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Oscillation and nonoscillation of perturbed half-linear Euler differential equations

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Abstract. Using general results for (non)oscillation of the second order half-linear differential equation

$$\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x,\ p > 1, \tag{$*$}$$

we establish new oscillation and nonoscillation criteria for the perturbed half-linear Euler differential equation

$$(\Phi(x'))' + \left[\frac{\gamma_p}{t^p} + c(t)\right]\Phi(x) = 0, \quad \gamma_p := \left(\frac{p-1}{p}\right)^p.$$

1. Introduction and preliminaries

The aim of this paper is to investigate oscillatory properties of solutions of the half-linear second order differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x,\ p > 1, \tag{1}$$

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where r, c are continuous functions, $r(t) > 0$, and its special case, the perturbed half-linear Euler differential equation

$$(\Phi(x'))' + \left[\frac{\gamma_p}{t^p} + c(t)\right]\Phi(x) = 0, \quad \gamma_p := \left(\frac{p-1}{p}\right)^p. \tag{2}$$

Recently, several papers (see [5], [6], [7], [4], [12], [13]) appeared, where equation (1) is viewed as a perturbation of the nonoscillatory equation of the same form

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0 \tag{3}$$

and (non)oscillation criteria are formulated in terms of the asymptotic behavior of the integrals

$$\int^t [c(s) - \tilde{c}(s)]h^p(s)\,ds, \quad \text{or} \quad \int_t^\infty [c(s) - \tilde{c}(s)]h^p(s)\,ds,$$

where h is a solution of (3).

Here we follow a slightly more general idea which is motivated by the fact that the exact solution of (3) is not known in many cases and only its asymptotic estimate is available. A typical example is the Euler equation with the critical coefficient

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0, \tag{4}$$

whose one solution is $x(t) = t^{\frac{p-1}{p}}$ and any linearly independent solution is known only asymptotically $x(t) \sim t^{\frac{p-1}{p}} \log^{\frac{2}{p}} t$ as $t \to \infty$. We are particularly motivated by the oscillation criterion given in [13], where it was shown that equation (2) is oscillatory provided

$$\liminf_{t\to\infty} \frac{1}{\log t} \int^t c(s)s^{p-1}\log^2 s\,ds > 2\left(\frac{p-1}{p}\right)^{p-1}, \tag{5}$$

and it was conjectured that the constant $2\left(\frac{p-1}{p}\right)^{p-1}$ in (5) can be replaced by the better constant $\frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$. At the same time it was conjectured (based on [5, Theorem 3]) that (2) is nonoscillatory provided

$$\limsup_{t\to\infty} \frac{1}{\log t} \int^t c(s)s^{p-1}\log^2 s\,ds < \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \tag{6}$$

and

$$\liminf_{t\to\infty} \frac{1}{\log t} \int^t c(s)s^{p-1}\log^2 s\,ds > -\frac{3}{2}\left(\frac{p-1}{p}\right)^{p-1}. \tag{7}$$

The aim of this paper is to prove both these conjectures. We use first an extension of [5, Theorem 2] to the situation when h *is not* a solution of (3) and then we apply the recently established oscillation criterion for the perturbed Euler–Weber half-linear differential equation

$$(\Phi(x'))' + \left[\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t} + c(t)\right]\Phi(x) = 0, \quad \mu_p := \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \tag{8}$$

coupled with the "oscillation constant improvement procedure" introduced for higher order linear differential equation in [2].

It is well known that the oscillation theory of half-linear equation (1) is very similar to that of the linear Sturm–Liouville differential equation (which is the special case $p = 2$ in (1))

$$(r(t)x')' + c(t)x = 0$$

even if the additivity of the solution space of (1) is lost and only homogeneity remains. In particular, similarly to the linear case, equation (1) can be classified as *oscillatory* or *nonoscillatory* according to whether every nontrivial solution has/does not have infinitely many zeros on every interval of the form $[T, \infty)$. For general background of the half-linear oscillation theory we refer to [1, Chap. 3], [3] or to [8].

