

Tensor product of proper contractions, stable and posinormal operators

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Abstract. It is shown that if a class of Hilbert space operators is closed under constant direct sums and ordinary products, then it is closed under tensors products. This leads to a proof that proper contractiveness is preserved by tensor products. Weak, strong and uniform stabilities of tensor products of operators are also investigated, and it is proved that the tensor product of power bounded operators is of class \mathcal{C}_{00} whenever one of the factors is a completely nonunitary contraction for which the intersection of the continuous spectrum with the unit circle has Lebesgue measure zero. Moreover, it is also shown that if a contraction has no nontrivial invariant subspace, then the tensor product with its adjoint is of class \mathcal{C}_{00} . Furthermore, it is verified that posinormality is preserved by tensor products as well.

1. Introduction

Notational preliminaries: Let \mathcal{H} and \mathcal{K} be nonzero complex Hilbert spaces. We shall consider the concept of tensor product space in terms of the single tensor product of vectors as a conjugate bilinear functional on the Cartesian product of \mathcal{H} and \mathcal{K} . (See e.g., [4] and [13] – for an abstract approach see e.g., [2] and [18].) The *single tensor product* of $x \in \mathcal{H}$ and $y \in \mathcal{K}$ is a conjugate bilinear functional $x \otimes y : \mathcal{H} \times \mathcal{K} \rightarrow \mathbb{C}$ defined by $(x \otimes y)(u, v) = \langle x; u \rangle \langle y; v \rangle$ for every $(u, v) \in \mathcal{H} \times \mathcal{K}$, where the first inner product is on \mathcal{H} and the second on \mathcal{K} . The collection of all (finite) sums of single tensors, denoted by $\mathcal{H} \otimes \mathcal{K}$, is a linear space (over the same complex field \mathbb{C}) that admits an inner product $\langle ; \rangle : (\mathcal{H} \otimes \mathcal{K}) \times (\mathcal{H} \otimes \mathcal{K}) \rightarrow \mathbb{C}$

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defined, for arbitrary $\sum_{i=1}^N x_i \otimes y_i$ and $\sum_{j=1}^M w_j \otimes z_j$ in $\mathcal{H} \otimes \mathcal{K}$, by

$$\left\langle \sum_{i=1}^N x_i \otimes y_i; \sum_{j=1}^M w_j \otimes z_j \right\rangle = \sum_{i=1}^N \sum_{j=1}^M \langle x_i; w_j \rangle \langle y_i; z_j \rangle$$

(the same notation for the inner products on \mathcal{H} , \mathcal{K} and $\mathcal{H} \otimes \mathcal{K}$). By an operator we mean a bounded linear transformation of a normed space into itself. Let $\mathcal{B}[\mathcal{H}]$, $\mathcal{B}[\mathcal{K}]$ and $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ be the normed algebras of all operators on \mathcal{H} , \mathcal{K} and $\mathcal{H} \otimes \mathcal{K}$. The *tensor product* on $\mathcal{H} \otimes \mathcal{K}$ of two operators A in $\mathcal{B}[\mathcal{H}]$ and B in $\mathcal{B}[\mathcal{K}]$ is the operator $A \otimes B : \mathcal{H} \otimes \mathcal{K} \rightarrow \mathcal{H} \otimes \mathcal{K}$ defined by

$$(A \otimes B) \sum_{i=1}^N x_i \otimes y_i = \sum_{i=1}^N A x_i \otimes B y_i \quad \text{for every} \quad \sum_{i=1}^N x_i \otimes y_i \in \mathcal{H} \otimes \mathcal{K},$$

which in fact lies in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$. The completion of the inner product space $\mathcal{H} \otimes \mathcal{K}$, denoted by $\mathcal{H} \widehat{\otimes} \mathcal{K}$, is the *tensor product space* of \mathcal{H} and \mathcal{K} . The extension of $A \otimes B$ over the Hilbert space $\mathcal{H} \widehat{\otimes} \mathcal{K}$, denoted by $A \widehat{\otimes} B$, is the *tensor product* of A and B on the tensor product space, which lies in $\mathcal{B}[\mathcal{H} \widehat{\otimes} \mathcal{K}]$.

Some properties of A and B are preserved when taking the tensor product. For instance, *if A and B are either self-adjoint, unitary, nonnegative, normal, quasinormal, subnormal, hyponormal, quasihyponormal, semi-quasihyponormal, or normaloid, then so is $A \widehat{\otimes} B$* (see e.g., [9]). The converse to many of these statements also holds true. *If $A \widehat{\otimes} B$ is either normal, quasinormal, subnormal or hyponormal, then so are A and B* (if they are nonzero) [15]. Preservation by tensor product has also been verified for other classes of close to normal operators (see e.g., [3], [5] and [17]). However, such a preservation may fail for some important classes. Indeed, there exist paranormal or spectraloid operators A and B for which $A \widehat{\otimes} B$ is not paranormal or spectraloid: the properties of being paranormal or spectraloid are not preserved when taking tensor products [14, pp. 629, 631].

