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A stability property of the octahedron and the icosahedron

By KÁROLY BÖRÖCZKY (Budapest) and KÁROLY J. BÖRÖCZKY (Budapest)

Abstract. According to a recent result, for $r = \sqrt{3}$ or $r = \sqrt{15 - 6\sqrt{5}}$, the convex body of minimal volume or of minimal surface area in $\mathbb{E}^3$ that contains a unit ball, and the extreme points are of distance at least r from the centre of the unit ball is the regular octahedron and icosahedron, respectively. In this paper we prove corresponding stability results.

1. Notation and known results

We write B^3 to denote the unit Euclidean ball centred at the origin o in $\mathbb{E}^3$, and S^2 to denote the boundary of B^3. As usual a convex body C in $\mathbb{E}^3$ is a compact convex set with non-empty interior, and $V(C)$ and $S(C)$ denotes its volume and surface area, respectively. The two-dimensional Hausdorff measure of a measurable subset C of the boundary of some convex body in $\mathbb{E}^3$ is called the area $A(C)$ of C.

Answering a conjecture of J. Molnár [11], K. Böröczky and K. Böröczky, Jr. [2] proved the following (see [2] for the history of the problem, and for related results and conjectures).

Theorem 1.1 (K. Böröczky, K. Böröczky, Jr.)**.** *Given $r = \sqrt{3}$ or $r = \sqrt{15 - 6\sqrt{5}}$, let M_r be the octahedron, or the icosahedron, respectively, circumscribed around B^3. If P is any polytope in $\mathbb{E}^3$ containing B^3, and each vertex of P is of distance at least r from o then $V(P) \geq V(M_r)$ and $S(P) \geq$*

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$S(M_r)$. Moreover equality holds in either inequalities if and only if P is congruent to M_r, and its circumcentre is o.

In this paper our goal is to provide a stability version of Theorem 1.1. To define the corresponding class of convex bodies in $\mathbb{E}^3$, we recall that x is an extreme point of a convex body C if it does not lie in the relative interior of any segment contained in C. Actually the extreme points form the minimal subset of C whose convex hull is C.

Definition. Given $r > 1$, we write $\mathcal{F}_r$ to denote the family of convex bodies in $\mathbb{E}^3$, which contain B^3, and whose extreme points are of distance at least r from o. Moreover let $P_r \in \mathcal{F}_r$ have minimal volume, and let $Q_r \in \mathcal{F}_r$ have minimal surface area.

The minima do exist according to the Blaschke Selection Theorem, and all extreme points of P_r and Q_r lie on rS^2 by the monotonicity of the volume and surface area. Theorem 1.1 states that if $r = \sqrt{3}$ or $r = \sqrt{15 - 6\sqrt{5}}$ then P_r is an octahedron, or an icosahedron, respectively, circumscribed around B^3. Moreover the analogous statement holds for Q_r.

To state a stability version of Theorem 1.1, we say that the compact convex sets M and N are ε-close for $\varepsilon > 0$ if there exist congruent copies M' and N' of M and N, respectively, satisfying

$$\frac{1}{1+\varepsilon} N' \subset M' \subset (1+\varepsilon) N'.$$

Naturally in this case the dimensions of M and N coincide.

Theorem 1.2. *Given $r = \sqrt{3}$ or $r = \sqrt{15 - 6\sqrt{5}}$, if $M \in \mathcal{F}_r$ satisfies $V(M) = (1+\varepsilon)V(P_r)$ for small ε then M is $c\sqrt{\varepsilon}$-close to some octahedron or to some icosahedron, respectively, where c is a positive absolute constant. Moreover the analogous statement holds for Q_r.*

The order of the error term $c\sqrt{\varepsilon}$ is optimal in Theorem 1.2 (see Example 5.1). Actually the statement concerning the surface area yields the statement concerning the volume in Theorem 1.2.

We note that if r is close to 1 then it seems to be out of reach to determine P_r or Q_r. However K. Böröczky, K.J. Böröczky and G. Wintsche [3] prove that in this case most part of the boundaries of P_r and of Q_r are the union of triangles that are almost regular.

The paper is structured in the following way: Section 2 discusses polytopal approximation from our point of view, and Section 3 introduces orthoschemes.

Section 4 proves that if the contribution of a face is close to be optimal then the face is close to be a suitable regular triangle. This section relies heavily on results and methods in K. Böröczky and K. Böröczky, Jr. [2]. Finally the proof of Theorem 1.2 is presented in Section 5.

2. Hausdorff distance and polytopal approximation

Let us introduce the notation used throughout the paper. For any notions related to convexity in this paper, consult R. Schneider [14]. We write $\langle \cdot, \cdot \rangle$ to denote the scalar product in $\mathbb{E}^3$, and $\|\cdot\|$ to denote the corresponding Euclidean norm. In addition for non-collinear points u, v, w, the angle of the half lines vu and vw is denoted by $\angle uvw$. Given a set $X \subset \mathbb{E}^3$, the affine hull and the convex hull of X are denoted by $\operatorname{aff} X$ and $\operatorname{conv} X$, respectively, moreover the interior of X is denoted by $\operatorname{int} X$. If X is compact convex then we write ∂X to denote the relative boundary of X with respect to $\operatorname{aff} X$.

