Publ. Math. Debrecen
43 / 3–4 (1993), 351–357

On the oscillatory behavior of solutions of second order nonlinear differential equations

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Abstract. In this paper we study the problem of oscillation of Emden-Fowler equations of the form

$$(a(t)\acute{x}(t))' + q(t)(|x(t)|^c)\operatorname{sgn} x(t) = 0\,, \tag{E}$$

where $a, q : [t_0, \infty) \to R$ are continuous, $a(t) > 0$ and $\int_t^\infty q(s)ds$ converges for $t \geq t_0$. The obtained results are applicable to equation (E) for all $c > 0$, while all known oscillation criteria for equation (E) with integrally small coefficient q are presented when $c = 1$.

1. Introduction

Consider the second order differential equation

$$(a(t)\acute{x}(t))' + q(t)(|x(t)|^c)\operatorname{sgn} x(t) = 0, \quad c > 0 \text{ and } (\ ' \ = \frac{d}{dt})\,, \tag{E}$$

where $a, q : [t_0, \infty) \to \mathbb{R} = (-\infty, \infty)$ are continuous and $a(t) > 0$ for $t \geq t_0$. We assume that:

$$\int^\infty \frac{1}{a(s)} ds = \infty, \quad \text{and} \tag{1}$$

$$\int_t^\infty q(s)ds \quad \text{converges}\,. \tag{2}$$

We consider only such solutions of equation (E) which exist on some interval $[t_x, \infty)$, $t_x \geq t_0 \geq 0$. A solution of equation (E) is said to be oscillatory if it has arbitrarily large zeros, and otherwise, it is said to be nonoscillatory. Equation (E) is called oscillatory if all of its solutions are oscillatory.

Of particular interest is the problem of finding criteria for the oscillation of the differential equation (E) when q is allowed to take on negative values and condition (2) is satisfied. Many criteria have been found which involve convergence of the improper integral of the alternating coefficient and which are motivated by the results of Kamenev [5], for the special case $c = 1$ and $a(t) = 1$ (or almost linear equations of type (E)). For such results on second order oscillation, we choose to refer to papers [1–7] and the references cited therein.

It seems that all extensions of Kamenev's results [5] are applicable to equations of type (E) with $c = 1$. Therefore, the purpose of this paper is to proceed further in this direction to present some new oscillation criteria for equation (E) for all $c > 0$.

As long as the improper integral (2) converges we can define

$$g_0(t) = (h(t))^{1/2} \int_t^\infty q(s)ds\,, \quad g_1(t) = \int_t^\infty (g_0(s)_+)^2 ds\,,$$

and

$$g_{n+1}(t) = \int_t^\infty ((g_0(s) + M(h(s))^{1/2} g_n(s))_+)^2 ds, \text{ for } n = 1, 2, 3, \ldots,$$

where $h(t) = \frac{1}{a(t)}$ if $c = 1$ and $h(t) = \frac{1}{a(t)} \left(\int_T^t \frac{1}{a(s)} ds \right)^{-1}$ if $c \neq 1$, and for $t \geq T$ for some $T \geq t_0$, M is any positive constant, $M = 1$ if $c = 1$ and $g_0(t)_+ = \max\{g(t), 0\}$.

In the following theorem we make use of the following condition: For any constant $M > 0$, there exists a positive integer N such that

(3) $\quad g_n$ exists for $n = 0, 1, \ldots, N-1$ and g_N does not exist.

Theorem 1. *Suppose that conditions* (1)–(3) *hold. Then equation* (E) *is oscillatory for all* $c > 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (E), say $x(t) > 0$ for $t \geq t_0 \geq 0$. Furthermore, we define

$$W(t) = \frac{a(t)\acute{x}(t)}{(x(t))^c} \quad \text{for} \quad t \geq t_0\,.$$

Then for $t \geq t_0$, we have

$$\acute{W}(t) = -q(t) - ca(t)\frac{(\acute{x}(t))^2}{(x(t))^{c+1}}\,. \tag{4}$$

Thus, for $t \ge t_0$, we get

$$W(t) + c\int_{t_0}^{t} a(s)\left(\frac{\acute{x}(s)}{x^m(s)}\right)^2 ds = W(t_0) - \int_{t_0}^{t} q(s)ds\,, \tag{5}$$

where $m = \frac{c+1}{2}$.

