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Limit theorems for normalized nearly critical branching processes with immigration

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Abstract. Functional limit theorems are proved for a sequence of Galton–Watson processes with immigration, where the offspring mean tends to its critical value 1 under weak conditions for the variances of offspring and immigration processes. In the limit theorems the norming factors depend on these variances.

1. Introduction

In this paper we consider a *sequence of branching processes with immigration* (SBPI) $(X_k^{(n)})_{k\in\mathbb{Z}_+}$, $n \in \mathbb{N}$, given by the recursion

$$X_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \xi_{k,j}^{(n)} + \varepsilon_k^{(n)} \quad \text{for } k, n \in \mathbb{N}, \qquad X_0^{(n)} = 0, \tag{1}$$

where $\{\xi_{k,j}^{(n)}, \varepsilon_k^{(n)} : k, j, n \in \mathbb{N}\}$ are independent, nonnegative, integer valued random variables such that $\{\xi_{k,j}^{(n)} : k, j \in \mathbb{N}\}$ and $\{\varepsilon_k^{(n)} : k \in \mathbb{N}\}$ for each $n \in \mathbb{N}$ are identically distributed. For a fixed $n \in \mathbb{N}$ we can interpret $X_k^{(n)}$ as the size of the k^{th} generation of a population, where $\xi_{k,j}^{(n)}$ is the number of offsprings of the j^{th} individual in the $(k-1)^{\text{st}}$ generation and $\varepsilon_k^{(n)}$ is the number of immigrants contributing to the k^{th} generation. Athreya and Vidyashankar [3] provides a

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short overview concerning these processes. A number of new developments and applications can be found in Athreya and Jagers [2], and Haccou et al. [5].

We assume that $m_n := \mathsf{E}\xi_{1,1}^{(n)}$, $\lambda_n := \mathsf{E}\varepsilon_1^{(n)}$, $\sigma_n^2 := \mathsf{Var}\xi_{1,1}^{(n)}$, $b_n^2 := \mathsf{Var}\varepsilon_1^{(n)}$ exist and finite for all $n \in \mathbb{N}$. The cases when the offspring mean is less, equal or larger than one are referred to *subcritical, critical* or *supercritical*, respectively. If m_n tends to 1 as $n \to \infty$ then the SBPI is called nearly critical. This concept is introduced in the next more precise definition.

Definition 1.1. A SBPI defined by (1) is called *nearly critical* with rate $\alpha \in \mathbb{R}$ if $m_n = 1 + \alpha n^{-1} + o(n^{-1})$ as $n \to \infty$.

This definition has been suggested by Chan and Wei [4] for the first time in case of AR(1) models.

Introduce the random step functions

$$\mathcal{X}^{(n)}(t) := X_{\lfloor nt \rfloor}^{(n)} \quad \text{for } t \in \mathbb{R}_+,\ n \in \mathbb{N},$$

where $\lfloor \cdot \rfloor$ denotes the lower integer part. We investigate, after appropriate normalization, the asymptotic behaviour of the processes $\mathcal{X}^{(n)}$ as $n \to \infty$. One can see that the necessary norming factor and the possible limit process strongly depend on the variance conditions that are supposed to hold for the offspring and immigration processes. In order to cover as many cases as possible such normalizing factors are used which depend on the variance of the offsprings or immigrations. If both the offspring and the immigration variances tend to non-zero finite limits then we say that the SBPI fulfills the standard variance conditions. This case has been investigated by Wei and Winnicki [16], and Sriram [15].

In this paper, some non-standard cases, where these variances are asymptotically small or large, are investigated. In Theorem 2.1, where the first two moments of the immigration are under the control of the offspring variance, we prove that the limit process is a square-root type diffusion process defined by (4), the norming factor being $(n\sigma_n^2)^{-1}$. If the offspring variances behave like a power function then we have Theorem 2.5 as a corollary of Theorem 2.1, and we obtain a similar limiting diffusion process in (6). Finally, if the offspring variances are asymptotically small, more precisely $\sigma_n^2 = \sigma^2 n^{-1} + o(n^{-1})$ as $n \to \infty$ with some $\sigma^2 \geq 0$, then with the norming factor $(nb_n^2)^{-1/2}$ the limit process will be an Ornstein–Uhlenbeck type process defined by (8), see Theorem 2.9 and its consequence, Theorem 2.12.

