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Maps on M_n preserving Lie products

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Abstract. Let M_n be the Lie algebra of all $n \times n$ complex matrices with the Lie product $[A, B] = AB - BA$ and let $\phi : M_n \to M_n$ satisfy $\phi([A, B]) = [\phi(A), \phi(B)]$, $A, B \in M_n$. Then $\phi(M_n)$ is a commutative subset of M_n or there exist an invertible matrix $T \in M_n$, a function $\varphi : M_n \to \mathbb{C}$ satisfying $\varphi(C) = 0$ for every trace zero matrix $C \in M_n$, and a homomorphism f of the complex field, such that $\phi([a_{ij}]) = T[f(a_{ij})]T^{-1} + \varphi([a_{ij}])I$ for all $[a_{ij}] \in M_n$, or $\phi([a_{ij}]) = -T[f(a_{ij})]^t T^{-1} + \varphi([a_{ij}])I$ for all $[a_{ij}] \in M_n$.

1. Introduction and statement of the result

Let M_n be the space of all $n \times n$ complex matrices. There are three standard products on M_n which induce the structure of an algebra, matrix multiplication, the Jordan product, and the Lie product $[A, B] = AB - BA$, $A, B \in M_n$. Maps which preserve matrix multiplication were characterized by Jodeit and Lam [3], maps which preserve the Jordan product were studied by Molnár [4] and it is the aim of this paper to characterize the maps, which are multiplicative for the Lie product, that is $\phi([A, B]) = [\phi(A), \phi(B)]$. We do not assume that ϕ is either linear or bijective.

Theorem. *Let $\phi : M_n \to M_n$ be a map satisfying*

$$\phi([A, B]) = [\phi(A), \phi(B)], \quad A, B \in M_n. \tag{1}$$

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Then $\phi(M_n)$ is a commutative subset of M_n or there exist an invertible matrix $T \in M_n$, a function $\varphi : M_n \to \mathbb{C}$ satisfying $\varphi(C) = 0$ for every trace zero matrix $C \in M_n$, and a homomorphism f of the complex field, such that

$$\phi([a_{ij}]) = T[f(a_{ij})]T^{-1} + \varphi([a_{ij}])I, \quad [a_{ij}] \in M_n,$$

or

$$\phi([a_{ij}]) = -T[f(a_{ij})]^t T^{-1} + \varphi([a_{ij}])I, \quad [a_{ij}] \in M_n.$$

A similar statement has recently been proved by Šemrl [5] under the strong additional assumption of bijectivity.

When considering homomorphisms of matrix algebras, Jordan algebras, and Lie algebras we assume that such maps are linear and multiplicative with respect to the corresponding product. So all mentioned results are non-linear extensions of classical structural results for homomorphisms of matrix algebras, Jordan algebras, and Lie algebras.

Recently the author also characterized bijective maps preserving Lie products on upper triangular matrices over an arbitrary field with characteristic zero [1].

2. Proof

We will distinguish the higher dimensional case $n \geq 3$ and the case $n = 2$. The case $n = 1$ is trivial.

Let $n \geq 3$. We begin with some easy observations. First, notice that $\phi(0) = \phi([A, A]) = [\phi(A), \phi(A)] = 0$. Second, recall that a matrix $A \in M_n$ has trace zero if and only if it can be written as $A = BC - CB = [B, C]$ for some $B, C \in M_n$ (see for example [2, p. 288, Theorem 4.5.2]). If $\operatorname{tr} A = 0$ and $A = [B, C]$, then $\phi(A) = \phi([B, C]) = [\phi(B), \phi(C)]$ and therefore $\operatorname{tr}\phi(A) = 0$. So, ϕ maps the set of trace zero matrices into the set of trace zero matrices. Furthermore,

$$\phi(-A) = \phi(-[B, C]) = \phi([C, B]) = -[\phi(B), \phi(C)] = -\phi(A) \tag{2}$$

for every trace zero matrix A.

In order to prove the theorem we will consider the two cases when ϕ maps all trace zero matrices into zero and when this is not the case.