Next, we consider the following two cases:

Case 1. The integral

$$\int_{t_0}^{\infty} a(s)\left(\frac{\acute{x}(s)}{x^m(s)}\right)^2 ds \tag{6}$$

is finite. In this case, there exists a positive constant A so that

$$\int_{t_0}^{t} a(s)\left(\frac{\acute{x}(s)}{x^m(s)}\right)^2 ds \le A \quad \text{for} \quad t \ge t_0\,.$$

By the Schwarz inequality,

$$\begin{aligned}\left|\int_{t_0}^{t} \frac{\acute{x}(s)}{x^m(s)} ds\right|^2 &\le \left(\int_{t_0}^{t} \frac{ds}{a(s)}\right)\left(\int_{t_0}^{t} a(s)\left(\frac{\acute{x}(s)}{x^m(s)}\right)^2 ds\right) \\ &\le A\int_{t_0}^{t} \frac{ds}{a(s)}\,,\end{aligned}$$

or

$$|x^{1-m}(t) - x^{1-m}(t_0)| \le |1-m|\left(A\int_{t_0}^{t} \frac{ds}{a(s)}\right)^{1/2}.$$

There exist $t_1 > t_0$ and a constant $B > 0$ so that

$$|x^{1-m}(t)| \le B\left(\int_{t_0}^{t} \frac{ds}{a(s)}\right)^{1/2} \quad \text{for all} \quad t \ge t_1\,. \tag{7}$$

Using (7) in (4), we get

$$\acute{W}(t) \le -q(t) - Mh(t)W^2(t) \quad \text{for} \quad t \ge t_1\,, \tag{8}$$

where $M = c/B^2$ and $h(t) = \frac{1}{a(t)}\left(\int\limits_{t_0}^{t} \frac{ds}{a(s)}\right)^{-1}$ if $c \neq 1$, $M = 1$ and $h(t) = 1/a(t)$ if $c = 1$. From (6) and (7) we see that

$$\int\limits_{t}^{\infty} h(s)W^2(s)ds < \infty\,. \tag{9}$$

Thus, for $z \geq t \geq t_1$, we have

$$W(z) + M\int\limits_{t}^{z} h(s)W^2(s)ds \leq W(t) - \int\limits_{t}^{z} q(s)ds\,, \tag{10}$$

and hence, one can easily check that

$$\lim_{t\to\infty} W(t) = 0 \tag{11}$$

and

$$W(t) \geq \int\limits_{t}^{\infty} q(s)ds + M\int\limits_{t}^{\infty} h(s)W^2(s)ds, \quad t \geq t_1\,. \tag{12}$$

Now,

$$W(t) \geq (h(t))^{-1/2}g_0(t) + M\int\limits_{t}^{\infty} h(s)W^2(s)ds, \quad t \geq t_1\,. \tag{13}$$

From (13),

$$W(t) \geq (h(t))^{-1/2}g_0(t)\,,$$

which implies that

$$W^2(t) \geq (h(t))^{-1}(g_0(t)_+)^2\,. \tag{14}$$

If $N = 1$, then (9) and (14) imply that

$$g_1(t) = \int\limits_{t}^{\infty} (g_0(s)_+)^2 ds < \infty\,,$$

which cotradicts the nonexistence of $g_N(t) = g_1(t)$. If $N = 2$, then from (13) and (14) we get

$$\begin{aligned} W(t) &\geq (h(t))^{-1}g_0(t) + M\int\limits_{t}^{\infty} (g_0(s)_+)^2 ds \\ &= (h(t))^{-1}g_0(t) + Mg_1(t)\,, \end{aligned}$$

so

$$(h(t))^{1/2}W(t) \geq g_0(t) + Mg_1(t)(h(t))^{1/2}\,,$$

from which it follows that

$$h(t)W^2(t) \geq ((g_0(t) + Mg_1(t)(h(t))^{1/2})_+)^2\,.$$

Then, in view of (9), an integration of the last inequality leads to a contradiction of the nonexistence of $g_N = g_2$. Similar arguments lead to contradiction for any integer $N > 2$.

Case 2. The integral

$$\int^{\infty} a(s)\left(\frac{\acute{x}(s)}{x^m(s)}\right)^2 ds$$

is infinite. Using (2) in (5) we have, for $t \geq T$ for some $T \geq t_0$

$$-a(t)\acute{x}(t)/x^{-c}(t) \geq L + c\int_T^t a(s)(\acute{x}(s))^2/x^{1+c}(s)ds\,, \tag{15}$$

where L is a constant. By the assumption, we can choose a $T_1 > T$ so that

$$c\int_T^{T_1} a(s)(\acute{x}(s))^2/x^{1+c}(s)ds = 1 + L$$

and then for any $t \geq T_1$, we get

$$\frac{-a(t)(\acute{x}(t)/x^c(t))(-c\acute{x}(t)/x(t))}{-L + \int_t^T a(s)(\acute{x}(s))^2/x^{1+c}(s)ds} \geq -c\acute{x}(t)/x(t)\,.$$

