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On characterizing permutability in direct products

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Abstract. This paper extends the author's earlier results regarding permutable subgroups of direct products. More specifically, a prior article characterizes when a subgroup of a direct product of finite groups is permutable, and this article improves that characterization.

1. Introduction

Over the last few years, several authors have been carrying out a project of examining how subgroups with various embedding properties can be characterized in a direct product of two groups. This article is a contribution to that project.

In particular, it has long been known that a subgroup N of the direct product $G \times H$ is normal if and only if $\pi_G(N)/(N \cap G) \leq Z(G/(N \cap G))$ and $\pi_H(N)/(N \cap H) \leq Z(H/(N \cap H))$ where π_G and π_H are the natural projections of $G \times H$ onto G and H respectively. Yet other embedding properties are also well understood in direct products. In [6], P. Hauck uses the Fitting subgroup to characterize subnormality. More recently, in his Ph.D. dissertation J. Petrillo [10] has studied the cover-avoidance property (p. 263 of [12] is a reference on cover-avoidence). B. Brewster, A. Martinez Pastor and M. D. Perez-Ramos [2] provide necessary and sufficient conditions for subgroups of direct products to be normally embedded, and extend their results to characterize system permutability in direct products of finite solvable groups.

This article focuses on permutable subgroups. Recall that a subgroup M of

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a group G is *permutable* if for all subgroups X of G, we have $MX = XM$. Permutable subgroups were first studied by Ore [9], who called them quasinormal, in 1939. Ore proved that in a finite group, a permutable subgroup is subnormal.

The author has examined permutability in direct products in a series of papers [3], [4], and [5]. This paper concludes these investigations. In [3], the permutability of a subgroup of $G \times H$ that is a direct product of subgroups of the direct factors is explored. In [4], we investigate subgroups of $G \times H$ whose intersections with the direct factors are normal. It is in [5] that we put these results together to provide necessary and sufficient conditions for a subgroup of a direct product of finite groups to be permutable.

Theorem 1.1 (Theorem 4.2 [5])**.** *Let M be a subgroup of the finite p-group $G \times H$. Without loss of generality, assume $\exp(G/\operatorname{Core}_G(M \cap G)) \geq \exp(H/\operatorname{Core}_H(M \cap H))$. Then, M is a permutable subgroup of $G \times H$ if and only if for all $(g, h) \in G \times H$, one of the following is true:*

1) *$M \leq N_{G\times H}(((M \cap G) \times (M \cap H))\langle(g, h)\rangle)$, or*

(2) *$|\langle g\rangle/(\langle g\rangle \cap M)| > \exp(H/\operatorname{Core}_H(M \cap H))$, and there is a nonnegative integer i so that $h^{p^i} \in \pi_H((\langle g\rangle \times \langle h\rangle) \cap M)$ and $M \leq N_{G\times H}(((M \cap G) \times (M \cap H))\langle(1, h^{p^i})\rangle\langle(g, h)\rangle)$.*

Although stated for p-groups, this theorem characterizes permutability in any direct product of finite groups. To observe this, apply the permutability of M in $G \times H$ together with the Maier–Schmid Theorem [8] to conclude $\pi_G(M)/(\operatorname{Core}_G(M \cap G))$ and $\pi_H(M)/(\operatorname{Core}_H(M \cap H))$ are hypercentral in $G/\operatorname{Core}_G(M \cap G)$ and $H/\operatorname{Core}_H(M \cap H)$.

Despite the presence of this theorem, interesting questions remain. Specifically we present in Conjecture 5.1 of [5] a much more natural condition that is sufficient for a subgroup of a direct product to be permutable. Unfortunately, Example 5.4 in [5] demonstrates that this condition, presented here as Condition A, is not in general necessary for permutability of a subgroup M of a direct product $G \times H$.

Condition A. (1) *The subgroup $(M \cap G) \times (M \cap H)$ is a permutable subgroup of $G \times H$, and* (2) *for all $S \leq G \times H$ such that $(M \cap G) \times (M \cap H) \leq S$, we have, $M \leq N_{G\times H}(S)$.*

One would like to have a better understanding of the relationship between Condition A and permutability in direct products, and that is what is explored in this paper. Section 2 provides examples that are critical for understanding Condition A. Example 2.3 establishes a substantial difference between even and odd group orders when studying permutability in direct products, and Example 2.5

reveals a significant distinction between the parts of Condition A. Specifically, previous examples of permutable subgroups of direct products, like Example 5.4 in [5], satisfy Part (2) of Condition A, but not Part (1). Example 2.5 shows it is possible for a permutable subgroup of a direct product to satisfy Part (1) of Condition A but not Part (2).