The first case is trivial. Assume that ϕ maps the set of trace zero matrices into 0. Then we obtain $[\phi(A), \phi(B)] = \phi([A, B]) = 0$ for every $A, B \in M_n$, since $[A, B]$ is a trace zero matrix which is mapped to 0 by ϕ. So, $\phi(A)$ and $\phi(B)$ commute for every $A, B \in M_n$, and $\phi(M_n)$ is therefore a commutative subset of M_n.

In the rest of the proof we will assume that there is a matrix $A_0 \in M_n$ such that $\operatorname{tr} A_0 = 0$ and $\phi(A_0) \neq 0$. Observe that A_0 is not a scalar matrix. Throughout the symbol N_0 will stand for the matrix $N_0 = \sum_{i=1}^{n-1} E_{i,i+1}$.

Lemma 1. *Let the map ϕ be as in the theorem. Suppose there exists a matrix $A_0 \in M_n$, such that $\operatorname{tr} A_0 = 0$ and $\phi(A_0) \neq 0$. Then $\phi(A)$ is a nonscalar matrix for every nonscalar matrix $A \in M_n$.*

Proof. Let us assume for the moment that A_0 is in the Jordan canonical form. We start by proving that every nonscalar diagonal matrix is mapped to a nonscalar matrix. We do this by induction on the number of pairs of equal neighboring elements on the diagonal.

Let $B = \sum_{i=1}^{n-1} b_i E_{i,i+1}$, where $b_i \neq 0$ for every $i = 1, \ldots, n-1$. Then it is easy to see that there exists a matrix $C = \operatorname{diag}\{d_1, \ldots, d_n\} + \sum_{i=1}^{n-1} c_i E_{i+1,i}$ such that $A_0 = [B, C]$. Since $\phi(A_0) = [\phi(B), \phi(C)] \neq 0$, it follows that $\phi(B)$ is a nonscalar matrix. So, if $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ with $d_i \neq d_{i+1}$, $i = 1, \ldots, n-1$, then $[D, N_0] = \sum_{i=1}^{n-1} (d_i - d_{i+1}) E_{i,i+1}$ and, because $\phi([D, N_0])$ is nonscalar, also $\phi(D)$ is nonscalar.

Let $0 \leq k \leq n-3$ and suppose that $\phi(D)$ is a nonscalar matrix for any matrix $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ with $d_i = d_{i+1}$ for at most k indices $i \in \{1, \ldots, n-1\}$.

It is not difficult to see that for any matrix $B = \sum_{i=1}^{n-1} b_i E_{i,i+1}$, where $b_i = 0$ for at most $k+1$ indices $i \in \{1, \ldots, n-1\}$, there exists a matrix $C = \sum_{i=1}^{n-1} c_i E_{i+1,i}$ such that

$$[B, C] = \operatorname{diag}\{d_1, \ldots, d_n\}$$

is a diagonal matrix with $d_i = d_{i+1}$ for at most k indices $i \in \{1, \ldots, n-1\}$. By the induction hypothesis, $\phi(\operatorname{diag}\{d_1, \ldots, d_n\})$ is nonscalar and therefore $\phi(B)$ is a nonscalar matrix as well.

Let $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ with $d_i = d_{i+1}$ for at most $k+1$ indices $i \in \{1, \ldots, n-1\}$. Then

$$[D, N_0] = \sum_{i=1}^{n-1} (d_i - d_{i+1}) E_{i,i+1} = \sum_{i=1}^{n-1} b_i E_{i,i+1}$$

where $b_i = 0$ for at most $k+1$ indices $i \in \{1, \ldots, n-1\}$. Hence $\phi([D, N_0]) \neq 0$ and therefore $\phi(D)$ is a nonscalar matrix.

It follows that every nonscalar diagonal matrix is mapped to a nonscalar matrix.

Finally, let A be an arbitrary nondiagonal matrix. Then $a_{ij} \neq 0$ for some indices $1 \leq i, j \leq n$, $i \neq j$. Since

$$a_{ij} E_{ii} - a_{ij} E_{jj} = [E_{ji}, [E_{jj}, [E_{ii}, A]]]$$

and $\phi(a_{ij}E_{ii} - a_{ij}E_{jj})$ is nonscalar, we see that $\phi(A)$ is a nonscalar matrix.