In many instances we will approximate convex bodies by polytopes (see the papers P. M. Gruber [7], [8], [9] and [10] for general surveys). A natural measure of closeness between convex bodies is the so-called *Hausdorff distance*. For a $x \in \mathbb{E}^3$ and a compact $X \subset \mathbb{E}^3$, we write $d(x, X)$ to denote the minimal distance between x and the points of X. If K and C are compact convex sets in $\mathbb{E}^3$ then their Hausdorff distance is

$$\delta_H(K, C) = \max\big\{\max_{x\in K} d(x, C), \max_{y\in C} d(y, K)\big\}.$$

Naturally the maximum of $d(x, C)$ among $x \in K$ is attained at some extreme point of K, and conversely. The Hausdorff distance is a metric, and we always consider the space of compact convex sets endowed with this metric. In particular we say that a sequence $\{K_m\}$ of compact convex sets tends to a compact convex set C if $\lim_{m\to\infty} \delta_H(K_m, C) = 0$. Actually the topology induced by the Hausdorff distance on the space of convex bodies coincides with the topology induced by ε-closeness. For the main properties of the Hausdorff distance, consult R. Schneider [14]. For example, the volume and surface area are continuous functions of convex bodies. More precisely, if K and C are convex bodies that contain B^3 then

$$(1 - \delta_H(K, C))^3 V(K) < V(C) < (1 + \delta_H(K, C))^3 V(K); \tag{1}$$

$$(1 - \delta_H(K, C))^2 S(K) < S(C) < (1 + \delta_H(K, C))^2 S(K). \tag{2}$$

According to the Blaschke Selection Theorem, if $\{K_m\}$ is a sequence of compact convex sets that are contained in a fixed ball then $\{K_m\}$ has a subsequence $\{K_{m'}\}$ that tends to some compact convex set C. For $r > 1$, we write $\widetilde{\mathcal{F}}_r$ to denote the family of convex bodies that contain B^3 and whose extreme points lie on rS^2.

Lemma 2.1. *Given $r \in (1, \sqrt{3}\,]$, if $k \geq \frac{576}{r-1}$ and $C \in \widetilde{\mathcal{F}}_r$ then there exists a polytope $M \in \widetilde{\mathcal{F}}_r$ that has at most k vertices, and satisfies*

$$\delta_H(M, C) < \frac{36\sqrt{r-1}}{\sqrt{k}}.$$

PROOF. Let $m = \lfloor \sqrt{k/6} \rfloor$, hence $m \geq \frac{\sqrt{k}}{2\sqrt{6}}$. Consider the edge to edge tiling of each face of cube $[-r, r]^3$ by m^2 congruent squares, and let $\mathcal{S}'$ denote the family of centres of the all together $6m^2 \leq k$ squares. For each boundary point x of $[-r, r]^3$, we have a $y \in \mathcal{S}'$ with $\|x - y\| \leq \frac{\sqrt{2}r}{m} \leq \frac{\sqrt{6}}{m} \leq \frac{12}{\sqrt{k}}$. We write $\mathcal{S}$ to denote the radial projection of $\mathcal{S}'$ into rS^2. Since radial projection from the complement of $\operatorname{int} rB^3$ onto rS^2 decreases distance, for any $x \in rS^2$, there exists a $y \in \mathcal{S}$ with $\|x - y\| \leq \frac{12}{\sqrt{k}}$. In addition $\mathcal{S}$ has at most k points.

We define M to be the convex hull of those $y \in \mathcal{S}$ whose distance from some extreme point of C is at most $\frac{24}{\sqrt{k}}$. First we show that M contains B^3; or in other words, for any $z \in S^2$, there exists $y \in M$ with $\langle y, z \rangle \geq 1$. Since $B^3 \subset C$, we have an extreme point x of C with $\langle x, z \rangle \geq 1$. Let $x' \in rS^2$ such that $\|x - x'\| = \frac{12}{\sqrt{k}}$, and either rz is contained in the shorter great circle arc on rS^2 connecting x and x' (if $\|rz - x\| \leq \frac{12}{\sqrt{k}}$), or x' is contained in the shorter great circle arc on rS^2 connecting x and rz (if $\|rz - x\| \geq \frac{12}{\sqrt{k}}$). Finally let y be a point of $\mathcal{S}$ with $\|y - x'\| \leq \frac{12}{\sqrt{k}}$, hence $y \in M$, and it is easy to see that $\|y - rz\| \leq \frac{24}{\sqrt{k}}$. Since $\frac{24}{\sqrt{k}} \leq \sqrt{r^2 - 1}$, we deduce $\langle y, z \rangle \geq 1$, as it is required.

Next we estimate the Hausdorff distance between M and C. If x is an extreme point of C then its distance from some vertex y of M is at most $\frac{24}{\sqrt{k}}$, hence

$$d(x, M) \leq d(x, \operatorname{conv}\{y, B^3\}) \leq \frac{24}{\sqrt{k}} \cdot \frac{\sqrt{r^2 - 1}}{r} < \frac{36}{\sqrt{k}} \cdot \sqrt{r - 1}.$$

The analogous argument for $d(y, C)$ where y is any vertex of M completes the proof of Lemma 2.1. □

3. Orthoschemes and the density of the surface area

Let us note that if $M \in \mathcal{F}_r$ then $\frac{S(B^3)}{S(M)}$ is maximal for $M = Q_r$. This observation suggests the following definition: We write $p_{S^2}(\cdot)$ to denote the radial projection onto S^2. If F is a convex domain in $\mathbb{E}^3$ whose affine hull avoids $\operatorname{int} B^3$ then we define the density of the surface area to be

$$d(F) = \frac{A(p_{S^2}(F))}{A(F)}.$$

If in addition $F \subset \sqrt{3}B^3$ then $d(F)$ satisfies

$$(\sqrt{3})^{-3} < d(F) < 1.$$

Next we say that a tetrahedron $S = \operatorname{conv}\{o, v_1, v_2, v_3\}$ is an *orthoscheme* if v_1 is orthogonal to $\operatorname{aff}\{v_1, v_2, v_3\}$, and $v_3 - v_2$ is orthogonal to $\operatorname{aff}\{o, v_1, v_2\}$. Naturally the order of vertices is important. If $\varrho_i = \|v_i\|$ for $i = 1, 2, 3$ then we call S a $(\varrho_1, \varrho_2, \varrho_3)$-orthoscheme. K. Bezdek [1] proved the following result:

Lemma 3.1 (Bezdek)**.** *For $i = 1, 2$, let $S_i = \operatorname{conv}\{o, v_{i1}, v_{i2}, v_{i3}\}$ be an orthoscheme in $\mathbb{E}^3$, and let $R_i = \operatorname{conv}\{v_{i1}, v_{i2}, v_{i3}\}$. If $1 \le \|v_{1j}\| \le \|v_{2j}\|$ for $j = 1, 2, 3$ then*

$$d(R_1) \ge d(R_2),$$

with equality if and only if S_1 and S_2 are congruent.