In Section 3 our goal is to determine a class of groups in which Condition A characterizes permutability. We focus on groups with modular subgroup lattices. A subgroup of a finite group is permutable if and only if it is subnormal and modular (Theorem 5.1.1 [13]), and so a finite p-group has a modular subgroup lattice if and only if all its subgroups are permutable. Classified by Iwasawa [7], these groups provide the most accessible examples of permutable subgroups that are not normal, and play an important role in exploring many questions. For example, they are crucial to the study of groups in which permutability is transitive, as seen in investigations by Beidleman, Brewster, and Robinson in [1] and Robinson's continuation [11]. Corollary 3.4 in this article concludes that when G and H are groups of odd order with modular subgroup lattices, a subgroup of $G \times H$ is permutable if and only if it satisfies Condition A. In light of remarks in the preceding paragraph, this result is especially noteworthy for those interested in modularity, since it characterizes modularity in the direct product of two finite p-groups of odd order with modular subgroup lattices. We extend this result by proving in Theorem 3.5 that if G and H are any finite groups with modular subgroup lattices, then a permutable subgroup of $G \times H$ must satisfy the first part of Condition A. The question of whether or not such a subgroup must satisfy the second part of Condition A remains open.

Notation is standard, as found in Robinson [12]. Special thanks are due to L. C. Kappe for her valuable suggestions and to the referee, whose careful reading and comments have greatly improved this article.

2. Examples

In this section, we construct examples that provide a deeper understanding of Condition A. These examples are constructed using permutations that compose left to right.

In Lemma 5.8 of [5], we prove that if $G \times H$ is a group of odd order, and M is a permutable subgroup of $G \times H$ whose intersections with the direct factors are cyclic, then M satisfies the first part of Condition A. Example 2.3 reveals that this cannot be extended to direct products of even order. Example 2.1 is used as a direct factor in Example 2.3.

Example 2.1. Suppose that $n \in \mathbb{N}$, and $n \geq 4$. Let g, y_i for $i \in 1,2,3,4$ be elements of S_{2^n} where $g = (1,2,3,\dots,2^n)$ and $y_i = (a_{i0}, a_{i1}, \dots, a_{ij}, \dots, a_{i(2^{n-2}-1)})$ with $a_{ij} = 4j + i$. Finally, let $m = y_2 y_4$. Then, $\langle m\rangle\langle g\rangle \leq S_{2^n}$, and $\langle m\rangle$ is a Core-free permutable subgroup of $\langle m\rangle\langle g\rangle$.

Proof. Observe that for $i \in \{1,2,3\}$, $g^{-1}y_i g = y_{i+1}$ and $g^{-1}y_4 g = y_1$. This fact is used throughout the example. First, we show that $\langle m\rangle\langle g\rangle$ is a group by demonstrating that $g^m \in \langle m\rangle\langle g^2\rangle$. Notice that $g^m = y_1y_3 = m^{-1}g^4$ since $g^4 = y_1y_2y_3y_4$, completing this argument. Next observe $\mathrm{Core}_{\langle m\rangle\langle g\rangle}(\langle m\rangle) = 1$ since otherwise $\langle y_2^{2^{n-3}} y_4^{2^{n-3}}\rangle \lhd \langle m\rangle\langle g\rangle$, which is false.

In order to demonstrate that $\langle m\rangle$ is permutable in $\langle m\rangle\langle g\rangle$, we must prove that for all $i, j \in \mathbf{N}$, $\langle m\rangle$ permutes with $\langle m^j g^i\rangle$. When $\gcd(2,i) = 2$, then $[m, m^j g^i] = 1$. So, suppose that $\gcd(2,i) \neq 2$. It follows that $\langle m, m^j g^i\rangle = \langle m\rangle\langle g\rangle$. Since $\mathrm{Core}_{\langle m\rangle\langle g\rangle}(\langle m\rangle) = 1$, we have $\langle m\rangle \cap \langle m^j g^i\rangle = 1$. This yields $|\langle m\rangle\langle m^j g^i\rangle| = |\langle m\rangle||\langle m^j g^i\rangle|$. So, by showing that $|\langle m^j g^i\rangle| = 2^n$, we conclude that $\langle m\rangle$ permutes with $\langle m^j g^i\rangle$.

Since $g^2 \in Z(\langle m\rangle\langle g\rangle)$, assume $i = 1$. Now $(m^j g^i)^2 = (y_1y_2y_3y_4)^j g^2 = g^{4j+2}$, which has order 2^{n-1}. As a result, $o(m^j g^i) = 2^n$. Therefore, $\langle m\rangle$ is permutable in $\langle m\rangle\langle g\rangle$. $\square$

Corollary 5.2 in [3] provides criteria for determining when a subgroup M of a group G permutes with cyclic subgroups of $G \times H$. It is applied in Example 2.3, and in subsequent results, and so we state it here.