If A_0 is not in the Jordan canonical form, then there exists an invertible matrix S such that SA_0S^{-1} is in the Jordan canonical form. As in the beginning of the proof we write $SA_0S^{-1} = [B, C]$ and therefore $A_0 = S^{-1}[B, C]S = [S^{-1}BS, S^{-1}CS]$. Since $\phi(A_0) = [\phi(S^{-1}BS), \phi(S^{-1}CS)] \neq 0$, it follows that $\phi(S^{-1}BS)$ is a nonscalar matrix. We proceed in the same way as above. First we prove that $\phi(S^{-1}DS)$ is not a scalar matrix for any nonscalar diagonal matrix D, and then that $\phi(S^{-1}AS)$ is a nonscalar matrix when A is not a diagonal matrix. $\square$

Lemma 2. *Let $D \in M_n$. Then $D = S\,\mathrm{diag}\{n, n-1, \ldots, 1\}S^{-1} + \lambda I$ for some invertible matrix $S \in M_n$ and $\lambda \in \mathbb{C}$ if and only if there exist matrices $N_1, N_2 \in M_n$, such that $[D, N_1] = N_1$, $[D, N_2] = N_2$, and the $(n-2)$-fold Lie product*

$$[\ldots[[N_2, N_1], N_1], \ldots, N_1]$$

is a nonscalar matrix.

Proof. Suppose $D = S\,\mathrm{diag}\{n, n-1, \ldots, 1\}S^{-1} + \lambda I$ for some invertible matrix $S \in M_n$ and $\lambda \in \mathbb{C}$. Then for $N_1 = S(\sum_{i=1}^{n-1} E_{i,i+1})S^{-1} = SN_0S^{-1}$ and $N_2 = SE_{12}S^{-1}$ we have $[D, N_1] = N_1$, $[D, N_2] = N_2$, and the $(n-2)$-fold Lie product

$$S[\ldots[[E_{12}, N_0], N_0], \ldots, N_0]S^{-1} = SE_{1n}S^{-1}$$

is nonscalar.

Suppose now that there exist matrices N_1 and N_2 such that $[D, N_1] = N_1$, $[D, N_2] = N_2$, and the $(n-2)$-fold Lie product

$$[\ldots[[N_2, N_1], N_1], \ldots, N_1]$$

is nonscalar. Without loss of generality we may assume that D is in the Jordan canonical form with its eigenvalues ordered $d_1 \geq d_2 \geq \cdots \geq d_n$. Then, since $[D, N_1] = N_1$ and $[D, N_2] = N_2$, it is easy to see that $N_1 = [p_{ij}]$ and $N_2 = [q_{ij}]$ are strictly upper triangular matrices and that $p_{i,i+1}(d_i - d_{i+1}) = p_{i,i+1}$, and $q_{i,i+1}(d_i - d_{i+1}) = q_{i,i+1}$ for every $i = 1, \ldots, n-1$. Let $[c_{ij}^k]$ denote the k-fold Lie product $[\ldots[[N_2, N_1], N_1], \ldots, N_1]$. Suppose there exists an index $i_0 \in \{1, \ldots, n-1\}$ such that $p_{i_0,i_0+1} = q_{i_0,i_0+1} = 0$. We will prove that in this case $[c_{ij}^{n-2}] = 0$ which contradicts the assumption that $[c_{ij}^{n-2}]$ is nonscalar. We distinguish four cases.

First, if $i_0 = 1$, then $c_{ij}^k = 0$ for $i + k \geq j$ and $c_{1,k+2}^k = 0$ for every $k = 1, \ldots, n-2$. Hence $[c_{ij}^{n-2}] = 0$.