4. The stability around the optimal face

K. Böröczky, K. Böröczky, Jr. [2] proved that the regular triangles touching B^3 maximize the density of the surface area among the faces of polytopes in $\widetilde{\mathcal{F}}_r$.

Lemma 4.1 ([2], Corollary 4.8)**.** *Given $r \in (1, \sqrt{3}]$, if F is any polygon whose vertices lie on rS^2 and $\operatorname{aff} F$ avoids $\operatorname{int} B^3$ then*

$$d(F) \le \frac{8 \arctan \frac{\sqrt{3}(r-1)}{3+r}}{\sqrt{3}(r^2 - 1)},$$

with equality if and only if F is a regular triangle touching B^3.

To prove Theorem 1.2, we will verify Corollary 4.11, which is a stability version of Lemma 4.1. But first we verify a stability version of Lemma 4.1 in the special case when F is a triangle and $\operatorname{aff} F$ touches B^3.

Since we are interested in the order of the error in Theorem 1.2, we only calculate the constants involved when it does not make the argument more complicated. Usually we also do not calculate how small exactly the positive ε should be chosen to make the estimates below work.

Lemma 4.2. *There exist positive absolute constants c_1 and c_2 with the following properties. Given $r \in (1, \sqrt{3}]$, let D be a circular disc of radius $\sqrt{r^2-1}$ that touches B^3 in the centre of D. If T is any triangle whose vertices lie on ∂D, and*

$$d(T) > [1 - \varepsilon\,(r-1)] \cdot \frac{8 \arctan \frac{\sqrt{3}(r-1)}{3+r}}{\sqrt{3}(r^2-1)}$$

for $\varepsilon \le c_1$, then T is $c_2\sqrt{\varepsilon}$-close to the regular triangle of circumradius $\sqrt{r^2-1}$.

The whole section is dedicated to the proof of Lemma 4.2 and Corollary 4.11. During the proof of Lemma 4.2, T always denotes some triangle whose vertices lie in ∂D. Moreover the sides of T and their lengths are denoted by a, b, c, and the distances of the midpoints of a, b and c from o are denoted by m_a, m_b and m_c, respectively. We plan to compare T to the regular triangle T_* inscribed into D. We write a_* to denote the common length of the sides of T_*, and m_* to denote the common distance of the midpoints of the sides of T_* from o.

In the course of the argument, we will consider T as part of a certain family $T(s)$ of triangles inscribed into D where the vertices of $T(s)$ are differentiable functions of the real parameter s. In all cases, s will be the length of a side of $T(s)$. Our main tool is to investigate the density of the variation; namely,

$$v(T(s)) = \frac{\frac{d}{d\,s}\,A(p_{S^2}(T(s)))}{\frac{d}{d\,s}\,A(T(s))}.$$

It is easy to see that reparametrization does not change the density of variation. When it is clear from the context what the family $T(s)$ is then we drop the reference to s. Let us explain the role of $v(T(s))$: If $A(T(s))$ is a strictly monotone function of s on an interval $[s_1, s_2]$ then the Cauchy Mean Value theorem provides $s \in (s_1, s_2)$ satisfying

$$d(T(s_2)) - d(T(s_1)) = [v(T(s)) - d(T(s_2))] \cdot \frac{A(T(s_2)) - A(T(s_1))}{A(T(s_1))}; \tag{3}$$

$$= [v(T(s)) - d(T(s_1))] \cdot \frac{A(T(s_2)) - A(T(s_1))}{A(T(s_2))}. \tag{4}$$

Fortunately the density of variation satisfies the simple formulae in Propositions 4.3 and 4.4.

The proof of Lemma 4.2 is prepared by Propositions 4.3 to 4.9. Actually Propositions 4.3, 4.4 and 4.5 were proved in K. Böröczky, K. Böröczky, Jr. [2].

Proposition 4.3 ([2], Proposition 4.3)**.** *If $a < b$, and T is deformed in a way that the side c is kept fixed and a is increased then $A(T)$ is strictly increasing, and*

$$v(T) = \frac{r}{m_a^2 m_b^2}.$$

Proposition 4.4 ([2], Proposition 4.4)**.** *Let $b = c$, and and let T be deformed in a way that T stays isosceles, and the side a is increased. If $a < a_*$ then $A(T)$ is strictly increasing, and if $a > a_*$ then $A(T)$ is strictly decreasing. Moreover*

$$v(T) = \frac{r}{m_a^2 m_b^2}.$$

Proposition 4.5 ([2], Proposition 4.5)**.** *Assuming that $b = c$ and T has no obtuse angle, let us parametrize T as $T(a)$ where $0 < a \leq 2\sqrt{r^2 - 1}$. Then*

(i) $d(T(a_*)) = d(T_*) = \dfrac{8 \arctan \frac{\sqrt{3}(r-1)}{3+r}}{\sqrt{3}(r^2 - 1)}$;

(ii) $\lim\limits_{a \to 0} d(T(a)) = \lim\limits_{a \to 0} v(T(a)) = \dfrac{1}{r}$;

(iii) $v\bigl(T\bigl(2\sqrt{r^2 - 1}\,\bigr)\bigr) = \dfrac{2r}{1 + r^2}$.