We investigate the preservation of three further properties by tensor products of Hilbert space operators. Although the preservation of plain contractiveness and strict contractiveness is trivially verified, the preservation of proper contractiveness is not, once this is not a property that can be separated when taking tensor products. In Section 2 we prove that closeness under constant direct sums and ordinary products implies closeness under tensor products (Theorem 1). This ensures that proper contractiveness is preserved by tensor products. Stability is considered in Section 3 leading to a spectral condition for a tensor product to be of class \mathcal{C}_{00} (Theorem 2). The role played by tensor products of class \mathcal{C}_{00} in

the invariant subspace problem is also explored (Theorem 3). Preservation of tensor products for classes of close to normal operators is extended in Section 4 by showing that A and B are posinormal if and only if $A \widehat{\otimes} B$ is (Theorem 4).

2. Proper contractions

An operator T is a contraction if $\|T\| \leq 1$ (i.e., $\|Tx\| \leq \|x\|$ for every x). It is a proper contraction if $\|Tx\| < \|x\|$ for every nonzero x , and a strict contraction if $\|T\| < 1$ (i.e., $\sup_{x \neq 0} (\|Tx\| / \|x\|) < 1$). These are related by proper inclusions:

Strict Contraction \subset Proper Contraction \subset Contraction.

Since $\|A \widehat{\otimes} B\| = \|A\| \|B\|$, it follows that $A \widehat{\otimes} B$ is a contraction (or a strict contraction) if and only if $\|A\| \|B\| \leq 1$ (or $\|A\| \|B\| < 1$). Thus it is trivially verified that if A in $\mathcal{B}[\mathcal{H}]$ and B in $\mathcal{B}[\mathcal{K}]$ are contractions, then so is $A \widehat{\otimes} B$ in $\mathcal{B}[\mathcal{H} \widehat{\otimes} \mathcal{K}]$ and, if in addition one of A or B is a strict contraction, then so is $A \widehat{\otimes} B$. However, the proof that *the tensor product of proper contractions is a proper contraction* does not follow at once by the above norm identity. Indeed, the italicized assertion is equivalent to saying that, for every nonzero $\sum_{i=1}^N x_i \otimes y_i$ in $\mathcal{H} \otimes \mathcal{K}$,

$$\sum_{i=1}^N \sum_{j=1}^N \langle Ax_i; Ax_j \rangle \langle By_i; By_j \rangle < \sum_{i=1}^N \sum_{j=1}^N \langle x_i; x_j \rangle \langle y_i; y_j \rangle$$

whenever $\|Ax\| < \|x\|$ and $\|By\| < \|y\|$ for every nonzero x and y in \mathcal{H} and \mathcal{K} . We show that the above statement is a corollary of the next theorem.

Theorem 1. *If \mathcal{C}' and \mathcal{C} are classes of operators (acting on separable Hilbert spaces) such that*

- (a) $\mathcal{C}' \subseteq \mathcal{C}$,
- (b) every operator unitary equivalent to an operator in \mathcal{C}' or in \mathcal{C} is an operator in \mathcal{C}' or in \mathcal{C} , respectively,
- (c) direct sum of countably many copies of an operator in \mathcal{C}' or in \mathcal{C} is an operator in \mathcal{C}' or in \mathcal{C} , respectively,
- (d) product (either left or right) of an operator in \mathcal{C}' with an operator in \mathcal{C} (acting on the same space) is an operator in \mathcal{C}' ,

then the tensor product of two operators of class \mathcal{C} , being one of them in class \mathcal{C}' , is an operator of class \mathcal{C}' .

PROOF. Let \mathcal{H} and \mathcal{K} be Hilbert spaces, take A in $\mathcal{B}[\mathcal{H}]$, B in $\mathcal{B}[\mathcal{K}]$, and recall that

$$A \widehat{\otimes} B = (A \widehat{\otimes} I)(I \widehat{\otimes} B) = (I \widehat{\otimes} B)(A \widehat{\otimes} I)$$

in $\mathcal{B}[\mathcal{H} \widehat{\otimes} \mathcal{K}]$, where the same notation I is being used for the identity on \mathcal{H} and on \mathcal{K} . Also recall that tensor product is unitarily equivalent commutative; that is,

$$A \widehat{\otimes} B \cong B \widehat{\otimes} A,$$

with \cong denoting unitary equivalence, and so $\mathcal{H} \widehat{\otimes} \mathcal{K} \cong \mathcal{K} \widehat{\otimes} \mathcal{H}$. No assumption on separability required so far (see e.g., [9]). Now suppose \mathcal{H} and \mathcal{K} are separable. This implies that the tensor products $I \widehat{\otimes} B$ on $\mathcal{H} \widehat{\otimes} \mathcal{K}$ and $I \widehat{\otimes} A$ on $\mathcal{K} \widehat{\otimes} \mathcal{H}$ are unitarily equivalent to the direct sums $\bigoplus_k B$ on $\bigoplus_k \mathcal{K}$ and $\bigoplus_k A$ on $\bigoplus_k \mathcal{H}$, respectively, and so $\mathcal{H} \widehat{\otimes} \mathcal{K} \cong \bigoplus_k \mathcal{K}$ and $\mathcal{K} \widehat{\otimes} \mathcal{H} \cong \bigoplus_k \mathcal{H}$ – recall that if one of \mathcal{H} or \mathcal{K} , say \mathcal{H} , is infinite-dimensional, then $\mathcal{H} \cong \ell_+^2$ and $\bigoplus_k \mathcal{K} = \ell_+^2(\mathcal{K})$. Therefore, with $U : \mathcal{H} \widehat{\otimes} \mathcal{K} \rightarrow \mathcal{K} \widehat{\otimes} \mathcal{H}$, $V : \mathcal{K} \widehat{\otimes} \mathcal{H} \rightarrow \bigoplus_k \mathcal{H}$, and $W : \mathcal{H} \widehat{\otimes} \mathcal{K} \rightarrow \bigoplus_k \mathcal{K}$ standing for the unitary transformations concerning the above-mentioned unitary equivalences,