Lemma 2.2. *Let G and H be finite groups. Suppose that M is a subgroup of G and $(g,h) \in G \times H$. Then M permutes with $\langle (g,h)\rangle$ if and only if $g \in N_G(\langle g^{o(h)}\rangle M)$.*

Example 2.3. Let $G = \langle m\rangle\langle g\rangle$ as in Example 2.1, $H = \langle h\rangle$ be a cyclic group of order 2^{n-1}, and $M = \langle (m,1)\rangle\langle (g^2,h)\rangle$. Then M is a permutable subgroup of $G \times H$, but $M \cap G$ is not permutable in $G \times H$.

Proof. Obviously, M is a subgroup of $G \times H$ since $(g^2, h) \in Z(G \times H)$. Let $r, t, i \in \mathbb{N}$. Clearly, if $\gcd(2,i) = 2$, then M and $(m^r g^i, h^t)$ satisfy Condition (1) of Theorem 1.1.

So, assume $\gcd(2,i) \neq 2$. Then, $o(m^r g^i) = 2^n$, and $\langle m^r g^i\rangle \cap \langle m\rangle = 1$, since $\mathrm{Core}_{\langle m\rangle\langle g\rangle}(\langle m\rangle)$ is trivial. Thus, $|\langle m^r g^i\rangle/(\langle (m^r g^i, 1)\rangle \cap M)| = 2^n$, which is greater than $\exp(H/\mathrm{Core}_H(M \cap H))$. Furthermore, $\pi_H(M \cap (\langle m^r g^i\rangle \times \langle h^t\rangle)) =$

$\langle h^t\rangle$. Notice M normalizes $(M\cap G)\langle(1,h^t)\rangle\langle(m^r g^i,h^t)\rangle$. Hence, M and $(m^r g^i,h^t)$ satisfy Condition (2) of Theorem 1.1, completing the proof that M is a permutable subgroup of $G\times H$.

Finally, observe that $g^{-1}mg\notin\langle m\rangle\langle g^8\rangle$. Therefore, $g\notin N_G((M\cap G)\langle g^{2^{n-1}}\rangle)$. By Lemma 2.2, $M\cap G$ does not permute with $\langle(g,h)\rangle$. $\square$

Until now, all examples of permutable subgroups M of direct products $G\times H$ that satisfy Part (1) of Condition A also satisfy Part (2). This is not the case in Example 2.5. Example 2.4 serves as a direct factor in Example 2.5.

Example 2.4. Suppose that p is an odd prime, and let g, x_i for $i\in\mathbb{N}$ such that $1\le i\le p$ be elements of S_{p^4}, where $g=(1,2,3,\dots,p^4)$ and $x_i=(a_{i0},a_{i1},a_{i2},\dots,a_{ij},\dots,a_{i(p^3-1)})$ with $a_{ij}=jp+i$. Let $m_1=(x_1x_2^2x_3^3\dots x_{p-1}^{p-1})^p$, and for $j\in\mathbb{N}$ such that $2\le j\le p$, let $m_j=x_j^{p^2}$. Finally, let $M=\langle m_i|i\in\mathbb{N}$ and $2\le i\le p\rangle$. Then:

(1) g normalizes $\langle g^{p^3}\rangle M$, and so $\langle g\rangle M\le S_{p^4}$;

(2) $m_1^{-1}gm_1=g^{p^2+1}\bar m$ for some $\bar m\in M$, and so $\langle m_1\rangle\langle g\rangle M\le S_{p^4}$;

(3) $\mathrm{Core}_{\langle m_1\rangle\langle g\rangle M}(M)=1$; and

(4) for $r,t\in\mathbb{N}$ with $\gcd(t,p)\ne p$ and $w\in M$, $\langle(m_1^r g^t w)^p\rangle=\langle g^p\rangle$, and M is a permutable subgroup of $\langle m_1\rangle\langle g\rangle M$.

Proof. First observe that for $i\in\{2,3,\dots,p-1\}$, $g^{-1}m_ig=m_{i+1}$. Furthermore, $g^{-1}m_pg=x_1^{p^2}$. But $x_1^{p^2}=(x_1x_2\dots x_p)^{p^2}(m_2m_3\dots m_p)^{-1}$. Since $(x_1x_2\dots x_p)^{p^2}=g^{p^3}$, we have $g^{-1}m_pg\in\langle g^{p^3}\rangle M$. It follows that g normalizes $\langle g^{p^3}\rangle M$, which proves (1).