Note that convergence of finite dimensional distributions of a SBPI has been investigated by Kawazu and Watanabe [11], and Aliev [1]. Functional limit theorems have been proved by Wei and Winnicki [16], Sriram [15], and Li [12].

The first attempts to deal with the non-standard case were Ispány et al. [7], [8], where conditions $\sigma_n^2 = \sigma^2 n^{-1} + o(n^{-1})$ and $b_n^2 = b^2 + o(1)$ as $n \to \infty$ with some $\sigma^2, b^2 \geq 0$ have been supposed. The cases of increasing or decreasing (in the mean) non-homogeneous immigration have been studied by Rahimov [14, Chapter III]. Theorem 2.5 and 2.12 can be extended by using regularly varying functions.

2. Limit theorems for nearly critical SBPI

In the sequel, let $(X_k^{(n)})_{k\in\mathbb{Z}_+}$, $n \in \mathbb{N}$, denote a nearly critical SBPI with parameters m_n, λ_n, σ_n^2, b_n^2, $n \in \mathbb{N}$, and rate $\alpha \in \mathbb{R}$. The first theorem covers the case where the offspring variances are strictly positive and the first two moments of the immigration are under the control of the offspring variance. Introduce the function

$$\mu(t) := \lambda \int_0^t \mathrm{e}^{\alpha s}\,\mathrm{d}s, \qquad t \in \mathbb{R}_+. \tag{2}$$

Theorem 2.1. *Suppose that $\sigma_n^2 > 0$ for all $n \in \mathbb{N}$, and*

(i) $\mathsf{E}\Big(|\xi_{1,1}^{(n)} - m_n|^2 \mathbb{1}_{\{|\xi_{1,1}^{(n)} - m_n| > \theta n \sigma_n^2\}}\Big) = o(\sigma_n^2)$ *as $n \to \infty$ for all $\theta > 0$,*

(ii) $\lambda_n = \lambda \sigma_n^2 + o(\sigma_n^2)$ *as $n \to \infty$ for some $\lambda \geq 0$,*

(iii) $b_n^2 = o(n\sigma_n^4)$ *as $n \to \infty$.*

Then

$$(n\sigma_n^2)^{-1}\mathsf{E}\mathcal{X}^{(n)}(t) \to \mu(t) \quad \text{as } n \to \infty \tag{3}$$

for all $t \in \mathbb{R}_+$, and

$$(n\sigma_n^2)^{-1}\mathcal{X}^{(n)} \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{X} \quad \text{as } n \to \infty,$$

that is, weakly in the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, where $(\mathcal{X}(t))_{t\in\mathbb{R}_+}$ is the unique solution of a stochastic differential equation (SDE)

$$\mathrm{d}\mathcal{X}(t) = (\lambda + \alpha\mathcal{X}(t))\,\mathrm{d}t + \sqrt{\mathcal{X}_+(t)}\,\mathrm{d}W(t), \quad t \in \mathbb{R}_+, \tag{4}$$

with initial condition $\mathcal{X}(0) = 0$, where $x_+ := \max\{x, 0\}$ and $(W(t))_{t\in\mathbb{R}_+}$ is a standard Wiener process. Moreover,

$$(n\sigma_n^2)^{-1}\big(\mathcal{X}^{(n)} - \mathsf{E}\mathcal{X}^{(n)}\big) \stackrel{\mathcal{D}}{\longrightarrow} \widetilde{\mathcal{X}} \quad \text{as } n \to \infty,$$

where $(\widetilde{\mathcal{X}}(t))_{t\in\mathbb{R}_+}$ is the unique solution of a SDE

$$\mathrm{d}\widetilde{\mathcal{X}}(t) = \alpha\widetilde{\mathcal{X}}(t)\,\mathrm{d}t + \sqrt{(\widetilde{\mathcal{X}}(t) + \mu(t))_+}\,\mathrm{d}W(t), \quad t \in \mathbb{R}_+, \tag{5}$$

with initial condition $\widetilde{\mathcal{X}}(0) = 0$.