Second, let $1 < i_0 \le \frac{n-1}{2}$. Notice that in this case $i_0 - 1 \le n - i_0 - 1$. We obtain $c^1_{ij} = 0$ for $i+1 \ge j$ and also $c^1_{i_0-1,i_0+1} = c^1_{i_0,i_0+2} = 0$. Inductively we see that $c^{i_0-1}_{ij} = 0$ for $i + (i_0 - 1) \ge j$ and also $c^{i_0-1}_{1,i_0+1} = c^{i_0-1}_{2,i_0+2} = \cdots = c^{i_0-1}_{i_0,2i_0} = 0$. So, after $n-i_0-1$ steps we obtain that $c^{n-i_0-1}_{1,n-i_0+1} = c^{n-i_0-1}_{2,n-i_0+2} = \cdots = c^{n-i_0-1}_{i_0,n} = 0$ and therefore $c^{n-i_0-1}_{ij} = 0$ for $i + (n - i_0 - 1) + 1 \ge j$. Since $n - i_0 - 1 < n - 2$, it follows that $[c^{n-2}_{ij}] = 0$.

Third, let $\frac{n-1}{2} < i_0 < n - 1$. Notice that $n - i_0 - 1 \le i_0 - 1$. It follows that $c^{n-i_0-1}_{ij} = 0$ for $i + (n - i_0 - 1) \ge j$ and also $c^{n-i_0-1}_{i_0,n} = c^{n-i_0-1}_{i_0-1,n-1} = \cdots = c^{n-i_0-1}_{2i_0-n+1,i_0+1} = 0$. Hence, after $i_0 - 1$ steps, $c^{i_0-1}_{ij} = 0$ for $i + (i_0 - 1) + 1 \ge j$. In this case $i_0 - 1 < n - 2$ and therefore again $[c^{n-2}_{ij}] = 0$.

Fourth, if $i_0 = n - 1$, then $c^k_{ij} = 0$ for $i + k \ge j$ and $c^k_{n-1-k,n} = 0$ for every $k = 1, \ldots, n - 2$. It follows that $[c^{n-2}_{ij}] = 0$.

So, for every index $i \in \{1, \ldots, n-1\}$ at least one of $p_{i,i+1}$ or $q_{i,i+1}$ is nonzero and therefore $d_i - d_{i+1} = 1$ for every $i \in \{1, \ldots, n - 1\}$. $\square$

Let us denote

$$D_0 = \operatorname{diag}\{n, n-1, \ldots, 1\} - \frac{n+1}{2} I = \operatorname{diag}\left\{\frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2}\right\}.$$

Observe that D_0 is a trace zero matrix. The map ϕ takes trace zero matrices to trace zero matrices, and by Lemma 1 nonscalar matrices to nonscalar matrices. By (1) and since $\phi(D_0)$ satisfies Lemma 2, we have $\phi(D_0) = TD_0T^{-1}$ for some invertible matrix T. Notice that if the map ϕ satisfies condition (1), then the map $A \mapsto T^{-1}\phi(A)T$ satisfy condition (1) as well. Without loss of generality we may therefore assume that

$$\phi(D_0) = D_0. \tag{3}$$

It is easy to see that the matrix D is diagonal if and only if $[D_0, D] = 0$, further, $B = \sum_{i=1}^{n-1} b_i E_{i,i+1}$ for some $b_1, \ldots, b_{n-1} \in \mathbb{C}$ if and only if $[D_0, B] = B$, and similarly, $C = \sum_{i=1}^{n-1} c_i E_{i+1,i}$ for some $c_1, \ldots, c_{n-1} \in \mathbb{C}$ if and only if $[C, D_0] = C$. It follows by (1) and (3) that ϕ maps diagonal matrices to diagonal matrices, $\phi(\sum_{i=1}^{n-1} b_i E_{i,i+1}) = \sum_{i=1}^{n-1} p_i E_{i,i+1}$, and $\phi(\sum_{i=1}^{n-1} c_i E_{i+1,i}) = \sum_{i=1}^{n-1} q_i E_{i+1,i}$, where $b_i, c_i, p_i, q_i \in \mathbb{C}$, $i = 1, \ldots, n - 1$.