Secondly, $m_1^{-1}gm_1=gg^{-1}(x_1^{p^3-p}x_2^{p^3-2p}\cdots x_{p-1}^{p^3-p(p-1)})g(x_1^px_2^{2p}\cdots x_{p-1}^{p(p-1)})$ $=g(x_1x_2\cdots x_{p-1})^p(x_p^{p^3-p(p-1)})$. Since $(x_1x_2\cdots x_p)^p=g^{p^2}$, we have $m_1^{-1}gm_1=g^{p^2+1}\bar m$ for $\bar m\in M$. Observe $[m_1,M]=1$, and it follows that $m_1\in N_{S_{p^4}}(\langle g\rangle M)$, proving (2).

Since $g^{-1}x_ig=x_{i+1}$ for $i\in\mathbb{N}$ such that $1\le i<p$, and $g^{-1}x_pg=x_1$, we conclude that (3) is true, and so it is left to prove (4). Let $r,t\in\mathbb{N}$, and let $w\in M$. In order to show M is permutable in $\langle m_1\rangle\langle g\rangle M$, it is sufficient to show that M permutes with $\langle m_1^rg^tw\rangle$. When $\gcd(p,t)=p$, we have $[\langle m_1\rangle M,g^t]=1$, and then M permutes with $\langle m_1^rg^tw\rangle$. So, assume $\gcd(p,t)\ne p$. We will first prove that $\langle(m_1^rg^tw)^p\rangle=\langle g^p\rangle$. Observe that there are $c_2,c_3,\dots c_p\in\mathbb{N}$ such that $w=x_2^{c_2p^2}x_3^{c_3p^2}\cdots x_p^{c_pp^2}$. Then, $(m_1^rg^tw)^p=((x_1x_2^2x_3^3\cdots x_{p-1}^{p-1})^{rp}g^t(x_2^{c_2}x_3^{c_3}\cdots x_p^{c_p})^{p^2})^p=g^{tp}(x_1x_2\cdots x_p)^{rp^2(p-1)/2}(x_1x_2\cdots x_p)^{p^2(c_2+\cdots+c_p)}$ due to the conjugation of $x_i's$ by g^t. But $g^p=x_1x_2\cdots x_p$, and since p is odd,

$(m_1^r g^t w)^p = (g^p)^t (g^{p^3})^{r((p-1)/2)+c_2+c_3\cdots+c_p}$, which is a generator for $\langle g^p\rangle$ since $\gcd(p,t) \neq p$. Thus, $\langle (m_1^r g^t w)^p\rangle = \langle g^p\rangle$.

Notice that $g^p \in Z(\langle m_1\rangle\langle g\rangle M)$. So as a consequence of (1) in this example, $m_1^r g^t w$ normalizes $\langle (m_1^r g^t w)^p\rangle M$, completing the proof of (4). $\square$

Example 2.5. Let $G = \langle m_1\rangle\langle g\rangle M$ from the previous example, and set $H = \langle h_1\rangle \times \langle h_2\rangle$ where $\langle h_1\rangle$ and $\langle h_2\rangle$ are cyclic groups of orders p^2 and p^3 respectively. Finally, let $S = \langle (m_1, h_1)\rangle\langle (g^p, h_2)\rangle M$. Then, $(S \cap G) \times (S \cap H)$ and S are permutable subgroups of $G \times H$, but S does not normalize $((S \cap G) \times (S \cap H))\langle (g, h_2)\rangle$.

Proof. Since $g^p \in Z(G)$, and $[M, m_1] = 1$, it follows that S is a subgroup of $G \times H$. Furthermore, as a result of (1) in Example 2.4, $\langle g^{p^3}\rangle M \lhd G$. So, by Lemma 2.2, we conclude that $M \times 1$, which is $(S \cap G) \times (S \cap H)$, is permutable in $G \times H$. In order to show that S is permutable in $G \times H$, let $(y, z) \in G \times H$ be arbitrary. It is sufficient to demonstrate that S and (y, z) satisfy one of the two conditions in Theorem 1.1.