$$A \widehat{\otimes} B = U^*(I \widehat{\otimes} A)U(I \widehat{\otimes} B) = U^*[V^*(\bigoplus_k A)V]UW^*(\bigoplus_k B)W$$

and

$$A \widehat{\otimes} B = (I \widehat{\otimes} B)U^*(I \widehat{\otimes} A)U = W^*(\bigoplus_k B)WU^*[V^*(\bigoplus_k A)V]U.$$

Let \mathcal{C}' and \mathcal{C} be classes of operators satisfying assumptions (a) to (d). If A and B are of class \mathcal{C} , with one of them being of class \mathcal{C}' , then $U^*[V^*(\bigoplus_k A)V]U$ and $W^*(\bigoplus_k B)W$ are of class \mathcal{C} , with one of them being of class \mathcal{C}' , by assumptions (b) and (c). Thus the above identities ensure that $A \widehat{\otimes} B$ is of class \mathcal{C}' if (d) holds. \square

Corollary 1. *Take $A \in \mathcal{B}[\mathcal{H}]$ and $B \in \mathcal{B}[\mathcal{K}]$ where \mathcal{H} and \mathcal{K} are separable Hilbert spaces. If one of A or B is a contraction and the other is a proper contraction, then $A \widehat{\otimes} B$ is a proper contraction.*

PROOF. It is readily verified that assumptions (a), (b) and (c) of Theorem 1 hold for contractions and proper contractions. That is, if \mathcal{C} denotes the class of all contractions and \mathcal{C}' the class of all proper contractions, both consisting of operators acting on separable Hilbert spaces, then assumptions (a), (b) and (c) hold true. Moreover, since the product (either left or right) of a contraction with a proper contraction is again a proper contraction [11], it follows that assumption (d) of Theorem 1 is also satisfied. Thus Theorem 1 ensures that $A \widehat{\otimes} B$ is a proper contraction whenever one of A or B is a contraction and the other is a plain contraction. \square

Remark 1. In particular, if $\mathcal{C}' = \mathcal{C}$, then Theorem 1 is rephrased as follows. Suppose a class of operators acting on separable Hilbert spaces is closed under unitary equivalence and under constant direct sums. If it is also closed under products, then it is closed under tensor products.

Thus, in this case, Corollary 1 is particularized accordingly:

If A and B are proper contractions acting on separable Hilbert spaces, then their tensor product $A \widehat{\otimes} B$ is again a proper contraction.

3. Stability

A Hilbert space operator T is uniformly stable if $\|T^n\| \rightarrow 0$, strongly stable if $\|T^n x\| \rightarrow 0$ for every x , and weakly stable if $\langle T^n x; y \rangle \rightarrow 0$ for every x and y (equivalently, if $\langle T^n x; x \rangle \rightarrow 0$ for every x). These are denoted by $T^n \xrightarrow{u} O$, $T^n \xrightarrow{s} O$ and $T^n \xrightarrow{w} O$, respectively. Moreover, T is power bounded if $\sup_n \|T^n\| < \infty$. With $r(T)$ denoting the spectral radius of T , it is well known that

$$r(T) < 1 \iff T^n \xrightarrow{u} O \implies T^n \xrightarrow{s} O \implies T^n \xrightarrow{w} O \implies \sup_n \|T^n\| < \infty \implies r(T) \leq 1.$$

The converses to the above one-way implications fail in general. We shall investigate how stability for A in $\mathcal{B}[\mathcal{H}]$ or B in $\mathcal{B}[\mathcal{K}]$ is transferred to $A \widehat{\otimes} B$ in $\mathcal{B}[\mathcal{H} \widehat{\otimes} \mathcal{K}]$. First, by the Gelfand–Beurling formula for the spectral radius we get

$$r(A \widehat{\otimes} B) = r(A)r(B).$$

Indeed, $\|(A \widehat{\otimes} B)^n\|^{\frac{1}{n}} = \|(A^n \widehat{\otimes} B^n)\|^{\frac{1}{n}} = \|A^n\|^{\frac{1}{n}} \|B^n\|^{\frac{1}{n}}$ for every nonnegative integer n . Thus $r(A \widehat{\otimes} B) < 1$ if and only if $r(A)r(B) < 1$, and therefore $A \widehat{\otimes} B$ is uniformly stable if and only if $r(A)r(B) < 1$.

Interesting results about strong stability for tensor products of operators on the same separable Hilbert space were presented in [3]. We extend some of those results (that will be required in the sequel) in Proposition 1 by considering distinct (not necessarily separable) Hilbert spaces and exhibiting a different proof for strong stability that can be carried through weak stability, thus also showing that weak stability for tensor products of operators is preserved as well.