We note that Proposition 4.5 (iii) corresponds to the case when the angle of T opposite to a is a right angle. Let us introduce some further notation that will be used through the proof of Lemma 4.2. We define

$$\begin{aligned} a_{**} &= 2\sqrt{r^2 - 1}\sqrt{\frac{5}{7}} < a_* - 0.01\sqrt{r - 1}; \\ m_{**} &= \sqrt{\frac{5 + 2r^2}{7}} > m_*; \end{aligned} \tag{5}$$

and compare the quantities in Proposition 4.5:

Proposition 4.6.

$$\frac{1}{r} + \frac{1}{28}(r - 1) < d(T_*) < \frac{r}{m_{**}^4} - \frac{1}{9}(r - 1);$$

$$\frac{r}{m_{**}^4} < \frac{r}{m_*^4} < \frac{2r}{1 + r^2}.$$

Proof. Since $\arctan t > t - \frac{1}{3}t^3$ and $1 \leq r \leq \sqrt{3}$, we obtain

$$d(T_*) - \frac{1}{r} > \frac{8}{\sqrt{3}(r^2-1)} \cdot \left[\frac{\sqrt{3}(r-1)}{3+r} - \frac{1}{3}\left(\frac{\sqrt{3}(r-1)}{3+r}\right)^3 \right] - \frac{1}{r}$$
$$= \frac{(27-10r-r^2)\cdot(r-1)}{r(r+3)^3} > \frac{r-1}{28}.$$

Next $\arctan t < t$ and $1 \le r \le \sqrt{3}$ yield

$$\frac{r}{m_{**}^4} - d(T_*) > \frac{49r}{(5+2r^2)^2} - \frac{8}{(3+r)(1+r)}$$
$$= \frac{(-32r^3+17r^2+53r+200)\cdot(r-1)}{(5+2r^2)^2(3+r)(1+r)} > \frac{r-1}{9}.$$

Finally the second set of inequalities follow as $m_{**} > m_*$, and as $m_* = \sqrt{r^2+3}/3$ yields $\frac{r}{m_*^4} < \frac{2r}{1+r^2}$. □

Next let $S_{**} = \text{conv}\{o, w_1, w_2, w_3\}$ be an $(1, m_{**}, r)$-orthoscheme, and let R_{**} be the face opposite to o. In particular, the side w_2w_3 of R_{**} is of length $a_{**}/2$.

Proposition 4.7. *There exists a positive absolute constant c with the following property. If $r \in (1, \sqrt{3}]$ then*

$$d(R_{**}) < d(T_*) - c(r-1).$$

Proof. During the argument, $\gamma_1, \gamma_2, \dots$ denote suitable positive absolute constants.

We choose w_2' in a way that $S_* = \text{conv}\{o, w_1, w_2', w_3\}$ is an $(1, m_*, r)$-orthoscheme, and w_2 and w_2' lie on the same side of $\text{aff}\{o, w_1, w_3\}$. Writing R_* to denote the face of S_* opposite to o, we have

$$d(R_*) = d(T_*).$$

We write p to denote the intersection point of the segments w_1w_2 and $w_2'w_3$, and q to denote the midpoint of the segment $w_2'p$. Further, let $R' = \text{conv}\{w_1, w_2', q\}$, $R'' = \text{conv}\{w_1, q, p\}$ and $R = \text{conv}\{w_1, p, w_3\}$. Our first goal is to prove

$$d(R) < d(R_*) - \gamma_1(r-1). \tag{6}$$

Since $A(R') > \gamma_2 A(R_*)$, it is sufficient to verify

$$d(R'') \le d(R) \quad \text{and} \tag{7}$$

$$d(R') \leq d(R) - \gamma_3(r-1). \tag{8}$$

We parametrize the side $w_2'w_3$ of R_* as $u(\omega)$ where ω is the angle $\angle w_2'w_1u(\omega)$ and $0 \leq \omega \leq \frac{\pi}{3}$. Moreover we define $\varphi(\omega) = \angle u(\omega)ow_1$, thus $\varphi(\omega)$ is an increasing function of ω, and $\|u(\omega) - v_1\| = \tan\varphi(\omega)$. Let ω' and ω'' be the parameters satisfying $q = u(\omega')$ and $p = u(\omega'')$, respectively, hence (5) yields

$$\omega' > \gamma_4, \quad \text{and} \quad \varphi(\omega') > \gamma_4\sqrt{r-1}, \quad \text{and} \quad \varphi(\omega'') - \varphi(\omega') > \gamma_4\sqrt{r-1}. \tag{9}$$

Rotating some plane about the line aff$\{o, w_1\}$ and using $\|u(\omega) - w_1\| = \tan\varphi(\omega)$ lead to

$$A(R') = \int_0^{\omega'} \frac{1}{2}\tan^2\varphi(\omega)\,d\omega \quad \text{and} \quad A(p_{S^2}(R')) = \int_0^{\omega'} (1 - \cos\varphi(\omega))\,d\omega.$$

In addition, we obtain the analogous formulae for R'' and R by integrating on the intervals $[\omega', \omega'']$ and $[\omega'', \frac{\pi}{3}]$, respectively. Now we define

$$g(\varphi) = \frac{1 - \cos\varphi}{\frac{1}{2}\tan^2\varphi} = \frac{2\cos^2\varphi}{1 + \cos\varphi}.$$

For $0 \leq \varphi < \psi \leq \frac{\pi}{3}$, we have

$$g(\varphi) - g(\psi) = 4\sin\frac{\psi - \phi}{2}\sin\frac{\psi + \phi}{2} \cdot \frac{\cos\varphi + \cos\psi + \cos\varphi\cos\psi}{(1 + \cos\varphi)(1 + \cos\psi)} > 0.$$

Since $g(\varphi(\omega))$ is increasing, we deduce (7) as

$$d(R'') < g(\varphi(\omega'')) < d(R).$$

Moreover (9) yields that $g(\omega'') - g(\omega') > \gamma_5(r-1)$, which in turn yields (8). Therefore we conclude (6).