The basic tools of the half-linear oscillation theory are the so-called *variational principle* and *Riccati technique*. The first one consists in the fact that (1) is oscillatory if and only if for every $T \in \mathbb{R}$ there exists a nontrivial function $y \in W_0^{1,p}(T, \infty)$ such that

$$\mathcal{F}(y; T, \infty) = \int_T^\infty \left[r(t)|y'|^p - c(t)|y|^p\right] dt \le 0.$$

The Riccati technique is based on the fact that if $x(t) \neq 0$ in an interval I is a solution of (1) then $w = r\Phi(x'/x)$ solves in I the Riccati type first order differential equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \tag{9}$$

where q is the conjugate number of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Actually, according to the Sturm comparison theory for half-linear equations, solvability of (9) can be replaced by the associated inequality as shows the following statement which can be found e.g. in [1, Theorem 3.7.1] or [8, Theorem 2.2.1].

Lemma 1. *Equation* (1) *is nonoscillatory if and only if there exists a differentiable function* w *such that*

$$R[w](t) := w'(t) + c(t) + (p-1)r^{1-q}(t)|w(t)|^q \le 0 \tag{10}$$

for large t.

We finish this section with an oscillation criterion for (8) which is proved in [4] using the variational principle and which we will need in the proof of Theorem 1 of the next section.

Proposition 1. *If*

$$\int^\infty c(t)t^{p-1}\log t\, dt = \infty, \tag{11}$$

then equation (8) *is oscillatory.*

2. Oscillation and nonoscillation criteria

We start with a technical result concerning the function $h(t) = t^{\frac{p-1}{p}} \log^{\frac{2}{p}} t$ which is asymptotically equivalent (in a certain sense) to the so-called nonprincipal solution of Euler equation (4). For more details concerning principal and nonprincipal solutions of (4) and of related equations we refer to [10].

Lemma 2. *Let* $h(t) = t^{\frac{p-1}{p}} \log^{\frac{2}{p}} t$, *denote* $\tilde{R}[w] := w' + \gamma_p/t^p + (p-1)|w|^q$ *the Riccati operator associated with* (4), *and let* $w_h = \Phi(h'/h)$. *Then*

$$h^p(t)\tilde{R}[w_h](t) = \frac{K}{t\log t}\left(1+o(1)\right), \quad \text{as } \ t\to\infty, \tag{12}$$

where K *is a real constant.*

Proof. By a direct computation we have

$$h'(t) = \left(\frac{p-1}{p}\right) t^{-\frac{1}{p}} \log^{\frac{2}{p}} t \left(1 + \frac{2}{(p-1)\log t}\right)$$

and hence (using the formula $(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3 + \dots$)

$$|w_h|^q = \left(\frac{p-1}{p}\right)^p \frac{1}{t^p}\left(1 + \frac{2}{(p-1)\log t}\right)^p$$

$$= \left(\frac{p-1}{p}\right)^p \frac{1}{t^p}\left(1 + \frac{2p}{(p-1)\log t} + \frac{2p}{(p-1)\log^2 t} + \frac{4p(p-2)}{3(p-1)^2\log^3 t} + \dots\right),$$

$$(p-1)|w_h|^q h^p = (p-1)\left(\frac{p-1}{p}\right)^p \frac{1}{t}\log^2 t$$
$$\times\left(1+\frac{2p}{(p-1)\log t}+\frac{2p}{(p-1)\log^2 t}+\frac{4p(p-2)}{3(p-1)^2\log^3 t}+\dots\right),$$

and

$$\begin{aligned} h^p w_h' &= -p\left(\frac{p-1}{p}\right)^p \frac{1}{t}\log^2 t\left(1+\frac{2}{(p-1)\log t}\right)^{p-2} \\ &\quad\times\left(1+\frac{2}{(p-1)\log t}+\frac{2}{(p-1)\log^2 t}\right) \\ &= -p\left(\frac{p-1}{p}\right)^p \frac{1}{t}\log^2 t\left(1+\frac{2\,(p-2)}{(p-1)\log t}+\frac{2\,(p-2)\,(p-3)}{(p-1)^2\log^2 t}\right. \\ &\quad\left.+\frac{4\,(p-2)\,(p-3)\,(p-4)}{3\,(p-1)^3\log^3 t}+\dots\right)\times\left(1+\frac{2}{(p-1)\log t}+\frac{2}{(p-1)\log^2 t}\right) \\ &= -p\left(\frac{p-1}{p}\right)^p \frac{1}{t}\log^2 t\left(1+\frac{2}{\log t}+\frac{2}{\log^2 t}+\frac{4p\,(p-2)}{3\,(p-1)^2\log^3 t}+\dots\right). \end{aligned}$$