Remark 2.2. It is well known that the SDE (4) has a unique global strong solution for every given initial value. Moreover, $X(t) \geq 0$ almost surely for all $t \in \mathbb{R}_+$. Thus, we can replace $\mathcal{X}_+(t)$ by $\mathcal{X}(t)$ under the square root. (See, e.g., Ikeda and Watanabe [6, Example IV.8.2].) The process $(\mathcal{X}(t))_{t\in\mathbb{R}_+}$ is called square-root process or Cox–Ingersoll–Ross model in financial mathematics, see Musiela and Rutkowski [13, p. 290].

Remark 2.3. Condition (i) is, in fact, the Lindeberg condition for the triangular system $\{\xi_{i,j}^{(n)}/(n\sigma_n^2) : n \in \mathbb{N},\ 1 \leq i \leq n,\ 1 \leq j \leq \lfloor n\sigma_n^2 \rfloor\}$. Note that the average size of the nth population at time n is $\mu(1)n\sigma_n^2$. Plainly, if there exists $\gamma > 0$ such that $n^{-\gamma}\sigma_n^{2(1-\gamma)}\mathsf{E}|\xi_{1,1}^{(n)} - m_n|^{2+\gamma} \to 0$ as $n \to \infty$, then condition (i) is satisfied. Note that no Lindeberg condition is needed for the immigration process.

Example 2.4. Let $\mathsf{P}(\xi_{1,1}^{(n)} = n) = \alpha n^{-2}$, $\mathsf{P}(\xi_{1,1}^{(n)} = k_n) = k_n^{-1}$, and $\mathsf{P}(\xi_{1,1}^{(n)} = 0) = 1 - k_n^{-1} - \alpha n^{-2}$ for all $n \in \mathbb{N}$, where $\alpha \in \mathbb{R}_+$ and $k_n \in \mathbb{N}$, $n \in \mathbb{N}$, such that $k_n \to \infty$ as $n \to \infty$. We have that $m_n = 1 + \alpha n^{-1}$ and $\sigma_n^2 = k_n + \alpha - (1 + \alpha n^{-1})^2$ for all $n \in \mathbb{N}$. Condition (i) fulfills since $\{|\xi_{1,1}^{(n)} - m_n| > \theta n\sigma_n^2\}$ is empty for all sufficiently large n if $\theta > 0$ is fixed. Moreover, suppose that $\varepsilon_1^{(n)}$ has Poisson distribution with parameter λ_n such that $\lambda_n = \lambda k_n + o(k_n)$ with some $\lambda \geq 0$, thus conditions (ii) and (iii) hold. Then the norming factor is $(nk_n)^{-1}$, and the limit process is given by (4).

In particular, if the variance of the offspring distribution is a power function we obtain the following limit theorem.

Theorem 2.5. *Suppose that there exists $\varrho \geq 0$ such that*

(i) $\sigma_n^2 = \sigma^2 n^\varrho + o(n^\varrho)$ *as* $n \to \infty$ *with some* $\sigma > 0$,

(ii) $\mathsf{E}\Big(|\xi_{1,1}^{(n)} - m_n|^2 \mathbb{1}_{\{|\xi_{1,1}^{(n)} - m_n| > \theta n^{1+\varrho}\}}\Big) = o(n^\varrho)$ *as* $n \to \infty$ *for all* $\theta > 0$,

(iii) $\lambda_n = \lambda n^\varrho + o(n^\varrho)$ *as* $n \to \infty$ *with some* $\lambda \geq 0$,

(iv) $b_n^2 = o(n^{1+2\varrho})$ *as* $n \to \infty$.

Then

$$n^{-(1+\varrho)}\mathcal{X}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{X} \quad \text{as } n \to \infty,$$

where $(\mathcal{X}(t))_{t\in\mathbb{R}_+}$ is the unique solution of a SDE

$$\mathrm{d}\mathcal{X}(t) = (\lambda + \alpha\mathcal{X}(t))\,\mathrm{d}t + \sigma\sqrt{\mathcal{X}_+(t)}\,\mathrm{d}W(t), \quad t \in \mathbb{R}_+, \tag{6}$$

with initial condition $\mathcal{X}(0) = 0$, where $(W(t))_{t\in\mathbb{R}_+}$ is a standard Wiener process.