Next we prove $d(R_{**}) \leq d(R)$. We write Φ to denote the linear transformation satisfying

$$\Phi w_1 = w_1, \quad \text{and} \quad \Phi w_2 = w_2' \quad \text{and} \quad \Phi w_3 = w_3,$$

and define $E = \Phi B^3$. Since $\langle \Phi w_i, \Phi w_j \rangle \leq \langle w_i, w_j \rangle$ for any $1 \leq i, j \leq 3$, we deduce that $E \cap S_* \subset B^3 \cap S_*$. Therefore

$$d(R_{**}) = \frac{V(B^3 \cap S_{**})}{V(S_{**})} = \frac{V(E \cap S_*)}{V(S_*)} \leq \frac{V(B^3 \cap S_*)}{V(S_*)} = d(R).$$

In turn we conclude Proposition 4.7. $\square$

We write T_{**} to denote the triangle inscribed into D that has two equal sides, and whose third (shortest) side is a_{**}. In addition we define

$$\Xi_{**} = \frac{1}{2}\left(d(T_*) + \max\left\{\frac{1}{r},\, d(T_{**})\right\}\right) < d(T_*)$$

(compare Lemma 4.1 and Proposition 4.6).

Proposition 4.8. *Let us assume that $b = c$ and T has no obtuse angle.*

(i) *If $a \geq a_{**}$ then*

$$\frac{r}{m_a^2 m_b^2} > d(T_*) + \frac{1}{9}(r-1).$$

(ii) *There exists positive $a_{00} < a_{**}$ with the following property:*

$$\text{if } a < a_{00} \text{ then } \frac{r}{m_a^2 m_b^2} < \Xi_{**};$$
$$\text{if } a > a_{00} \text{ then } \frac{r}{m_a^2 m_b^2} > \Xi_{**}.$$

Proof. Let 2ω be the angle of T opposite to a, and let $s = \sin^2\omega$. In particular $s \in (0, \frac{1}{2}]$ where $s = \frac{1}{2}$ and $s = \frac{1}{4}$ correspond to the cases when T has a right angle or is regular, respectively, and if s tends to zero then T approaches a diameter of D. Writing $\Omega = r^2 - 1$, we have

$$m_a^2 m_b^2 = f(s) \quad \text{for } f(s) = \left[1 + \Omega(1-2s)^2\right]\cdot(1+\Omega s).$$

We deduce by Proposition 4.6 that

$$\frac{r}{f(0)} < \Xi_{**} < d(T_*) \quad \text{and} \quad \frac{r}{f(\frac{1}{2})} > \frac{r}{f(\frac{1}{4})} > d(T_*). \tag{10}$$

Let us observe that

$$f'(s) = 12\Omega^2 s^2 + 8(\Omega - \Omega^2)s + \Omega^2 - 3\Omega, \tag{11}$$

hence $0 < \Omega \leq 2$ implies that $f'(0) < 0$ and $f'(\frac{1}{2}) = \Omega > 0$. Since f' is quadratic in s with positive main coefficient, we deduce that f is first decreasing then increasing on $[0, \frac{1}{2}]$. It follows that $\frac{r}{m_a^2 m_b^2} = \frac{r}{f(s)}$ is first an increasing, and afterwards a decreasing function of s, and hence of a. We define a_{00} to be the smallest positive a such that for the corresponding s_{00}, we have $\frac{r}{f(s_{00})} = \Xi_{**}$ (compare (10)). In particular (10) and $\frac{2r}{1+r^2} > \frac{r}{m_{**}^4} > d(T_*) + \frac{1}{9}(r-1)$ complete the proof of Proposition 4.8. $\square$

In order to apply (3) and (4) for stability statements, we need estimates on the variation of the area of a triangle.

Proposition 4.9. *There exist positive absolute constants $\tilde{c}_1$ and $\tilde{c}_2$ with the following properties. Let T be not ε_1-close but ε_2-close to T_* for $\varepsilon_2 > \varepsilon_1 > 0$.*

(i) *If $\varepsilon_1 < 0.05$ then*
$$A(T) < (1 - \tilde{c}_1 \varepsilon_1^2) A(T_*);$$

(ii) *if $\varepsilon_2 < 0.05$ then*
$$A(T) > (1 - \tilde{c}_2 \varepsilon_2^2) A(T_*).$$

Proof. If T has an angle that is at most $\frac{\pi}{4}$ or at least $\frac{5\pi}{12}$ then T has an angle that is at least $\frac{3\pi}{8}$, hence has a side whose length is at least $\frac{\sin\frac{3\pi}{8}}{\sin\frac{\pi}{3}}\, a_* > 1.06 a_*$. Moreover $A(T) \leq \frac{\sin\frac{3\pi}{4} + 2\sin\frac{5\pi}{8}}{3\sin\frac{2\pi}{3}}\, A(T_*) < 0.99 A(T_*)$ in this case. Therefore we may assume that all angles of T are between $\frac{\pi}{4}$ and $\frac{5\pi}{12}$.

Let θ_1, θ_2 and θ_3 be the angles enclosed by the radii that connect the centre of D to the vertices of T. We have $\theta_1 + \theta_2 + \theta_3 = 2\pi$, $\frac{\pi}{2} \leq \theta_i \leq \frac{5\pi}{6}$ for $i = 1, 2, 3$, and
$$A(T) = \frac{\sin\theta_1 + \sin\theta_2 + \sin\theta_3}{3\sin\frac{2\pi}{3}} \cdot A(T_*).$$

We note that if $\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6}$ then
$$\sin\frac{2\pi}{3} - \frac{1}{2}\left(\theta - \frac{2\pi}{3}\right) - \frac{1}{2}\left(\theta - \frac{2\pi}{3}\right)^2 < \sin\theta < \sin\frac{2\pi}{3} - \frac{1}{2}\left(\theta - \frac{2\pi}{3}\right) - \frac{1}{4}\left(\theta - \frac{2\pi}{3}\right)^2$$

according to the Taylor formula. Since one θ_i satisfies $|\theta_i - \frac{2\pi}{3}| \geq \gamma_1 \varepsilon_1$ in the case of (i), and all θ_i satisfy $|\theta_i - \frac{2\pi}{3}| \leq \gamma_2 \varepsilon_2$ in the case of (ii) where γ_1 and γ_2 are positive absolute constants, we conclude Proposition 4.9. □

To prove Lemma 4.2, we verify the following statement: If T is not ε-close to T_* for positive $\varepsilon < c_1'$ then
$$d(T) < d(T_*) - c_2'(r-1) \cdot \varepsilon^2 \tag{12}$$

where c_1' and c_2' are positive absolute constants. It is easy to see that (12) yields Lemma 4.2.