Proposition 1. Consider the tensor product $A \widehat{\otimes} B$ of $A \in \mathcal{B}[\mathcal{H}]$ and $B \in \mathcal{B}[\mathcal{K}]$.

- (a) If $A \widehat{\otimes} B$ is uniformly (strongly, weakly) stable, then so is one of A or B .
 (b) If one of A or B is uniformly (strongly, weakly) stable and the other is power bounded, then $A \widehat{\otimes} B$ is uniformly (strongly, weakly) stable.

PROOF. (a) If $A \widehat{\otimes} B$ is uniformly stable, then $r(A)r(B) < 1$ so that $r(A) < 1$ or $r(B) < 1$, which means that A or B is uniformly stable. This proves (a) for uniform stability. To verify assertion (a) for strong and weak stabilities proceed as follows. Take arbitrary $x \in \mathcal{H}$ and $y \in \mathcal{K}$ and observe that

$$\|(A \widehat{\otimes} B)^n x \otimes y\| = \|A^n x \otimes B^n y\| = \|A^n x\| \|B^n y\|$$

and

$$|\langle (A \widehat{\otimes} B)^n x \otimes y; x \otimes y \rangle| = |\langle A^n x \otimes B^n y; x \otimes y \rangle| = |\langle A^n x; x \rangle| |\langle B^n y; y \rangle|.$$

If $A \widehat{\otimes} B$ is strongly stable, then $\|A^n x\| \|B^n y\| \rightarrow 0$. If $\|B^n y\| \rightarrow 0$ for every $y \in \mathcal{K}$ then B is itself strongly stable. If $\inf_n \|B^n y\| > 0$ for some $y \in \mathcal{K}$, then $\|A^n x\| \rightarrow 0$ for every $x \in \mathcal{H}$. Thus A or B is strongly stable. If $A \widehat{\otimes} B$ is weakly stable, then $|\langle A^n x; x \rangle| |\langle B^n y; y \rangle| \rightarrow 0$. A similar argument ensures that $|\langle A^n x; x \rangle| \rightarrow 0$ for every $x \in \mathcal{H}$ or $|\langle B^n y; y \rangle| \rightarrow 0$ for every $y \in \mathcal{K}$, which means that A or B is weakly stable.

(b) Since $r(A \widehat{\otimes} B) = r(A)r(B)$, and since $r(B) \leq 1$ whenever $\sup_n \|B^n\| < \infty$, it follows that if $r(A) < 1$ and $\sup_n \|B^n\| < \infty$ (or vice versa), then $r(A \widehat{\otimes} B) < 1$, which proves (b) for uniform stability. To prove assertion (b) for strong and weak stabilities take an arbitrary $\sum_{i=1}^N x_i \otimes y_i$ in $\mathcal{H} \otimes \mathcal{K}$. Note that

$$\begin{aligned} \left\| (A \otimes B)^n \sum_{i=1}^N x_i \otimes y_i \right\| &= \left\| \sum_{i=1}^N A^n x_i \otimes B^n y_i \right\| \leq \sum_{i=1}^N \|A^n x_i \otimes B^n y_i\| \\ &\leq \sum_{i=1}^N \|A^n x_i\| \|B^n y_i\| \leq \left(\sum_{i=1}^N \|A^n x_i\| \right) \sup_n \|B^n\| \sum_{i=1}^N \|y_i\|. \end{aligned}$$

If A is strongly stable (so that $\sum_{i=1}^N \|A^n x_i\| \rightarrow 0$) and B is power bounded, then $\|(A \otimes B)^n \sum_{i=1}^N x_i \otimes y_i\| \rightarrow 0$. Thus $A \otimes B$ is strongly stable whenever one of A or B is strongly stable and the other is power bounded. Since strong stability is preserved under unitary equivalence, and is also preserved from a dense subspace of a normed space to the whole space, it follows that the extension $A \widehat{\otimes} B$ of $A \otimes B$ on the completion $\mathcal{H} \widehat{\otimes} \mathcal{K}$ of $\mathcal{H} \otimes \mathcal{K}$ is strongly stable whenever $A \otimes B$ is. This proves (b) for strong stability. Similarly, and applying the Schwarz inequality,

$$\left| \left\langle (A \otimes B)^n \sum_{i=1}^N x_i \otimes y_i; \sum_{i=1}^N x_i \otimes y_i \right\rangle \right| = \left| \left\langle \sum_{i=1}^N A^n x_i \otimes B^n y_i; \sum_{i=1}^N x_i \otimes y_i \right\rangle \right|$$

$$\begin{aligned}
 &= \left| \sum_{i=1}^N \sum_{j=1}^N \langle A^n x_i; x_j \rangle \langle B^n y_i; y_j \rangle \right| \\
 &\leq \left(\sum_{i=1}^N \sum_{j=1}^N |\langle A^n x_i; x_j \rangle| \right) \sup_n \|B^n\| \sum_{i=1}^N \sum_{j=1}^N \|y_i\| \|y_j\|.
 \end{aligned}$$