Lemma 3. *If $\sum_{i=1}^{n-1} b_i E_{i,i+1}$ is of rank $n - 1$, then also $\phi(\sum_{i=1}^{n-1} b_i E_{i,i+1}) = \sum_{i=1}^{n-1} p_i E_{i,i+1}$ is of rank $n - 1$.*

Proof. Suppose $\sum_{i=1}^{n-1} b_i E_{i,i+1}$ is of rank $n-1$. Then there exists a matrix $\sum_{i=1}^{n-1} c_i E_{i+1,i}$ such that

$$\left[\sum_{i=1}^{n-1} b_i E_{i,i+1}, \sum_{i=1}^{n-1} c_i E_{i+1,i}\right] = D_0.$$

It follows that

$$\left[\phi\left(\sum_{i=1}^{n-1} b_i E_{i,i+1}\right), \phi\left(\sum_{i=1}^{n-1} c_i E_{i+1,i}\right)\right] = \left[\sum_{i=1}^{n-1} p_i E_{i,i+1}, \sum_{i=1}^{n-1} q_i E_{i+1,i}\right]$$
$$= p_1 q_1 E_{11} + \sum_{i=2}^{n-1} (p_i q_i - p_{i-1} q_{i-1}) E_{ii} - p_{n-1} q_{n-1} E_{nn} = D_0.$$

Since $p_1 q_1 > 0$ and $p_i q_i - p_{i-1} q_{i-1} \geq 0$ for $2 \leq i \leq \frac{n+1}{2}$, we obtain inductively that $p_i q_i > 0$ for every $i \leq \frac{n+1}{2}$. Similarly, $p_{n-1} q_{n-1} > 0$ and $p_{i-1} q_{i-1} - p_i q_i \geq 0$ for $\frac{n+1}{2} \leq i \leq n-1$, therefore $p_i q_i > 0$ also for every $i \geq \frac{n-1}{2}$. Hence $p_i \neq 0$ for every $i = 1, \ldots, n-1$. □

Let $\phi(N_0) = \sum_{i=1}^{n-1} p_i E_{i,i+1}$ where, by Lemma 3, $p_i \neq 0$ for every $i = 1, \ldots, n-1$. If $P = \operatorname{diag}\{1, p_1, p_1 p_2, \ldots, p_1 p_2 \ldots p_{n-1}\}$, then $P D_0 P^{-1} = D_0$ and $P(\sum_{i=1}^{n-1} p_i E_{i,i+1}) P^{-1} = N_0$. Therefore we may assume without loss of generality that $\phi(N_0) = N_0$.

Because $[N_0, \sum_{i=1}^{n-1} b_i E_{i,i+1}] = 0$ if and only if $\sum_{i=1}^{n-1} b_i E_{i,i+1} = \alpha N_0$, it follows that

$$\phi(\alpha N_0) = f(\alpha) N_0,$$

where $f : \mathbb{C} \to \mathbb{C}$. Notice that $f(\alpha) = 0$ if and only if $\alpha = 0$, that $f(1) = 1$, and that $f(-\alpha) = -f(\alpha)$ by (2). We will prove that f is a homomorphism of the field $\mathbb{C}$.

For every $\alpha, \beta \in \mathbb{C}$ we have

$$[D_0, \alpha D_0 + \beta N_0] = \beta N_0,$$

and

$$[\alpha D_0 + \beta N_0, N_0] = \alpha N_0.$$

Hence

$$[D_0, \phi(\alpha D_0 + \beta N_0)] = f(\beta) N_0 \tag{4}$$

and

$$[\phi(\alpha D_0 + \beta N_0), N_0] = f(\alpha) N_0. \tag{5}$$

By (4) the matrix $\phi(\alpha D_0 + \beta N_0) - f(\beta) N_0$ is diagonal with trace zero because $\operatorname{tr}(\alpha D_0 + \beta N_0) = 0$, and therefore we obtain by (5) that $\phi(\alpha D_0 + \beta N_0) = f(\alpha) D_0 + f(\beta) N_0$.