Remark 2.6. In case of $\varrho = 0$ the theorem is a generalization of the SRIRAM's theorem, see [15, Theorem 3.1]. Moreover, condition (ii) is weaker than SRIRAM's one (see [15, Section 3]) where $n^{1/2}$ rate is supposed in the indicator function. Plainly, if there exists $\gamma > 0$ such that

$$n^{\varrho(1-\gamma)-\gamma}\mathsf{E}|\xi_{1,1}^{(n)} - m_n|^{2+\gamma} \to 0 \quad \text{as } n \to \infty, \tag{7}$$

then condition (ii) is satisfied. The centered sequence $n^{-(1+\varrho)}(\mathcal{X}^{(n)} - \mathsf{E}\mathcal{X}^{(n)})$, $n \in \mathbb{N}$, converges weakly to a process $(\widetilde{\mathcal{X}}(t))_{t\in\mathbb{R}_+}$ which is the unique solution of a SDE

$$\mathrm{d}\widetilde{\mathcal{X}}(t) = \alpha\widetilde{\mathcal{X}}(t)\,\mathrm{d}t + \sigma\sqrt{(\widetilde{\mathcal{X}}(t) + \mu(t))_+}\,\mathrm{d}W(t), \quad t \in \mathbb{R}_+, \quad \widetilde{\mathcal{X}}(0) = 0.$$

Example 2.7. Let $\mathsf{P}(\xi_{1,1}^{(n)} = 0) = 1 - n^{-1}$, $\mathsf{P}(\xi_{1,1}^{(n)} = n) = n^{-1}$ for all $n \in \mathbb{N}$, and suppose that $\varepsilon_1^{(n)}$ has Poisson distribution with parameter λn with some $\lambda > 0$. Since $\{|\xi_{1,1}^{(n)} - m_n| > \theta n^{1+\varrho}\}$ is empty for all sufficiently large n condition (ii) holds. Then the norming factor is n^{-2} and the limit process is the unique solution of a SDE

$$\mathrm{d}\mathcal{X}(t) = \lambda\,\mathrm{d}t + \sqrt{\mathcal{X}_+(t)}\,\mathrm{d}W(t), \quad \mathcal{X}(0) = 0.$$

Example 2.8. If $\xi_{1,1}^{(n)}$ has a Poisson distribution with parameter $1 + \alpha n^{-1}$, where $\alpha \in \mathbb{R}$, then $m_n = \sigma_n^2 = 1 + \alpha n^{-1}$. Thus, the model is nearly critical, the condition (i) holds with $\sigma^2 = 1$ and $\varrho = 0$, moreover the Lyapunov condition (7) fulfills with $\gamma = 2$ implying (ii). Let $\mathsf{P}(\varepsilon_1^{(n)} = 0) = 1 - n^{-1}\ln n$ and $\mathsf{P}(\varepsilon_1^{(n)} = \lfloor \lambda n \ln^{-1} n \rfloor) = n^{-1}\ln n$ for all $n \in \mathbb{N}$ with some $\lambda > 0$. It is easy to see that conditions (iii) and (iv) hold. However $b_n^2 \to \infty$ as $n \to \infty$, thus [15, Theorem 3.1] can not be applied. The limit process of this sequence of branching processes will be the square-root process (6).

If the offspring variances tend to 0 with speed n^{-1} then the norming factor depends on the immigration variance and we have Ornstein–Uhlenbeck process as a fluctuation limit.

Theorem 2.9. *Suppose that $b_n^2 > 0$ for all $n \in \mathbb{N}$ such that $nb_n^2 \to \infty$ as $n \to \infty$, and*

(i) $\sigma_n^2 = \sigma^2 n^{-1} + o(n^{-1})$ *as $n \to \infty$ with some $\sigma \geq 0$,*

(ii) $\mathsf{E}\Big(|\xi_{1,1}^{(n)} - m_n|^2 \mathbb{1}_{\{|\xi_{1,1}^{(n)} - m_n| > \theta\sqrt{nb_n^2}\}}\Big) = o(n^{-1})$ *as $n \to \infty$ for all $\theta > 0$,*

(iii) $\lambda_n = \lambda b_n^2 + o(b_n^2)$ *as $n \to \infty$ for some $\lambda \geq 0$,*

(iv) $\mathsf{E}\Big(|\varepsilon_1^{(n)} - \lambda_n|^2 \mathbb{1}_{\{|\varepsilon_1^{(n)} - \lambda_n| > \theta\sqrt{nb_n^2}\}}\Big) = o(b_n^2)$ *as* $n \to \infty$ *for all* $\theta > 0$.