Observe that $y = m_1^i g^j m$, where $i, j \in \mathbb{N}$ and $m \in M$. If $\gcd(p, j) = p$, then $[S, (y, z)] = 1$, and Condition 1 of Theorem 1.1 holds. On the other hand, if $\gcd(p, j) \neq p$, then $m_1^{-1} y m_1 = m_1^i (m_1^{-1} g^j m_1) m$ since $[M, m_1] = 1$. Recall from (2) in Example 2.4 that $m_1^{-1} g m_1 = g^{p^2+1}\bar{m}$ where $\bar{m} \in M$. Since $g^{p^2} \in Z(G)$, we have $m_1^{-1} g^j m_1 = (g^{p^2})^j (g\bar{m})^j$. As a result of (1) in Example 2.4, $(g\bar{m})^j = g^{cp^3} m^* g^j$ for some $m^* \in M$ and $c \in \mathbb{N}$. Now since $g^{p^2} \in Z(G)$ and $[M, m_1] = 1$, we have $m_1^{-1} y m_1 = m^* g^{c_1 p^2} y$, where $c_1 \in \mathbb{N}$ and $\gcd(c_1, p) = 1$. By (4) in Example 2.4, $\langle g^{p^2}\rangle = \langle y^{p^2}\rangle$. Hence, $m_1^{-1} y m_1 = m^* y^{c_2 p^2+1}$ for $c_2 \in \mathbb{N}$ with $\gcd(c_2, p) = 1$.

Suppose $o(z) \leq p^2$. Then, since $g^p \in Z(G)$ and M is permutable in $G \times H$, we have that S and (y, z) satisfy Condition 1 of Theorem 1.1. So, assume $o(z) = p^3$. First observe that $\langle y, m_1, M\rangle = G$. Thus, $\langle y\rangle \cap M = 1$, since M is Core-free by (3) in Example 2.4. Hence, it follows from (4) in Example 2.4 that $|\langle y\rangle/(\langle y\rangle \cap M)| = p^4$.

Let $z = h_1^k h_2^l$ for $k, l \in \mathbb{N}$. Then $\gcd(l, p) = 1$. Since $\langle g^{p^3}\rangle = \langle y^{p^3}\rangle$, $\langle (g^{p^3}, h_2^{p^2})\rangle \leq (S \cap (\langle y\rangle \times \langle z\rangle))$. But $(m_1, h_1)^{-1}(y, z)(m_1, h_1) = (m^* y^{c_2 p^2+1}, h_1^k h_2^l) = (m^*, 1)(1, (h_2^{p^2})^{-c_2 l})(y, z)^{c_2 p^2+1}$. Since $g^p \in Z(G)$ and M is permutable in $G \times H$ it follows that S and (y, z) satisfy Condition 2 of Theorem 1.1, completing the proof that S is permutable in $G \times H$. Yet (g, h_2) satisfies the conditions placed on (y, z), and so (m_1, h) fails to normalize $((S \cap G) \times (S \cap H))\langle (g, h_2)\rangle$. $\square$

3. Direct products of groups with modular subgroup lattices

In this section, we prove that Condition A characterizes permutability in a direct product of groups of odd order with modular subgroup lattices. This is achieved in Corollary 3.4, which follows from Theorem 3.3. Observe that due to Example 2.5, it is necessary to prove that each part of Condition A is satisfied by a permutable subgroup.

We begin here with two lemmas that are applied repeatedly throughout the section. Lemmas 3.1 and 3.2 are (13) and (11) on Page 202 of [13] respectively.

Lemma 3.1. *Let M and N be subgroups of a group G such that $N \leq M$ and $N \lhd G$. Then M is a permutable subgroup of G if and only if M/N is a permutable subgroup of G/N.*

Lemma 3.2. *If M is a permutable subgroup of a group G, and S is a subgroup of G, then $M \cap S$ is a permutable subgroup of S.*

Theorem 3.3. *Let G and H be groups of odd order, and let M be a subgroup of $G \times H$ such that every subgroup of $M \cap G$ is permutable in G and every subgroup of $M \cap H$ is permutable in H. Then, M is a permutable subgroup of $G \times H$ if and only if M satisfies Condition A.*

Proof. The converse is clearly true, and so we prove only the forward direction. Let $G \times H$ be a group with permutable subgroup M that fails to satisfy Part (1) of Condition A, and assume $G \times H$ has minimal order with respect to this property. By the minimality of $|G \times H|$ and Lemma 3.1, $\mathrm{Core}_G(M \cap G)$ and $\mathrm{Core}_H(M \cap H)$ are trivial. It then follows from Lemma 3.2 and the minimality of $|G \times H|$ that $G = (M \cap G)\langle g\rangle$ and $H = (M \cap H)\langle h\rangle$ where $(g, h) \in G \times H$ and $(M \cap G) \times (M \cap H)$ and $\langle (g, h)\rangle$ do not permute. By an argument similar to one used in the proof of Theorem 5.2.8(b) in [13], $\langle g\rangle \cap (M \cap G)$ is trivial. Specifically, $(\langle g\rangle \cap (M \cap G))^G = (\langle g\rangle \cap (M \cap G))^{\langle g\rangle(M \cap G)} = (\langle g\rangle \cap (M \cap G))^{M \cap G} \leq M \cap G$, and since $\mathrm{Core}_G(M \cap G) = 1$, it follows that $\langle g\rangle \cap (M \cap G) = 1$. Similarly, $\langle h\rangle \cap (M \cap H) = 1$. So by the minimality of $|G \times H|$ and Lemma 3.2, and since all subgroups of $M \cap G$ and $M \cap H$ are permutable in G and H respectively, $M \cap G$ and $M \cap H$ are cyclic, contradicting Lemma 5.8 in [5].

