n_k}(d_k f)\|_X > \rho\big) &\le \rho\lambda\Bigg(\Big(\sum_{k=0}^{\infty} \|s_{n_k}(d_k f)\|_X^q\Big)^{1/q} > \rho\Bigg) \\ &\le C_q \Bigg\|\Big(\sum_{k=0}^{\infty} \|d_k f\|_X^q\Big)^{1/q}\Bigg\|_1\end{aligned}$$

for all $\rho > 0$ and $1 < q < \infty$. If we take an atomic decomposition of f as in (2) then

$$d_k f = \sum_{j=0}^{\infty} \mu_j d_k a^j \qquad \text{a.e. in } X \text{ norm.}$$

It is easy to show that

$$\rho\lambda\big(\sup_k \|s_{n_k}(d_k f)\|_X > \rho\big) \le C_q \sum_{j=0}^{\infty} |\mu_j| E\Big(\sum_{k=0}^{\infty} \|d_k a^j\|_X^q\Big)^{1/q}. \tag{15}$$

Since every UMD lattice is superreflexive (see e.g. Rubio de Francia [16]), X is q-concave for some $1 < q < \infty$. We may suppose that $q > 2$. Hence X has cotype q (see Lindenstrauss and Tzafriri [10]). This means that

$$\left(\sum_{k=0}^{N}\|d_k a^j\|_X^q\right)^{1/q} \leq C\int_0^1 \left\|\sum_{k=0}^{N} r_k(t)d_k a^j\right\|_X dt$$

for every $N, j \in \mathbb{N}$, where r_k denote now the original Rademacher functions with $p_n = 2$ ($\in \mathbb{N}$). If I_j denotes the support of the atom a^j, then we obtain by the UMD property and by the definition of the atom that for each fixed t,

$$E\left\|\sum_{k=0}^{N} r_k(t)d_k a^j\right\|_X = \int_{I_j}\left\|\sum_{k=0}^{N} r_k(t)d_k a^j\right\|_X d\lambda$$

$$\leq \lambda(I_j)^{1/2}\left(\int_{I_j}\left\|\sum_{k=0}^{N} r_k(t)d_k a^j\right\|_X^2 d\lambda\right)^{1/2} \leq C\lambda(I_j)^{1/2}\left(\int_{I_j}\|a^j\|_X^2 d\lambda\right)^{1/2} \leq C.$$

Now Theorem 2, (14) and (15) finishes the proof of the theorem. □

By the usual density argument of Marcinkiewicz and Zygmund we obtain

Corollary 2. *Assume that X is a UMD lattice and $(n_k, k \in \mathbb{N})$ is a lacunary sequence of positive integers. If $f \in H_1(X)$ then $\lim_{k\to\infty} s_{n_k} f = f$ a.e. in X norm.*

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