We divide the proof of (12) into the four cases Cases 1–4 below. During the argument in Cases 1–4, we always assume that T is not ε-close to T_* for positive $\varepsilon < c_1'$ and for a suitable positive absolute constant c_1'. In addition $\gamma_1, \gamma_2, \ldots$ denote suitable positive absolute constants, and a, b and c denote the sides of T.

Case 1 The angle of T opposite to a is obtuse, and $c \le b \le a_{**}$.

We write q to denote the centre of D. Moreover let T_a, T_b and T_c denote the convex hulls of q on the one hand, and a, b and c, respectively, on the other hand. We observe that $\mathrm{conv}\{o, T_a\}$, $\mathrm{conv}\{o, T_b\}$ and $\mathrm{conv}\{o, T_c\}$ can be dissected into two $(1, m_a, r)$-orthoschemes, $(1, m_b, r)$-orthoschemes and $(1, m_c, r)$-orthoschemes, respectively. Now Lemma 3.1 and the inequalities $m_b, m_c \ge m_{**}$ imply that $d(T_b), d(T_c) \le d(R_{**})$. Since $m_a < m_b, m_c$, it follows by Lemma 3.1 that $d(T_a) > d(T_b), d(T_c)$. Now T is the difference of $T_b \cup T_c$ and T_a, thus Proposition 4.7 yields

$$d(T) < \max\{d(T_b), d(T_c)\} \le d(R_{**}) \le d(T_*) - \gamma_1(r-1).$$

Since $a_{**} > \sqrt{2}\sqrt{r^2-1}$, Case 1 covers all isosceles triangles with an obtuse angle.

Case 2 $b = c$, and the angle of T opposite to a is at most $\frac{\pi}{2}$.

We may parametrize T by a as the family $T(a)$, $0 < a \le 2\sqrt{r^2-1}$. If $a > a_*$ then applying (4) to $[a_*, a]$, and using Proposition 4.8 (i) and Proposition 4.9 yield (12).

If $a_{**} \le a < a_*$ then applying (3) to $[a, a_*]$, and using again Proposition 4.8 (i) and Proposition 4.9 imply (12). In particular $d(T_{**}) = d(T(a_{**})) < d(T_*) - \gamma_1(r-1)$ by (5), hence Proposition 4.6 yields

$$\Xi_{**} < d(T_*) - \gamma_2(r-1). \tag{13}$$

For $a < a_{**}$, we prove $d(T) \le \Xi_{**}$. If $a_{00} \le a < a_{**}$ then we apply (3) to $[a, a_{**}]$, and use Proposition 4.8 (ii). If $a < a_{00}$ then we choose $a_1 \in (0, a)$ satisfying $d(T(a_1)) < \Xi_{**}$. Therefore applying (3) to $[a_1, a]$ yields the existence of some $s \in (a_1, a)$ satisfying

$$d(T(a)) = \frac{A(T(a_1))}{A(T(a))} \cdot d(T(a_1)) + \left(1 - \frac{A(T(a_1))}{A(T(a))}\right) \cdot v(T(s)).$$

Since $v(T(a)) < \Xi_{**}$ according to Proposition 4.8 (ii), we deduce $d(T) \le \Xi_{**}$, and in turn (12) by(13).

Case 3 $b > a \ge a_{**}$.

Let $\tilde{c}_1$ and $\tilde{c}_2$ be the positive absolute constants in Proposition 4.9, and let $\gamma = \sqrt{\frac{\tilde{c}_1}{2\tilde{c}_2}} < 1$. We may assume that $\varepsilon < 0.05$.

We fix c, and deform $T = T(a)$ in a way that a increases until T becomes isosceles at $a = a_2$. Since $m_b < m_a \le m_{**}$ and $d(T(a_2)) \le d(T_*)$, we deduce by Proposition 4.6 and (3) that

$$d(T(a_2)) - d(T(a)) \ge \frac{r-1}{9} \cdot \frac{A(T(a_2)) - A(T)}{A(T)}. \tag{14}$$

If $T(a_2)$ is not $\gamma\varepsilon$-close to T_* then Case 2 verifies (12). If $T(a_2)$ is $\gamma\varepsilon$-close to T_* then Proposition 4.9 yields that $A(T(a_2)) - A(T) \geq \frac{1}{2}\, c_1\varepsilon^2 A(T_*)$, hence we conclude (12) by (14).

Case 4 $c \leq b \leq a$ *and* $c \leq a_{**}$.

It is sufficient to prove that

$$d(T) \leq d(T_*) - \gamma_3(r-1). \tag{15}$$

If the angle of T opposite to a is obtuse then (15) holds by Case 3 if $b \geq a_{**}$, and by Case 1 if $b \leq a_{**}$.

Therefore we assume that the angle of T opposite to a is at most $\frac{\pi}{2}$. In this case $a > a_* > a_{**}$, and we claim that

$$\frac{r}{m_a^2 m_b^2} > d(T_*). \tag{16}$$

If b' is the common side of the isosceles triangle inscribed into D whose one side is a, and the angle opposite to a is at most $\frac{\pi}{2}$ then $m_b \leq m_{b'}$, hence Proposition 4.8 (i) yields (16).

Now we fix c, and deform $T = T(b)$ in a way that b increases until $b = a$ at some $b = b_0$, hence $A(T(b))$ strictly increases. Since $v(T(b)) > d(T_*)$ for any b according to (16), and $d(T(b_0)) < d(T^*)$ according to Case 2, we conclude by (3) that $d(T) \leq d(T(b_0))$. Therefore (15) follows by Case 2 and by (5), completing the proof of (12) in Case 4.