If A is weakly stable (so that $\sum_{i=1}^N \sum_{j=1}^N |\langle A^n x_i; x_j \rangle| \rightarrow 0$) and B is power bounded, then $\langle (A \otimes B)^n \sum_{i=1}^N x_i \otimes y_i; \sum_{i=1}^N x_i \otimes y_i \rangle \rightarrow 0$. Thus $A \otimes B$ is weakly stable whenever one of A or B weakly stable and the other is power bounded. Again, as weak stability is preserved under unitary equivalence, and is also preserved from a dense subspace of a inner product space to the whole space, the extension $A \widehat{\otimes} B$ of $A \otimes B$ on the completion $\mathcal{H} \widehat{\otimes} \mathcal{K}$ of $\mathcal{H} \otimes \mathcal{K}$ is weakly stable whenever $A \otimes B$ is. \square

Remark 2. If A and B are power bounded, then $A \widehat{\otimes} B$ is uniformly (strongly, weakly) stable if and only if one of A or B is (cf. Proposition 1). Note that power boundedness is not required in part (a) of Proposition 1, and is sufficient but not necessary in part (b). Indeed, put $A = \frac{1}{3}I$ and $B = 2I$ so that $A \widehat{\otimes} B = \frac{2}{3}(I \widehat{\otimes} I)$. Thus $r(A) = \frac{1}{3}$ and $r(A \widehat{\otimes} B) = \frac{2}{3}$, which means that A and $A \widehat{\otimes} B$ are uniformly (thus strongly and so weakly) stable, although B is not power bounded.

If T is a contraction, then the sequence of nonnegative numbers $\{\|T^n x\|\}$ is decreasing (thus convergent) for every vector x . A contraction T is of class \mathcal{C}_0 . if it is strongly stable; that is, if $\{\|T^n x\|\}$ converges to zero for every x , and of class \mathcal{C}_1 . if $\{\|T^n x\|\}$ does not converge to zero for every nonzero vector x . It is of class \mathcal{C}_0 or of class \mathcal{C}_1 if its adjoint T^* is of class \mathcal{C}_0 . or \mathcal{C}_1 ., respectively. All combinations are possible, leading to the Nagy–Foiaş classes of contractions \mathcal{C}_{00} , \mathcal{C}_{01} , \mathcal{C}_{10} and \mathcal{C}_{11} [16, p. 72]. A contraction is completely nonunitary if it has no unitary direct summand. A uniformly stable contraction is of class \mathcal{C}_{00} , and hence completely nonunitary. Note that the classes \mathcal{C}_0 . and \mathcal{C}_0 ., originally defined for contractions, can be naturally extended to power bounded operators.

Tensor products of operators comprise a most useful way for exhibiting examples and counterexamples (e.g., see [14, Section 6]), which in part is due to the fact that $\sigma(A \widehat{\otimes} B) = \sigma(A)\sigma(B)$ [1], where $\sigma(T)$ stands for the spectrum of T . In particular, an example of a strongly stable operator that is not similar to any contraction was obtained in [6] by means of the tensor product $S^* \widehat{\otimes} F$ on $\ell_+^2 \widehat{\otimes} (\ell_+^2 \oplus \ell_+^2)$ of the adjoint S^* of the canonical unilateral shift on ℓ_+^2 with the Foguel operator F on $\ell_+^2 \oplus \ell_+^2$. However, $S^* \widehat{\otimes} F$ is not of class \mathcal{C}_{00} – it is of class \mathcal{C}_0 . but not of class \mathcal{C}_0 .. In fact, it is still unknown whether every operator of class

\mathcal{C}_{00} is similar to a contraction (see [8, Section 8.2]). Next we provide a sufficient spectral condition for a tensor product of operators to be of class \mathcal{C}_{00} .

Theorem 2. *Let $A \in \mathcal{B}[\mathcal{H}]$ and $B \in \mathcal{B}[\mathcal{K}]$ be power bounded. Suppose one of them is a completely nonunitary contraction such that the intersection of its continuous spectrum with the unit circle either has Lebesgue measure zero or is empty. In the former case $A \widehat{\otimes} B$ is of class \mathcal{C}_{00} , in the latter case $A \widehat{\otimes} B$ is uniformly stable.*

PROOF. Consider the classical partition $\{\sigma_P(T), \sigma_R(T), \sigma_C(T)\}$ of the spectrum $\sigma(T)$ of any Hilbert space operator T , where $\sigma_P(T)$ is the point spectrum (the set of all eigenvalues of T), $\sigma_R(T) = \sigma_P(T^*)^* \setminus \sigma_P(T)$ is the residual spectrum, and $\sigma_C(T) = \sigma(T) \setminus (\sigma_P(T) \cup \sigma_R(T))$ is the continuous spectrum. (We are using the standard notation $\Lambda^* = \{\lambda \in \mathbb{C} : \bar{\lambda} \in \Lambda\}$.) Let μ denote the Lebesgue measure on the unit circle Γ . Suppose T is a completely nonunitary contraction. We split the proof into two parts. In part (a) it is assumed that $\mu(\sigma_C(T) \cap \Gamma) = 0$ and in part (b) it is assumed that $\sigma_C(T) \cap \Gamma = \emptyset$.