Summarizing the previous computations,

$$h^p\tilde{R}[w_h] = K\frac{1}{t\log t}(1+o(1)), \quad \text{as } t\to\infty, \quad K := -\left(\frac{p-1}{p}\right)^p \frac{4p\,(p-2)}{3\,(p-1)^2},$$

□

Using the previous lemma and Proposition 1 coupled with the “oscillation constant improvement procedure” introduced in [2], we obtain the following oscillation criterion.

Theorem 1. *If*

$$\liminf_{t\to\infty}\frac{1}{\log t}\int_T^t c(s)s^{p-1}\log^2 s\,ds > \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \tag{13}$$

for some (and hence every) $T\in\mathbb{R}$ sufficiently large, then equation (2) *is oscillatory.*

Proof. We rewrite (2) into the form

$$(\Phi(x'))' + \left[\frac{\gamma_p}{t^p}+\frac{\mu_p}{t^p\log^2 t}+\left(c(t)-\frac{\mu_p}{t^p\log^2 t}\right)\right]\Phi(x) = 0$$

and we regard this equation as a perturbation of the so-called half-linear Euler–Weber differential equation

$$(\Phi(x'))' + \left[\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}\right] \Phi(x) = 0. \tag{14}$$

According to Proposition 1, it is sufficient to show that

$$\int_T^\infty \left[c(t) - \frac{\mu_p}{t^p \log^2 t}\right] t^{p-1} \log t \, dt = \infty. \tag{15}$$

To this end, we proceed as follows. According to (13) there exists $\varepsilon > 0$ such that still

$$\liminf_{t\to\infty} \frac{1}{\log t} \int_T^t c(s) s^{p-1} \log^2 s \, ds > \mu_p + \varepsilon \tag{16}$$

for t sufficiently large, say $t > \tilde{T}$. From (16) we have that

$$\frac{1}{t} \int_T^t c(s) s^{p-1} \log^2 s \, ds > (\mu_p + \varepsilon) \frac{\log t}{t} \tag{17}$$

for $t > \tilde{T}$. At the same time, using integration by parts and (17)

$$\begin{aligned}
&\int_T^b \left[c(t) - \frac{\mu_p}{t^p \log^2 t}\right] t^{p-1} \log t \, dt = \int_T^b c(t) t^{p-1} \log t \, dt - \mu_p \int_T^b \frac{1}{t \log t} \, dt \\
&= \left[\frac{1}{\log t} \int_T^t c(s) s^{p-1} \log^2 s \, ds\right]_T^b \\
&\quad + \int_T^b \frac{1}{\log^2 t} \frac{\int_T^t c(s) s^{p-1} \log^2 s \, ds}{t} \, dt - \mu_p \log\left(\frac{\log b}{\log T}\right) \\
&= \left[\frac{1}{\log t} \int_T^t c(s) s^{p-1} \log^2 s \, ds\right]_T^b + \int_T^{\tilde{T}} \frac{1}{\log^2 t} \frac{\int_T^t c(s) s^{p-1} \log^2 s \, ds}{t} \, dt \\
&\quad + \int_{\tilde{T}}^b \frac{1}{\log^2 t} \frac{\int_T^t c(s) s^{p-1} \log^2 s \, ds}{t} \, dt - \mu_p \log\left(\frac{\log b}{\log T}\right) \\
&\geq \frac{1}{\log b} \int_T^b c(s) s^{p-1} \log^2 s \, ds + K + (\mu_p + \varepsilon) \int_T^b \frac{1}{t \log t} \, dt - \mu_p \log\left(\frac{\log b}{\log T}\right) \\
&> (\mu_p + \varepsilon) + K + \varepsilon \log\left(\frac{\log b}{\log T}\right) \to \infty
\end{aligned}$$

as $b \to \infty$, where $K = \int_T^{\tilde{T}} t^{-1} \log^{-2} t \big(\int_T^t c(s) s^{p-1} \log^2 s \, ds\big) \, dt$. Consequently, (2) is oscillatory by Proposition 1. □

Next we prove a general nonoscillation criterion for (1) where this equation is viewed as a perturbation of (3). In contrast to [5, Theorem 3], we do not suppose that h is a solution of (3).