Let us prove that f is multiplicative. Since

$$[\alpha D_0, \beta N_0] = \alpha\beta N_0$$

it follows that

$$[f(\alpha)D_0, f(\beta)N_0] = f(\alpha\beta)N_0$$

and therefore $f(\alpha\beta) = f(\alpha)f(\beta)$ for every pair of complex numbers α and β.

In order to prove that ϕ is additive we write the equation

$$[D_0 - \alpha N_0, D_0 + \beta N_0] = (\alpha + \beta)N_0$$

and obtain

$$[D_0 - f(\alpha)N_0, D_0 + f(\beta)N_0] = f(\alpha + \beta)N_0.$$

So, $f(\alpha + \beta) = f(\alpha) + f(\beta)$, $\alpha, \beta \in \mathbb{C}$.

And since f is a nontrivial homomorphism of the complex field, $f(r) = r$ for every rational number r.

Furthermore,

$$\left[\frac{1}{n-1}D_0, E_{1n}\right] = E_{1n},$$

hence

$$\left[\frac{1}{n-1}D_0, \phi(E_{1n})\right] = \phi(E_{1n})$$

and $\phi(E_{1n}) = \eta E_{1n}$, where η is a nonzero constant. If we write

$$\left[\frac{\alpha}{n-1}D_0, E_{1n}\right] = \alpha E_{1n},$$

we see that

$$\left[\frac{f(\alpha)}{n-1}D_0, \eta E_{1n}\right] = \phi(\alpha E_{1n}),$$

so

$$\phi(\alpha E_{1n}) = f(\alpha)\eta E_{1n}.$$

Similarly the equation

$$\left[E_{n1}, \frac{1}{n-1}D_0\right] = E_{n1}$$

yields in the same way

$$\phi(\alpha E_{n1}) = f(\alpha)\nu E_{n1}$$

for some nonzero constant ν.

Since $[[E_{1n}, E_{n1}], E_{1n}] = 2E_{1n}$, it follows that $[[\eta E_{1n}, \nu E_{n1}], \eta E_{1n}] = 2\eta E_{1n}$, hence

$$\eta\nu = 1. \tag{6}$$

Let $C_0 = \sum_{i=1}^{n-1} c_i E_{i+1,i}$ be such that $[N_0, C_0] = D_0$. Then $c_1 = c_{n-1} = \frac{n-1}{2}$. Since $[N_0, \phi(\alpha C_0)] = f(\alpha)D_0$, it follows that $\phi(\alpha C_0) = f(\alpha)C_0$. Now,

$$\left[E_{1n}, \frac{2}{n-1}C_0\right] = E_{1,n-1} - E_{2n}$$

so

$$\left[\eta E_{1n}, \frac{2}{n-1}C_0\right] = \phi(E_{1,n-1} - E_{2n})$$

and

$$\phi(E_{1,n-1} - E_{2n}) = \eta(E_{1,n-1} - E_{2n}).$$

Lemma 4. *Suppose E is a diagonal matrix. Then $E = E_{11} + \lambda I$ or $E = -E_{nn} + \lambda I$ for some $\lambda \in \mathbb{C}$ if and only if*

$$[E, [E, N_0]] = [E, N_0]$$

and

$$[[E, E_{1,n-1} - E_{2n}], N_0] = E_{1n}.$$

Proof. If $E = E_{11} + \lambda I$ or $E = -E_{nn} + \lambda I$ for some $\lambda \in \mathbb{C}$, then it is easy to check that E fulfills the two conditions.

Suppose $E = \operatorname{diag}\{e_1, \ldots, e_n\}$ and $[E, [E, N_0]] = [E, N_0]$. Then $(e_i - e_{i+1})^2 = e_i - e_{i+1}$ for every $i = 1, \ldots, n-1$. So $e_i - e_{i+1}$ is equal to 0 or 1. Without loss of generality we may assume that $e_1 = 1$. Then $e_1 \geq e_2 \geq \cdots \geq e_n$ and e_i is an integer for every $i = 1, \ldots, n$. Since $[[E, E_{1,n-1} - E_{2n}], N_0] = E_{1n}$, we obtain the equation $e_1 + e_2 - e_{n-1} - e_n = 1$. Because $e_1 = 1 \geq e_2 \geq \cdots \geq e_{n-1} \geq e_n$ are integers and $e_1 - e_2$ is equal to 0 or 1, and also $e_{n-1} - e_n$ is equal to 0 or 1, this equation has only two solutions, $e_1 = \cdots = e_{n-1} = 1$, $e_n = 0$, and $e_1 = 1$, $e_2 = \cdots = e_n = 0$. □