(a) If $\mu(\sigma_C(T) \cap \Gamma) = 0$, then T is of class \mathcal{C}_{00} (cf. [16, p. 85]). Moreover, every completely nonunitary contraction is weakly stable, and every weakly stable contraction T is such that $\sigma_P(T) \cup \sigma_R(T)$ is included in the open unit disc (cf. [8, pp. 106, 114]). Thus if $\mu(\sigma_C(T) \cap \Gamma) = 0$, then T is of class \mathcal{C}_{00} . Therefore, if A and B are power bounded and one of them is a completely nonunitary contraction whose intersection of the continuous spectrum with the unit circle has Lebesgue measure zero, then this one is of class \mathcal{C}_{00} , and hence $A \widehat{\otimes} B$ is of class \mathcal{C}_{00} by Proposition 1(b).

(b) Every completely nonunitary contraction is weakly stable, and an operator T is uniformly stable if and only if it is weakly stable and $\sigma_C(T) \cap \Gamma = \emptyset$ (cf. [8, p. 115]). Thus, if A and B are power bounded and one of them is a completely nonunitary contraction whose intersection of the continuous spectrum with the unit circle is empty, then this one is uniformly stable, and so is $A \widehat{\otimes} B$ by Proposition 1(b). \square

Remark 3. There exist tensor products of class \mathcal{C}_{00} for which both factors are completely nonunitary contractions whose continuous spectra coincide with the whole unit circle. Indeed, if V is any isometry, then Proposition 1 ensures that

$$(V \widehat{\otimes} B)^n \xrightarrow{s} O \iff (B \widehat{\otimes} V)^n \xrightarrow{s} O \iff B^n \xrightarrow{s} O.$$

In particular, if $\mathcal{K} = \mathcal{H}$ is separable and S is a unilateral shift on \mathcal{H} , then $S \widehat{\otimes} S^*$ and $S^* \widehat{\otimes} S$ are strongly stable because S^* is strongly stable. (The only isometry

with a strongly stable adjoint is a unilateral shift – see e.g., [8, Lemma 6.1].) Therefore, $S \widehat{\otimes} S^*$ is a \mathcal{C}_{00} -contraction on $\mathcal{H} \widehat{\otimes} \mathcal{H}$.

The invariant subspace problem is the classical open question that asks whether there exists an operator (equivalently, a contraction) on an infinite-dimensional complex separable Hilbert space that does not have a nontrivial invariant subspace. By a subspace \mathcal{M} of a Hilbert space \mathcal{H} we mean a closed linear manifold of \mathcal{H} , which is nontrivial if $\{0\} \neq \mathcal{M} \neq \mathcal{H}$, and T -invariant (i.e., invariant for an operator T on \mathcal{H}) if $T(\mathcal{M}) \subseteq \mathcal{M}$. A subspace \mathcal{M} is T -invariant if and only if \mathcal{M}^\perp is T^* -invariant, where \mathcal{M}^\perp is the orthogonal complement of \mathcal{M} . If \mathcal{M} and \mathcal{M}^\perp are both invariant for T (equivalently, if \mathcal{M} is invariant for both T and T^*), then \mathcal{M} is a reducing subspace for T . Contractions A for which the tensor product $A \widehat{\otimes} A^*$ is of class \mathcal{C}_{00} play an important role in the invariant subspace problem as we shall see in Theorem 3 below. First we need the following auxiliary result, which shows how nontrivial invariant or reducing subspaces are transferred when taking tensor products.

Proposition 2. *Let A and B be nonzero operators on \mathcal{H} and \mathcal{K} , and let \mathcal{M} and \mathcal{N} be subspaces of \mathcal{H} and \mathcal{K} , respectively.*

- (a) *\mathcal{M} is invariant (reducing) for A and \mathcal{N} is invariant (reducing) for B if and only if $\mathcal{M} \widehat{\otimes} \mathcal{N}$ is an invariant (reducing) subspace for $A \widehat{\otimes} B$.*
- (b) *Moreover, one of \mathcal{M} or \mathcal{N} is nontrivial and the other is nonzero if and only if $\mathcal{M} \widehat{\otimes} \mathcal{N}$ is nontrivial.*

PROOF. A subspace \mathcal{R} of a Hilbert space is invariant (or reducing) for a nonzero operator T if and only if there exists an orthogonal projection E such that $ETE = TE$ (or such that $TE = ET$) where $\mathcal{R} = \text{range}(E)$. Moreover, since the orthogonal projection with range \mathcal{R} is unique, \mathcal{R} is nontrivial if and only if E is nontrivial (i.e., $0 \neq E \neq I$). We work out the proof for the invariant subspace case only. The proof for the reducing subspace case is similar, where the relation $ETE = TE$ is replaced with the commuting assumption $TE = ET$.