Theorem 2. *Let $h \in C^1$ be a positive function such that $h'(t) > 0$ for large t, say $t > T$, $\int^\infty r^{-1}(t)h^{-2}(t)(h'(t))^{2-p}\,dt < \infty$, and denote*

$$G(t) := \int_t^\infty \frac{ds}{r(s)h^2(s)(h'(s))^{p-2}}. \tag{18}$$

Suppose that

$$\lim_{t\to\infty} G(t)r(t)h(t)\Phi(h'(t)) = \infty \tag{19}$$

and

$$\lim_{t\to\infty} G^2(t)r(t)h^3(t)(h'(t))^{p-2}\left[(r(t)\Phi(h'(t)))' + \tilde{c}(t)\Phi(h(t))\right] = 0. \tag{20}$$

If

$$\limsup_{t\to\infty} G(t)\int_T^t [c(s) - \tilde{c}(s)]h^p(s)\,ds < \frac{1}{2q}, \tag{21}$$

and

$$\liminf_{t\to\infty} G(t)\int_T^t [c(s) - \tilde{c}(s)]h^p(s)\,ds > -\frac{3}{2q} \tag{22}$$

for some $T \in \mathbb{R}$ sufficiently large, then (1) is nonoscillatory.

Proof. Denote

$$v(t) = r(t)h(t)\Phi(h'(t)) - \frac{1}{2qG(t)}, \quad C(t) = \int_T^t [c(s) - \tilde{c}(s)]h^p(s)\,ds$$

and let $w(t) = h^{-p}(t)[v(t) - C(t)]$. To prove that (1) is nonoscillatory, according to Lemma 1 it suffices to show that w satisfies (10) and this happens if v satisfies the inequality (suppressing the argument t)

$$v' - rh'^p + \tilde{c}h^p + prh'^p Q\left(\frac{v - C}{rh\Phi(h')}\right) \le 0, \quad Q(s) := \frac{|s|^q}{q} - s + \frac{1}{p}. \tag{23}$$

Indeed, suppose that (23) holds, then

$$\begin{aligned} w' &= h^{-p}(v' - ch^p + \tilde{c}h^p) - p(v - C)h'h^{-p-1} \\ &\le h^{-p}\left[rh'^p - ch^p - prh'^p\left(\frac{1}{q}\left|\frac{v - C}{rh\Phi(h')}\right|^q - \frac{v - C}{rh\Phi(h')} + \frac{1}{p}\right) - p\frac{h'(v - C)}{h}\right] \\ &= -c - (p-1)r^{1-q}|w|^q. \end{aligned}$$

To verify (23) let us first estimate (again suppressing the argument t)

$$\frac{v-C}{rh\Phi(h')} = \frac{rh\Phi(h') - \frac{1}{2qG} - C}{rh\Phi(h')} = 1 - \frac{1+2qGC}{2qGrh\Phi(h')} \to 1 \quad \text{as } t \to \infty$$

since the numerator of the last fraction is bounded by (21) and (22), while the denominator tends to ∞ by (19). Now, let $\varepsilon > 0$ be such limsup in (21) is less than $\frac{1}{2q} - 2\varepsilon$ and liminf in (22) is greater than $-\frac{3}{2q} + 2\varepsilon$. This means that the expression in these limits satisfies

$$-\frac{3}{2q} + \varepsilon < G(t)C(t) < \frac{1}{2q} - \varepsilon \quad \Longleftrightarrow \quad |1 + 2qG(t)C(t)| < 2 - \varepsilon$$