Since $\phi(E_{1n}) = \eta E_{1n}$ and $\phi(E_{1,n-1} - E_{2n}) = \eta(E_{1,n-1} - E_{2n})$, it follows by Lemma 4 that $\phi(E_{11}) = E_{11} + \lambda I$ or $\phi(E_{11}) = -E_{nn} + \lambda I$ for some $\lambda \in \mathbb{C}$.

If ϕ satisfies condition (1), then the map $A \mapsto -T^{-1}\phi(A)^t T$, where $T = \sum_{i=1}^{n}(-1)^i E_{i,n+1-i}$, satisfies condition (1) as well. Also, since $-T^{-1}D_0^t T = D_0$, $-T^{-1}N_0^t T = N_0$, and $-T^{-1}(-E_{nn} + \lambda I)^t T = E_{11} - \lambda I$, we may assume without loss of generality that $\phi(E_{11}) - E_{11}$ is a scalar matrix.

Let us find the image of the matrix E_{1k}. Since the k-fold Lie product $[\ldots[[E_{11}, \alpha N_0], N_0], \ldots, N_0]$ equals $\alpha E_{1,k+1}$, $1 \leq k \leq n-1$, it follows that

$$\phi(\alpha E_{1k}) = f(\alpha)E_{1k}$$

for every $k = 2, \ldots, n$. In particular for $k = n$ this implies that $\eta = 1$ and therefore by (6) also $\nu = 1$.

To find the image of E_{k1} we inductively prove that

$$[N_0, \ldots, [N_0, [N_0, \alpha E_{n1}]] \ldots] = \alpha \sum_{i=1}^{n-1} (-1)^{i-1} \binom{n-2}{i-1} E_{i+1,i}$$

where the Lie product is applied $(n-2)$-times. If $X_0 = \sum_{i=1}^{n-1} (-1)^{i-1} \binom{n-2}{i-1} E_{i+1,i}$, then

$$\phi(\alpha X_0) = f(\alpha) X_0.$$

Now we express

$$\left[\frac{(-1)^{k-1}}{\binom{n-2}{k-1}} X_0, \ldots, \left[\frac{-1}{n-2} X_0, [\alpha X_0, E_{11}] \right] \ldots \right] = \alpha E_{k+1,1},$$

where the Lie product is applied k-times, $1 \le k \le n-1$, and therefore

$$\phi(\alpha E_{k1}) = f(\alpha) E_{k1}$$

for every $k = 2, \ldots, n$.

Let $i, j \in \{1, \ldots, n\}$, $i \ne j$, and $\alpha \in \mathbb{C}$. Then

$$\alpha E_{ij} = [\alpha E_{i1}, E_{1j}],$$

hence

$$\phi(\alpha E_{ij}) = f(\alpha) E_{ij}.$$

Furthermore,

$$[E_{ij}, [E_{ji}, \alpha E_{ii}]] = [\alpha E_{ij}, E_{ji}].$$

We know that ϕ maps diagonal matrices to diagonal matrices, so $\phi(\alpha E_{ii}) = \mathrm{diag}\{e_1, \ldots, e_n\}$ is diagonal. Thus

$$(e_i - e_j)(E_{ii} - E_{jj}) = f(\alpha)(E_{ii} - E_{jj}).$$

Therefore $\phi(\alpha E_{ii}) - f(\alpha) E_{ii}$ is a scalar matrix for every $i = 1, \ldots, n$.