(a) If $\mathcal{M} = \text{range}(P)$ is an invariant subspace for A where P is the orthogonal projection on \mathcal{H} such that $PAP = AP$ and if $\mathcal{N} = \text{range}(Q)$ is an invariant subspace for B where Q is the orthogonal projection on \mathcal{K} such that $QBQ = BQ$, then

$$(P \widehat{\otimes} Q)(A \widehat{\otimes} B)(P \widehat{\otimes} Q) = PAP \widehat{\otimes} QBQ = AP \widehat{\otimes} BQ = (A \widehat{\otimes} B)(P \widehat{\otimes} Q),$$

where $P \widehat{\otimes} Q$ is an orthogonal projection on $\mathcal{H} \widehat{\otimes} \mathcal{K}$ (since it is idempotent and self-adjoint whenever P and Q are) with

$$\text{range}(P \widehat{\otimes} Q) = \text{range}(P) \widehat{\otimes} \text{range}(Q) = \mathcal{M} \widehat{\otimes} \mathcal{N},$$

which is a subspace of $\mathcal{H} \widehat{\otimes} \mathcal{K}$ (since it is the range of an orthogonal projection). Therefore, $\mathcal{M} \widehat{\otimes} \mathcal{N}$ is an invariant subspace for the tensor product $A \widehat{\otimes} B$. Conversely, take arbitrary vectors $u \in \mathcal{M}$ and $v \in \mathcal{N}$ so that the single tensor $u \otimes v$ lies in $\mathcal{M} \widehat{\otimes} \mathcal{N}$. If $\mathcal{M} \widehat{\otimes} \mathcal{N}$ is an invariant subspace for $A \widehat{\otimes} B$ then, in particular, $(A \widehat{\otimes} B)(u \otimes v) = Au \otimes Bv$ lies in $\mathcal{M} \widehat{\otimes} \mathcal{N}$ so that $Au \in \mathcal{M}$ and $Bv \in \mathcal{N}$. Hence \mathcal{M} is A -invariant and \mathcal{N} is B -invariant.

(b) One of \mathcal{M} or \mathcal{N} is nontrivial and the other is nonzero if and only if one of the orthogonal projections P or Q is nontrivial and the other is nonzero (since \mathcal{M} and \mathcal{N} are the ranges of P and Q), which means that the orthogonal projection $P \widehat{\otimes} Q$ is nontrivial or, equivalently, that the subspace $\mathcal{M} \widehat{\otimes} \mathcal{N}$ (which is the range of $P \widehat{\otimes} Q$) is nontrivial. \square

Does a contraction not in \mathcal{C}_{00} have a nontrivial invariant subspace? Equivalently, is a contraction without a nontrivial invariant subspace necessarily of class \mathcal{C}_{00} ? This is a classical open question in operator theory (see [7] for equivalent versions of it). We show next that a contraction A for which $A \widehat{\otimes} A^$ is not in \mathcal{C}_{00} has a nontrivial invariant subspace. This ensures that if A is a contraction and $A \widehat{\otimes} A^*$ has no nontrivial invariant subspace, then $A \widehat{\otimes} A^*$ is necessarily of class \mathcal{C}_{00} . Recall that \mathcal{H} is separable if and only if $\mathcal{H} \widehat{\otimes} \mathcal{H}$ is separable.*

Theorem 3. *Let A be a contraction on a separable Hilbert space \mathcal{H} , let \mathcal{M} and \mathcal{N} be subspaces of \mathcal{H} , and consider the following assertions.*

- (a) $A \widehat{\otimes} A^*$ has no nontrivial invariant subspace.
- (b) $A \widehat{\otimes} A^*$ has no nontrivial invariant subspace of the form $\mathcal{M} \widehat{\otimes} \mathcal{N}$.
- (c) A has no nontrivial invariant subspace.
- (d) $A \widehat{\otimes} A^*$ is a \mathcal{C}_{00} -contraction.

Claim: (a) \implies (b) \iff (c) \implies (d).

PROOF. Assertion (a) trivially implies (b). Now Proposition 2 ensures that $A \widehat{\otimes} A^*$ has a nontrivial invariant subspace of the form $\mathcal{M} \widehat{\otimes} \mathcal{N}$ if and only if \mathcal{M} and \mathcal{N} are invariant subspaces for A and A^* , being one of them nontrivial. Thus, since A has a nontrivial invariant subspace if and only if A^* has, the denial of (b) is equivalent to the denial of (c), and so (b) and (c) are equivalent assertions. Finally, if (c) holds true, then A is either a \mathcal{C}_{00} , a \mathcal{C}_{01} or a \mathcal{C}_{10} -contraction [7, Proposition 1]. Hence, $A \widehat{\otimes} A^*$ on $\mathcal{H} \widehat{\otimes} \mathcal{H}$ is of class \mathcal{C}_{00} by Proposition 1(b), which is a contraction since $\|A \widehat{\otimes} A^*\| = \|A\| \|A^*\| = \|A\|^2$. Thus (c) implies (d). \square

Therefore, if A does not have a nontrivial invariant subspace, then $A \widehat{\otimes} A^*$ is a \mathcal{C}_{00} -contraction, and there is no subspaces \mathcal{M} and \mathcal{N} of \mathcal{H} , with one of them

being nontrivial, such that $\mathcal{M} \widehat{\otimes} \mathcal{N}$ is $A \widehat{\otimes} A^*$ -invariant; in particular, there is no nontrivial subspace \mathcal{M} of \mathcal{H} such that $\mathcal{M} \widehat{\otimes} \mathcal{M}^\perp$ is $A \widehat{\otimes} A^*$ -invariant.

4. Posinormality

Posinormal operators were introduced in [12] as the class of Hilbert space operators T for which $TT^* = T^*QT$ for some nonnegative operator Q . This is a very large class that includes the hyponormal (in fact all dominant operators) as well as the invertible operators (in fact every injective operator with a closed range). We show below that posinormality is preserved by tensor products.