Now let $G_1 \times H_1$ be a group with a permutable subgroup S that fails to satisfy Part (2) of Condition A, and assume that $G_1 \times H_1$ is of minimal order with this property. By Lemma 3.1 and the minimality of $|G_1 \times H_1|$, both $\mathrm{Core}_{G_1}(S \cap G_1)$ and $\mathrm{Core}_{H_1}(S \cap H_1)$ are trivial. It then follows from 3.7 in [5] that $\pi_{G_1}(S)$ and $\pi_{H_1}(S)$ are hypercentral in G_1 and H_1 respectively. So G_1 and H_1 are p-groups for the same odd prime p. As a consequence of Lemma 4.1 in [5], we may assume without loss of generality that $S \cap H_1 = 1$ and $\exp(H_1) \leq \exp(G_1)$. By Lemma 5.1 in [4], for all $(g_1, h_1) \in G_1 \times H_1$, we have $\pi_{G_1}(S)$ and $\pi_{H_1}(S)$ contained in $N_{G_1}(\langle g_1\rangle(S \cap G_1))$ and $N_{H_1}(\langle h_1\rangle(S \cap H_1))$ respectively. Thus it follows from Lemma 3.2 and the minimality of $|G_1 \times H_1|$ that $G_1 = \langle s_1\rangle(S \cap G_1)\langle x\rangle$ and $H_1 = \langle s_2\rangle\langle y\rangle$ where $(x, y) \in G_1 \times H_1$, $(s_1, s_2) \in S$, and $(s_1, s_2) \notin N_{G_1\times H_1}(((S\cap G_1)\times(S\cap H_1))\langle(x, y)\rangle)$.

Yet observe that $s_1^{-1}xs_1 = wx^i$ for $w \in S\cap G_1$ and $i \in \mathbb{N}$ where $\gcd(i, p) = 1$, and additionally, $s_2^{-1}ys_2 = y^j$ for $j \in \mathbb{N}$ with $\gcd(j, p) = 1$. We have already proved that (1) of Condition A holds. But every subgroup of $S\cap G_1$ is permutable in G_1, and so $\langle w\rangle\langle x\rangle$ is a group. Furthermore, by Lemma 3.2, $(S \cap G_1) \cap \langle w\rangle\langle x\rangle$ is permutable in $\langle w\rangle\langle x\rangle \times H_1$. Notice that $(\langle x\rangle \cap (S \cap G_1))^{G_1} = (\langle x\rangle \cap (S \cap G_1))^{\langle x\rangle(S\cap G_1)\langle s_1\rangle} = (\langle x\rangle\cap(S\cap G_1))^{(S\cap G_1)\langle s_1\rangle} \leq S\cap G_1$ since $S\cap G_1 \lhd \pi_{G_1}(S)$ and $s_1 \in \pi_{G_1}(S)$. Since $\mathrm{Core}_{G_1}(S \cap G_1) = 1$, it follows that $\langle x\rangle \cap (S \cap G_1) = 1$. Thus $\langle w\rangle$ is permutable in $\langle w\rangle\langle x\rangle \times H_1$. As a result of Theorem 1.1, $o(x) > o(y)$. Let $o(y) = p^k$ where $k \in \mathbb{N}$. Having established that $\langle w\rangle$ is permutable in $\langle w\rangle\langle x\rangle\times H_1$, we know that for any $x_0 \in \langle x\rangle$, the subgroups $\langle w\rangle$ and $\langle(x_0, y)\rangle$ permute. So by Lemma 2.2, $x_0 \in N_{\langle w\rangle\langle x\rangle}(\langle w\rangle\langle(x_0)^{p^k}\rangle)$. Hence, $x^iwx^{-i} = w^a(x^i)^{c_1p^k}$, where $a, c_1 \in \mathbb{N}$. Since $\langle w^p\rangle\langle x\rangle \leq G_1$, we have $\gcd(p,(a-1)) = p$.