Integrating the above inequality from T_1 to t we obtain

$$\ln(-L + c\int_{T_1}^t a(s)(\acute{x}(s))^2/x^{1+c}(s)ds) \geq c\int_{T_1}^t -\acute{x}(s)/x(s)ds =$$
$$= \ln(x(T_1)/x(t))^c\,,$$

which together with (15) yields

$$-a(t)(\acute{x}(t)/x^c(t)) \geq (x(T_1)/x(t))^c\,,$$

from which it follows that

$$\acute{x}(t) \leq -(-x(T_1))^c(1/a(t)) < 0 \quad \text{for} \quad t \geq T_1\,,$$

or

$$x(t) \leq x(T_1) - (x(T_1))^c \int_{T_1}^{t} 1/a(s)ds \to -\infty \quad \text{as} \quad t \to \infty\,,$$

contradicting the fact that $x(t) > 0$ for $t \geq t_0$. This completes the proof.

The following criterion removes the condition that g_n must fail to exist for some $n = N$:

Theorem 2. *Assume that conditions* (1) *and* (2) *hold. If* g_n *exists for all* $n = 1, 2, \ldots$ *and there exists an increasing sequence* $s_j \to \infty$ *as* $j \to \infty$ *such that* $g_n(s_j) \to \infty$ *as* $n \to \infty$ *for each* j*, then equation* (E) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (E), say $x(t) > 0$ for $t \geq t_0 \geq 0$. Proceeding as in the proof of Theorem 1 (case 1) we again obtain (11) and (12), so that (13) and (14) hold. Since g_n exists for each n, an argument similar to the one used in Theorem 1 shows that

$$W(t) \geq (h(t))^{-1/2} g_0(t) + M g_n(t) \quad \text{for} \quad n \geq 1\,.$$

Hence there exists $s_J > t_1$ such that

$$W(s_J) \geq (h(s_J))^{-1/2} g_0(s_J) + M g_n(s_J) \to \infty \quad \text{as} \quad n \to \infty\,,$$

which contradicts (12). The rest of the proof is similar to the proof of Theorem 1 and hence is omitted.

There are numerous known criteria related to Theorem 1; see for example [1–7] and the references contained therein. None of these appear to contain Theorem 1 with $c \neq 1$. In particular, it follows from Theorem 1 that all solutions of the differential equation

$$(E^*) \quad (e^{-t}\acute{x}(t)) + t^{-3/2}(\cos(\ln t) - \sin(\ln t))\ (|x(t)|^c) \operatorname{sgn} x(t) = 0\,, \qquad c > 0, \quad t \geq t_0 = 1\,,$$

are oscillatory for all $c > 0$, while none of the results cited above apply to this equation. To check that equation (E^*) satisfies the conditions of Theorem 1, observe that

$$\int_{t}^{\infty} q(s)ds = t^{-1/2}\sin(\ln t), \quad h(t) = \frac{e^t}{e^t - 1}\,,$$

$$g_0(t) = (h(t))^{1/2} \int_t^\infty q(s)ds = \left(\frac{e^t}{e^t - 1}\right) t^{-1/2} \sin(\ln t), \text{ and}$$

$$g_1(t) = \int_t^\infty (g_0(s)_+)^2 ds = \frac{1}{4} \int_t^\infty \frac{e^s}{e^s - 1} \frac{1}{s} (\sin(\ln s) + |\sin(\ln s)|)^2 ds$$

$$\geq \frac{1}{2} \int_t^\infty \frac{1}{s} (\sin^2(\ln s) + \sin(\ln s)|\sin(\ln s)|) ds$$

$$\geq \frac{1}{2} \sum_{n=k}^\infty \int_{\exp(n\pi)}^{\exp(n+1)\pi} \frac{1}{s} (\sin^2(\ln s) + \sin(\ln s)\cdot$$

$$\cdot |\sin(\ln s)|) ds, \quad k = [t]$$

$$\geq \sum_{n=k}^\infty \int_{\exp(2n\pi)}^{\exp(2n+1)\pi} \frac{\sin^2(\ln s)}{s} ds = \sum_{n=k}^\infty \int_0^\pi \sin^2(\ln s) ds = \infty \,.$$

Thus, g_1 does not exist, i.e. condition (3) holds with $N = 1$.

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(Received April 24, 1992)