Let $A = [a_{ij}] \in M_n$ be an arbitrary matrix. For $i, j, k \in \{1, \ldots, n\}$, $i \ne j$, $j \ne k$, $k \ne i$, we have

$$[E_{jk}, [E_{jj}, [E_{ii}, A]]] = a_{ij} E_{ik}$$

and therefore

$$[E_{jk}, [E_{jj}, [E_{ii}, \phi(A)]]] = f(a_{ij})E_{ik}.$$

Let $i \in \{2, \ldots, n\}$. Then

$$[E_{ii}, [E_{11}, [E_{1i}, A]]] = (a_{11} - a_{ii})E_{1i},$$

and hence

$$[E_{ii}, [E_{11}, [E_{1i}, \phi(A)]]] = f(a_{11} - a_{ii})E_{1i} = (f(a_{11}) - f(a_{ii}))E_{1i}.$$

It follows that

$$\phi([a_{ij}]) - [f(a_{ij})]$$

is a scalar matrix for every matrix $[a_{ij}] \in M_n$ and this concludes the proof of the theorem for $n \geq 3$.

In the case $n = 2$ the proof is the same as in the higher dimensional case with the exception of three steps, which must be proved separately.

First, let $D_0 = \frac{1}{2}(E_{11} - E_{22})$. The map ϕ preserves the trace, so $\phi(D_0) = SE_{12}S^{-1}$ or $\phi(D_0) = \alpha SD_0S^{-1}$ for some invertible matrix S and a nonzero complex number α. Assume that $\phi(D_0) = SE_{12}S^{-1}$. Because $[D_0, E_{12}] = E_{12}$, we obtain $[SE_{12}S^{-1}, \phi(E_{12})] = \phi(E_{12})$. Hence, $\phi(E_{12}) = 0$, a contradiction. If $\phi(D_0) = \alpha SD_0S^{-1}$, again because $[D_0, E_{12}] = E_{12}$ we obtain $[\alpha SD_0S^{-1}, \phi(E_{12})] = \phi(E_{12})$. If we solve the last equation, we see that α must be equal to 1 or -1. So, $\phi(D_0)$ is similar to D_0.

Second, because $\phi(E_{11})$ is diagonal and $E_{12} = \phi(E_{12}) = \phi([E_{11}, E_{12}]) = [\phi(E_{11}), E_{12}]$, it follows that $\phi(E_{11}) - E_{11}$ is a scalar matrix.

Third, in the same way as we proved that $\phi(\alpha D_0 + \beta E_{12}) = f(\alpha)D_0 + f(\beta)E_{12}$, we can prove also that $\phi(\alpha D_0 + \beta E_{21}) = f(\alpha)D_0 + f(\beta)E_{21}$. In order to complete the proof in the case $n = 2$ it remains to solve the equations

$$\left[\phi\left(\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\right), \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right] = \begin{bmatrix}-f(\gamma) & f(\alpha-\delta)\\ 0 & f(\gamma)\end{bmatrix}$$

and

$$\left[\phi\left(\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\right), \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right] = \begin{bmatrix}f(\beta) & 0\\ f(\delta-\alpha) & -f(\beta)\end{bmatrix}.$$

Notice that $-A^t = (E_{21} - E_{12})A(E_{21} - E_{12})^{-1} - \mathrm{tr}(A)I$ for every matrix $A \in M_2$, so in the case $n = 2$ the statement of the theorem can be simplified.

References

[1] G. Dolinar, Maps on upper triangular matrices preserving Lie products, *Linear and Multilinear Algebra* **55** (2) (2007), 191–198.

[2] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, *Cambridge University Press*, 1991.

[3] M. Jodeit Jr. and T. Y. Lam, Multiplicative maps of matrix semi-groups, *Arch. Math.* **20** (1969), 10–16.

[4] L. Molnár, Jordan Maps on Standard Operator Algebras, Functional Equations – Results and Advances, Vol. 3, Advances in Mathematics, (Z. Daróczy and Zs. Páles, eds.), *Kluwer Acad. Publ.*, *Dordrecht*, 2002, 305–320.

[5] P. Šemrl, Non-linear commutativity preserving maps, *Acta Sci. Math. (Szeged)* **71** (2005), 781–819.

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