In this paragraph we prove that $\langle w^p\rangle\langle x^{p^{k+1}}\rangle \lhd \langle w\rangle\langle x\rangle$. This is applied in calculations in the ensuing paragraphs. First, $\langle w^p\rangle\langle x^{p^{k+1}}\rangle = \langle w^p\rangle\langle x^{p^k}\rangle\cap\langle w\rangle\langle x^{p^{k+1}}\rangle$. Since both $\langle w^p\rangle\langle x^{p^k}\rangle$ and $\langle w\rangle\langle x^{p^{k+1}}\rangle$ have index p in $\langle w\rangle\langle x^{p^k}\rangle$, each of these subgroups is normal in $\langle w\rangle\langle x^{p^k}\rangle$, and so $\langle w^p\rangle\langle x^{p^{k+1}}\rangle \lhd \langle w\rangle\langle x^{p^k}\rangle$ with abelian factor group. From our work in the preceding paragraph we know that $x \in N_{\langle w\rangle\langle x\rangle}(\langle w\rangle\langle x^{p^k}\rangle)$. Therefore, there are $\bar{a}, \bar{c} \in \mathbb{N}$ so that $x^{-1}w^px = (x^{-1}wx)^p = (w^{\bar{a}}x^{\bar{c}p^k})^p \in \langle w^p\rangle\langle x^{p^{k+1}}\rangle$. This shows that x normalizes $\langle w^p\rangle\langle x^{p^{k+1}}\rangle$, and w normalizes $\langle w^p\rangle\langle x^{p^{k+1}}\rangle$ since $\langle w^p\rangle\langle x^{p^{k+1}}\rangle \lhd \langle w\rangle\langle x^{p^k}\rangle$, completing this proof.

We will now prove by induction that for all $l \in \mathbb{N}$, there are $r, t \in \mathbb{N}$ such that $(wx^i)^l = w^lw^{rp}x^{itp^{k+1}}(x^i)^{c_1(0+1+2+\cdots+(l-1))p^k}(x^i)^l$. Clearly this is true when $l = 1$. Assume it is true for $l = u$, where $u \in \mathbb{N}$. Then, $(wx^i)^{u+1} = (wx^i)w^uw^{r_1p}x^{it_1p^{k+1}}(x^i)^{c_1(0+1+2+\cdots+(u-1))p^k}(x^i)^u$ where $r_1, t_1 \in \mathbb{N}$. If one can show there are $c_2, c_3 \in \mathbb{N}$ so that $x^iw^u = w^uw^{c_2p}(x^i)^{c_1up^k}(x^i)^{c_3p^{k+1}}x^i$, then since $\langle w^p\rangle\langle x^{p^{k+1}}\rangle \lhd \langle w\rangle\langle x\rangle$, the induction proof is complete.

We proceed to prove that there are $c_2, c_3 \in \mathbb{N}$ such that
$x^iw^u = w^uw^{c_2p}(x^i)^{c_1up^k}(x^i)^{c_3p^{k+1}}x^i$ by induction on u. Since $\gcd(p,(a-1)) = p$, this is clear when $u = 1$. Assume it is true when $u = q$, for $q \in \mathbb{N}$. Then there are $c_4, c_5 \in \mathbb{N}$ such that $x^iw^{q+1} = w^qw^{c_4p}(x^i)^{c_1qp^k}(x^i)^{c_5p^{k+1}}x^iw$. Also $x^iw = w^a(x^i)^{c_1p^k+1}$, and applying that $\langle w^p\rangle\langle x^{p^{k+1}}\rangle \lhd \langle w\rangle\langle x\rangle$ and $\gcd(p,(a-1)) = p$, we have $x^iw^{q+1} = w^{q+1}w^{c_7p}(x^i)^{(q+1)c_1p^k}(x^i)^{c_6p^{k+1}}x^i$ for $c_6, c_7 \in \mathbb{N}$, completing the induction proof.

At this point, $s_1^{-1}x^p s_1 = (wx^i)^p = w^b (x^i)^{vp^{k+1}} (x^i)^p$ for some $b, v \in \mathbb{N}$ since p divides $1 + \cdots + (p-1)$. Furthermore, $w^b (x^i)^{vp^{k+1}} (x^i)^p = w^b (x^p)^{ivp^k + i}$. But by the minimality of $|G_1 \times H_1|$, we have $(s_1, s_2) \in N_{G_1 \times H_1}((S \cap G_1)\langle (x^p, y)\rangle)$. Since $o(x) > o(y)$ and $o(y) = p^k$, it follows that $ivp^k + i \equiv j \mod p^k$. Therefore, $i \equiv j \mod p^k$, contradicting that $(s_1, s_2) \notin N_{G_1 \times H_1}((S \cap G_1)\langle (x, y)\rangle)$. □

In order to prove Corollary 3.4, notice that a minimal counterexample would need to reduce to a p-group of odd order in which every subgroup is permutable. Theorem 3.3 guarantees that such a counterexample does not exist.