The arguments in Cases 1–4 prove (12) because T has two sides whose lengths are either both at least a_{**}, or both at most a_{**}. In turn we conclude Lemma 4.2. $\square$

Reversing the analysis of Cases 2 and 3 above shows that Lemma 4.2 is essentially optimal:

Remark 4.10. There exist positive absolute constants c_1 and c_2 with the following properties. Given $1 < r \leq \sqrt{3}$, if T is a triangle whose vertices lie on rS^2, whose affine hull touches B^3, and that is ε-close to the regular triangle of circumradius $\sqrt{r^2-1}$ for positive $\varepsilon < c_1$ then

$$d(T) \geq [1 - c_2\varepsilon^2 \cdot (r-1)] \cdot \frac{8 \arctan \frac{\sqrt{3}(r-1)}{3+r}}{\sqrt{3}(r^2-1)}.$$

Corollary 4.11. *There exist positive absolute constants c_1, c_2 and c_3 with the following properties. Given $1 < r \leq \sqrt{3}$, let F be a polygon with $\operatorname{ext} F \subset rS^2$*

such that aff F *avoids* int B^3. *If*

$$d(F) > [1-\varepsilon\,(r-1)]\cdot \frac{8\arctan\frac{\sqrt{3}(r-1)}{3+r}}{\sqrt{3}(r^2-1)} \tag{17}$$

for positive $\varepsilon < c_1$ *then* F *is* $c_2\sqrt{\varepsilon}$*-close to the regular triangle of circumradius* $\sqrt{r^2-1}$*, and* aff F *is of distance at most* $c_3\varepsilon(r-1)$ *from* B^3.

Proof. During the argument, $\gamma_1,\gamma_2,\ldots$ denote suitable positive absolute constants.

First we provide the stability version (19) of Lemma 3.1 about orthoschemes in a special case. For $1\le\varrho<r\le\sqrt{3}$, we write $T_*(r,\varrho)$ to denote a regular triangle that touches ϱB^3 in its centroid, and whose vertices lie on rS^2. In particular $T_*(r,1)$ is the extremal triangle of Lemmas 4.1 and 4.2, and

$$d(T_*(r,\varrho)) = \frac{d(T_*(\frac{r}{\varrho},1))}{\varrho^2} = \frac{8\arctan\frac{\sqrt{3}(r-\varrho)}{3\varrho+r}}{\sqrt{3}(r^2-\varrho^2)} \tag{18}$$

(compare Proposition 4.5 (i)). We observe that

$$\frac{\partial}{\partial\varrho}\,d(T_*(r,\varrho)) = \frac{-32(6\varrho^2+3\varrho r+r^2)}{(r+\varrho)^2} < -\frac{80}{3}.$$

Therefore if $1\le\varrho\le\frac{1}{2}(r+1)$ then

$$d(T_*(r,\varrho)) \le [1-\gamma_1(\varrho-1)]\cdot d(T_*(r,1)). \tag{19}$$

Now we prove Corollary 4.11 in the case when the polygon F is a triangle T. Let ϱ be the distance of aff T from B^3. If $T'=\frac{1}{\varrho}T$ then $d(T)=\frac{d(T')}{\varrho^2}$, hence $d(T)\le d(T_*(r,\varrho))$ according to Lemma 4.1. Since $d(T_*(r,\varrho))$ is a decreasing function of ϱ (see Lemma 3.1), (19 yields that $\varrho\le 1+\gamma_2\varepsilon(r-1)$, and

$$d(T') \ge [1-\varepsilon\cdot(r-1)]\cdot d\left(T_*\left(\frac{r}{\varrho},1\right)\right) \ge \left[1-\gamma_3\varepsilon\cdot\left(\frac{r}{\varrho}-1\right)\right]\cdot d\left(T_*\left(\frac{r}{\varrho},1\right)\right).$$

As $T_*(r,\varrho)$ is $\gamma_4\varepsilon$-close to $T_*(r,1)$, we conclude Corollary 4.11 for $F=T$ by Lemma 4.2.

Finally we assume that the polygon F in Corollary 4.11 has at least four sides. Let D be the section of rB^3 by aff F. We triangulate F into the triangles $T_1,\ldots,T_k$, $k\ge 2$, such that any vertex of some T_i is a vertex of F. In particular

$$d(F) = \frac{1}{A(F)}\sum_{i=1}^{k} A(T_i)\cdot d(T_i).$$

We observe that if some T_i does not contain the centre of D in its relative interior then T_i is not γ_5-close to $T_*(r,1)$. Therefore (17) and the case of triangles in Corollary 4.11 yield that one T_i, say T_1, contains the centre of D in its relative interior. It also follows that T_1 is $\gamma_6\sqrt{\varepsilon}$-close to $T_*(r,1)$, and the total area of $T_2,\dots,T_k$ is at most $\gamma_6\varepsilon$. In particular any point of F is of distance at most $\gamma_7\varepsilon\sqrt{r-1}$ from T_1, hence we conclude Corollary 4.11. □

Remark 4.10 and the argument for Corollary 4.11 yield the following converse of Corollary 4.11:

Remark 4.12. There exist positive absolute constants c_1, c_2 and c_3 with the following properties. Given $1 < r \leq \sqrt{3}$, if T is a triangle whose vertices lie on rS^2, whose affine hull avoids $\operatorname{int} B^3$, and that is ε-close to the regular triangle of circumradius $\sqrt{r^2-1}$ for positive $\varepsilon < c_1$ then

$$d(T) \geq [1 - c_2\varepsilon\cdot(r-1)]\cdot\frac{8\arctan\frac{\sqrt{3}(r-1)}{3+r}}{\sqrt{3}(r^2-1)}, \tag{20}$$

and the distance of $\operatorname{aff} T$ from B^3 is at most $c_3\varepsilon(r-1)$.

5. Proof of Theorem 1.2

First we consider the case of the surface area, and the optimality of the octahedron; namely, let

$$r = \sqrt{3}.$$

We write O^3 to denote the regular octahedron circumscribed around the unit ball, and T_* to denote one of its faces. In particular $A(p_{S^2}(T_*)) = \frac{1}{8}A(S^2)$, and $S(O^3) = \frac{A(S^2)}{d(T_*)}$. Now we choose a positive absolute constant γ such that if T is any triangle that avoids $\operatorname{int} B^3$, is γ-close to T_*, and whose vertices lie on rS^2 then

$$A(p_{S^2}(T)) \geq \frac{1}{8.5}\,A(S^2). \tag{21}$$

During the argument, $\gamma_1, \gamma_2, \dots$ denote further positive absolute constants.