for large t, i.e.,

$$(1 + 2qG(t)C(t))^2 < (2-\varepsilon)^2 \tag{24}$$

for large t. Now, since $(v-C)/(rh\Phi(h')) \to 1$ as $t \to \infty$ and $Q(1) = 0 = Q'(1)$, by the second degree Taylor formula, to $\varepsilon(q-1)/4 > 0$ there exists $\hat{T}$, such that

$$\begin{aligned} Q\left(\frac{v(t)-C(t)}{r(t)h(t)\Phi(h'))}\right) &\le \left(\frac{q-1}{2} + \frac{(q-1)\varepsilon}{4}\right)\left(\frac{v(t)-C(t)}{r(t)h(t)\Phi(h')} - 1\right)^2 \\ &= \frac{q-1}{2}\left(1+\frac{\varepsilon}{2}\right)\frac{(1+2qG(t)C(t))^2}{4q^2G^2(t)r^2(t)h^2(t)(h'(t))^{2p-2}} \\ &< \frac{q-1}{2}\left(1+\frac{\varepsilon}{2}\right)\frac{(2-\varepsilon)^2}{4q^2G^2(t)r^2(t)h^2(t)(h'(t))^{2p-2}} \end{aligned}$$

for $t > \hat{T}$. Using these estimate we have

$$\begin{aligned} &v' - rh'^p + \tilde{c}h^p + prh'^p Q\left(\frac{v-C}{rh\Phi(h')}\right) \\ &= (r\Phi(h'))'h + rh'^p + \frac{G'}{2qG^2} - rh'^p + \tilde{c}h^p + prh'^p Q\left(\frac{v-C}{rh\Phi(h')}\right) \\ &\le (r\Phi(h'))'h + \tilde{c}h^p - \frac{1}{2qG^2rh^2(h')^{p-2}} + \frac{q}{2}\left(1+\frac{\varepsilon}{2}\right)rh'^p\left(\frac{v-C}{rh\Phi(h')} - 1\right)^2 \\ &\le h[(r\Phi(h'))' + \tilde{c}\Phi(h)] + \frac{1}{2qG^2rh^2(h')^{p-2}}\left[-1 + \frac{(2-\varepsilon)^2}{4}\left(1+\frac{\varepsilon}{2}\right)\right] \\ &< \frac{1}{2qG^2rh^2(h')^{p-2}}\left\{2qG^2rh^3(h')^{p-2}[(r\Phi(h'))' + \tilde{c}\Phi(h)] - \frac{\varepsilon}{2}\right\} < 0 \end{aligned}$$

for large t since the first term in braces tends to zero according to (20) and $-1 + \frac{(2-\varepsilon)^2}{4}\left(1+\frac{\varepsilon}{2}\right) < -\frac{\varepsilon}{2}$, if $\varepsilon > 0$ is sufficiently small. □

The previous theorem and Lemma 2 applied to (2) prove the conjecture mentioned at the beginning of the paper.

Corollary 1. *Suppose that* (6) *and* (7) *hold, then* (2) *is nonoscillatory.*

Proof. We will use the previous theorem where equation (4) plays the role of (3). We take $h(t) = t^{\frac{p-1}{p}} \log^{\frac{2}{p}} t$. Then

$$G(t) = \int_t^\infty \frac{ds}{h^2(s)(h'(s))^{p-2}} \sim \left(\frac{p}{p-1}\right)^{p-2} \int_t^\infty \frac{ds}{s\log^2 s} = \left(\frac{p}{p-1}\right)^{p-2} \frac{1}{\log t},$$

$$G(t)h(t)\Phi(h'(t)) \sim \frac{p-1}{p} \log t \to \infty \quad \text{as} \quad t \to \infty.$$

and using the the fact that $h^p(t)\tilde{R}[w_h](t) = h(t)[(\Phi(h'(t)))' + \frac{\gamma_p}{t^p}\Phi(h(t))]$, from Lemma 2 we have

$$G^2(t)h^3(t)(h'(t))^{p-2}\left[(\Phi(h'(t)))' + \frac{\gamma_p}{t^p}\Phi(h(t))\right] \sim \frac{\text{const}}{t} \to 0, \quad \text{as } t \to \infty.$$

Consequently, all assumptions of Theorem 2 are satisfied and we have the required statement. □

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