Corollary 3.4. *Let G and H be groups of odd order with modular subgroup lattices. Then M is a permutable subgroup of $G \times H$ if and only if M satisfies Condition A.*

Due to Example 2.3, the hypothesis that G and H have odd order cannot be removed from Theorem 3.3. In Theorem 3.5, we are able to show that when this hypothesis is removed from Corollary 3.4, a permutable subgroup must still satisfy the first part of Condition A. It remains an open question as to whether such a subgroup must satisfy Part (2) of Condition A.

Theorem 3.5. *Let G and H be finite groups with modular subgroup lattices. If M is a permutable subgroup of $G \times H$, then M satisfies Part 1 of Condition A.*

Proof. It is sufficient to consider the case where G and H are 2-groups. Let $G \times H$ be a group with a permutable subgroup M that fails to satisfy Part (1) of Condition A, and assume that $G \times H$ is of minimal order with this property.

By Lemma 3.1 and the minimality of $|G \times H|$, both $\mathrm{Core}_G(M \cap G)$ and $\mathrm{Core}_H(M \cap H)$ are trivial. As a result of Lemma 4.1 in [5], we may assume without loss of generality that $M \cap H = 1$ and $\exp(H) < \exp(G)$. Again by the minimality of $|G \times H|$ and Lemma 3.2, $G = (M \cap G)\langle g \rangle$ and $H = \langle h \rangle$ where $M \cap G$ does not permute with $\langle (g, h)\rangle$. It then follows from Lemma 2.2 that $g \notin N_G(\langle g^{o(h)}\rangle (M \cap G))$.

By Theorem 5.2.8 in [13], $o(g) = \exp(G)$. So, let $o(g) = 2^n$ and $o(h) = 2^k$ with $n > k$. Apply the minimality of $|G \times H|$ and Lemma 3.2 to see that $M \cap G$ is permutable in $G \times \langle h^2 \rangle$. Thus, by Lemma 2.1, $g \in N_G(\langle g^{2^{k-1}}\rangle (M \cap G))$. Let $M \cap G = \langle m \rangle$. Then $g^{-1} m g = g^{c2^{k-1}} m^i$, where $i, c \in \mathbb{N}$, but $\gcd(c, 2) = 1$.

Once more, apply the minimality of $|G \times H|$ and Lemma 3.2 to conclude that $M \cap G$ is permutable in $(M \cap G)\langle g^2 \rangle \times \langle h \rangle$. As a result of Lemma 2.2, $g^2 \in N_G(\langle g^{2^{k+1}}\rangle (M \cap G))$. Now $g^{-2} m g^2 = g^{-1}(g^{c2^{k-1}} m^i) g = g^{c2^{k-1}} (g^{c2^{k-1}} m)^i = g^{c(1+i)2^{k-1}} g^{l2^{k+1}} m^j$ for $j, l \in \mathbb{N}$. Since $\langle g \rangle \cap \langle m \rangle = 1$ by Theorem 5.2.8 in [13], 4 divides $1 + i$, and so, $i \equiv 3 \mod 4$.

We will now show that $\langle m^4\rangle\langle g^2\rangle$ is a normal subgroup of G. Since $\langle m\rangle\langle g\rangle$ is a 2-group, $\langle m^2\rangle\langle g^2\rangle$ is a normal subgroup of G. Furthermore, $\langle m^4\rangle\langle g\rangle$ is a subgroup of G, and recall that $\langle m\rangle \cap \langle g\rangle = 1$. So, $g^{-1}m^4g \in \langle m^4\rangle\langle g^2\rangle$. Next apply the fact that $\langle m^2\rangle\langle g\rangle \lhd \langle m\rangle\langle g\rangle$, together with the normality of $\langle m^4\rangle\langle g\rangle$ in $\langle m^2\rangle\langle g\rangle$, to conclude $m^{-1}g^2m \in \langle m^4\rangle\langle g^2\rangle$. This completes the argument that $\langle m^4\rangle\langle g^2\rangle \lhd G$. Finally, observe that $G/\langle m^4\rangle\langle g^2\rangle$ is isomorphic to the dihedral group of order 8, contradicting that G is a group with a modular subgroup lattice. □

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