Theorem 4. *Take nonzero operators $A \in \mathcal{B}[\mathcal{H}]$ and $B \in \mathcal{B}[\mathcal{K}]$. The tensor product $A \widehat{\otimes} B$ is posinormal if and only if both A and B are posinormal.*

PROOF. Suppose A and B are both nonzero (otherwise $A \widehat{\otimes} B$ is trivially posinormal). It is known that an operator T is posinormal if and only if there is a $\gamma > 0$ for which $TT^* \leq \gamma^2 T^*T$ [12] (see also [10]). Thus, if A and B are posinormal, then there are positive constants α and β such that $AA^* \leq \alpha^2 A^*A$ and $BB^* \leq \beta^2 B^*B$. Since the operators involved in the above inequalities are nonnegative, it follows that

$$AA^* \widehat{\otimes} BB^* \leq \alpha^2 \beta^2 (A^*A \widehat{\otimes} B^*B)$$

(see e.g., [9]), and therefore

$$(A \widehat{\otimes} B)(A \widehat{\otimes} B)^* \leq \alpha^2 \beta^2 (A \widehat{\otimes} B)^*(A \widehat{\otimes} B)$$

so that $A \widehat{\otimes} B$ is posinormal. Conversely, if $A \widehat{\otimes} B$ is posinormal, then there exists a positive γ such that

$$(A \widehat{\otimes} B)(A \widehat{\otimes} B)^* \leq \gamma^2 (A \widehat{\otimes} B)^*(A \widehat{\otimes} B),$$

which means

$$AA^* \widehat{\otimes} BB^* \leq (\gamma A^*A) \widehat{\otimes} (\gamma B^*B).$$

Since the operators involved in the above inequalities are all nonzero and nonnegative, it follows by [15, Proposition 2.2] that there is a positive number δ such that $AA^* \leq \delta(A^*A)$ and $BB^* \delta^{-1} \leq (B^*B)$ so that both A and B are posinormal. \square

Recall the following standard definitions. A Hilbert space operator T is hyponormal if $TT^* \leq T^*T$ or, equivalently, if $(\lambda I - T)(\bar{\lambda}I - T^*) \leq (\bar{\lambda}I - T^*)(\lambda I - T)$ for every scalar λ in \mathbb{C} . It is M -hyponormal if there exists a constant $M \geq 0$ such that $(\lambda I - T)(\bar{\lambda}I - T^*) \leq M(\bar{\lambda}I - T^*)(\lambda I - T)$ for every λ in \mathbb{C} (actually, hyponormal means 1-hyponormal), and dominant if for each λ in \mathbb{C} there exists an $M_\lambda \geq 0$ such that $(\lambda I - T)(\bar{\lambda}I - T^*) \leq M_\lambda(\bar{\lambda}I - T^*)(\lambda I - T)$. Thus, as we saw in the proof of Theorem 4, *an operator T is dominant if and only if $(\lambda I - T)$ is posinormal for every λ in \mathbb{C}* . These classes are related by proper inclusions:

$$\text{Hyponormal} \subset M\text{-Hyponormal} \subset \text{Dominant} \subset \text{Posinormal}.$$

Hyponormality is preserved by tensor products (as commented in Section 1) and so does posinormality (Theorem 4) but the classes in between are defined in terms of translations, and translation is not a property that can be separated when taking tensor products. Moreover, these classes are not closed under ordinary products, which prevent an application of Theorem 1. The next result gives a sufficient condition for preserving dominance by tensor products – recall that if A and B are dominant, then they are posinormal.

Corollary 2. *Suppose $A \in \mathcal{B}[\mathcal{H}]$ and $B \in \mathcal{B}[\mathcal{K}]$ are posinormal. If one of them is quasinilpotent, then the tensor product $A \widehat{\otimes} B$ is dominant.*

PROOF. Recall that an operator has a one-point spectrum if its spectrum is a singleton (i.e., has exactly one element). In particular, a quasinilpotent is a one-point spectrum operator whose spectrum is $\{0\}$ (i.e., an operator with zero spectral radius). Take an arbitrary Hilbert space operator T . If it is a one-point spectrum, say $\sigma(T) = \{\nu\}$, then $T' = \nu I - T$ is quasinilpotent (by the Spectral Mapping Theorem). Thus every nonzero complex number lies in the resolvent set of T' so that $\lambda I - T'$ is invertible for every $\lambda \neq 0$ in \mathbb{C} . Since invertible operators are posinormal [12] (see also [10]), it follows that $\lambda I - T'$ is posinormal for every λ in \mathbb{C} whenever T' is itself posinormal, which means that T' is dominant. But it is clear by the very definition of a dominant operator that if $T' = \nu I - T$ is dominant, then so is $T = \nu I - T'$. Outcome: *if a one-point spectrum T with $\sigma(T) = \{\nu\}$ is such that $\nu I - T$ is posinormal, then T is dominant*. In particular, *a posinormal quasinilpotent operator is dominant*. Now suppose A and B are posinormal so that $A \widehat{\otimes} B$ is posinormal by Theorem 4. If one of A or B is quasinilpotent, then so is the tensor product $A \widehat{\otimes} B$ since $r(A \widehat{\otimes} B) = r(A)r(B)$. Thus $A \widehat{\otimes} B$ is a posinormal quasinilpotent, thus dominant. \square

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