The heart of the proof is the following claim: Let N be a convex body containing B^3 such that all extreme points of N lie on rS^2, and

$$S(N) < (1+\varepsilon)\cdot\frac{\sqrt{3}(r^2-1)}{8\arctan\frac{\sqrt{3}(r-1)}{3+r}}\cdot A(S^2) = (1+\varepsilon)\cdot\frac{A(S^2)}{d(T_*)}. \tag{22}$$

Then N has eight two-dimensional faces $F_1, \ldots, F_8$ such that aff $F_1, \ldots,$ aff F_8 bound a polytope N' that satisfies

$$(1 - \gamma_1\sqrt{\varepsilon})O^3 \subset N \subset N' \subset (1 + \gamma_1\sqrt{\varepsilon})O^3 \tag{23}$$

possibly after rotating O^3.

To prove (23), we may assume that N is a polytope according to Lemma 2.1. Writing $F_1, \ldots, F_k$ to denote the faces of N, we have

$$S(N) = \sum_{i=1}^{k} \frac{A(p_{S^2}(F_i))}{d(F_i)}.$$

Since $d(F_i) \leq d(T_*)$ holds for any face F_i according to Lemma 4.1, we deduce that the total area of the faces of N that are not γ-close to T_* is at most $\gamma_2\varepsilon$ by (22) and Lemma 4.2. Therefore (21) yields that if ε is small then N has exactly eight faces that are γ-close to T_*, say $F_1, \ldots, F_8$. It also follows that each of $F_1, \ldots, F_8$ is actually $\gamma_3\sqrt{\varepsilon}$-close to T_* by Corollary 4.11, and any spherical circular disc of radius $\gamma_4\sqrt{\varepsilon}$ intersects one of $p_{S^2}(F_1), \ldots, p_{S^2}(F_8)$.

We observe that if $x \in F_i$, and $u_i \in S^2$ is the exterior unit normal to F_i for $i = 1, \ldots, 8$ then $\|x\|/\langle \frac{x}{\|x\|}, u_i\rangle \leq 3$. Thus writing $\widetilde{F}_i$ to denote the $4\gamma_4\sqrt{\varepsilon}$ neighbourhood of F_i in aff F_i, the projections $p_{S^2}(\widetilde{F}_i)$ cover S^2 for $i = 1, \ldots, 8$. Therefore aff $F_1, \ldots,$ aff F_8 bound a polytope N', and the face F_i' of N' containing F_i is $\gamma_5\sqrt{\varepsilon}$-close to T_* for $i = 1, \ldots, 8$.

Next let $q_1, \ldots, q_8 \in S^2$ be the points where the faces of O^3 touch B^3 where q_1 and q_2 belong to faces with common edge. It follows that if F_i' shares a common edge with F_j' and F_k' for $i, j, k \leq 8$ then

$$(1 - \gamma_6\sqrt{\varepsilon}) \cdot \angle q_1 o q_2 \leq \angle u_i o u_j \leq (1 + \gamma_6\sqrt{\varepsilon}) \cdot \angle q_1 o q_2,$$

and the angle of the spherical arcs $u_j u_i$ and $u_k u_i$ is between $\frac{2\pi}{3} + \gamma_7\sqrt{\varepsilon}$ and $\frac{2\pi}{3} - \gamma_7\sqrt{\varepsilon}$. Thus we may rotate O^3 in a way that the distance of q_i and u_i is at most $\gamma_8\sqrt{\varepsilon}$ for $i = 1, \ldots, 8$. In turn we conclude (23).

Now let $M \in \mathcal{F}_r$ satisfy that $S(M) \leq (1+\varepsilon)S(O^3)$ for small ε, hence $S(N) \leq (1+\varepsilon)S(O^3)$ holds for $N = M \cap rB^3$. Since $M \subset N'$ for the N' in (23), we deduce Theorem 1.2 when $r = \sqrt{3}$ and the surface area is maximized.

Still keeping $r = \sqrt{3}$, the case of the volume follows from the case of the surface area because $S(O^3) = 3V(O^3)$ and $S(M) \leq 3V(M)$ for $M \in \mathcal{F}_r$. Finally the case $r = \sqrt{15 - 6\sqrt{5}}$ can be handled analogously to the case $r = \sqrt{3}$, completing the proof of Theorem 1.2. □

Example 5.1. Given small positive α, let P' be the polyhedron in $\mathbb{E}^3$ with vertices

$$(\pm\sqrt{3}\cos\alpha, 0, \sqrt{3}\sin\alpha) \qquad (0, \pm\sqrt{3}\cos\alpha, -\sqrt{3}\sin\alpha) \qquad (0, 0, \pm\sqrt{3}).$$

Then elementary calculations yield that $P = (1 + c_1\alpha^2)P' \in \mathcal{F}_{\sqrt{3}}$, $V(P) \leq (1 + c_2\alpha^2)V(O^3)$ and P is not $c_3\alpha$-close to O^3 where c_1, c_2 and c_3 are positive absolute constants.

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KÁROLY BÖRÖCZKY
DEPARTMENT OF GEOMETRY
ROLAND EÖTVÖS UNIVERSITY
PÁZMÁNY PÉTER SÉTÁNY 1/C
H-1117, BUDAPEST
HUNGARY

E-mail: boroczky@cs.elte.hu

KÁROLY BÖRÖCZKY, JR.
ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS
P.O. BOX 127
H-1364, BUDAPEST
HUNGARY
AND
DEPARTMENT OF GEOMETRY
ROLAND EÖTVÖS UNIVERSITY
PÁZMÁNY PÉTER SÉTÁNY 1/C
H-1117, BUDAPEST
HUNGARY

E-mail: carlos@renyi.hu

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