Then

$$(nb_n^2)^{-1/2}(\mathcal{X}^{(n)} - \mathsf{E}\mathcal{X}^{(n)}) \xrightarrow{\mathcal{D}} \widetilde{\mathcal{X}} \quad \text{as } n \to \infty,$$

where $(\widetilde{\mathcal{X}}(t))_{t\in\mathbb{R}_+}$ *is an Ornstein–Uhlenbeck type process defined by the SDE*

$$\mathrm{d}\widetilde{\mathcal{X}}(t) = \alpha\widetilde{\mathcal{X}}(t)\,\mathrm{d}t + \sqrt{\sigma^2\mu(t) + 1}\,\mathrm{d}W(t), \quad \widetilde{\mathcal{X}}(0) = 0, \tag{8}$$

where $(W(t))_{t\in\mathbb{R}_+}$ *is a standard Wiener process, and* μ *is defined by* (2).

Remark 2.10. If $b_n^2 \to b^2$ as $n \to \infty$ with some $b \geq 0$, then we have that $n^{-1/2}(\mathcal{X}^{(n)} - \mathsf{E}\mathcal{X}^{(n)}) \xrightarrow{\mathcal{D}} \widetilde{\mathcal{X}}$ as $n \to \infty$, where $(\widetilde{\mathcal{X}}(t))_{t\in\mathbb{R}_+}$ is an Ornstein–Uhlenbeck type process defined by the SDE

$$\mathrm{d}\widetilde{\mathcal{X}}(t) = \alpha\widetilde{\mathcal{X}}(t)\,\mathrm{d}t + b\sqrt{\sigma^2\mu(t) + 1}\,\mathrm{d}W(t), \quad \widetilde{\mathcal{X}}(0) = 0.$$

Thus, Theorem 2.9 is a generalization of [8, Theorem 2.2]. Moreover, if the offspring distributions are Bernoulli distributions with mean $1 - \alpha n^{-1}$ then conditions (i) and (ii) are fulfilled, see Ispány et al. [7] for details.

Remark 2.11. Conditions (ii) and (iv) are the Lindeberg conditions for the triangular systems $\big\{\xi_{i,j}^{(n)}/\sqrt{nb_n^2} : n \in \mathbb{N},\ 1 \leq i,\ j \leq n\big\}$ and $\big\{\varepsilon_j^{(n)}/\sqrt{nb_n^2} : n \in \mathbb{N},\ 1 \leq j \leq n\big\}$, respectively. Plainly, if there exists $\gamma > 0$ such that $n^{1-\gamma/2}b_n^{-(2+\gamma)}\mathsf{E}|\xi_{1,1}^{(n)} - m_n|^{2+\gamma} \to 0$ and $n^{-\gamma/2}b_n^{-(2+\gamma)}\mathsf{E}|\varepsilon_1^{(n)} - \lambda_n|^{2+\gamma} \to 0$ as $n \to \infty$ then conditions (ii) and (iv) are satisfied.

In particular, if the variance of the immigration distribution is a power function we obtain the next limit theorem.

Theorem 2.12. *Suppose that there exists* $\varrho > -1$ *such that*

(i) $\sigma_n^2 = \sigma^2 n^{-1} + o(n^{-1})$ *as* $n \to \infty$ *with some* $\sigma \geq 0$,

(ii) $\lambda_n = \lambda n^\varrho + o(n^\varrho)$ *as* $n \to \infty$ *with some* $\lambda \geq 0$,

(iii) $b_n^2 = b^2 n^\varrho + o(n^\varrho)$ *as* $n \to \infty$ *with some* $b > 0$,

(iv) $\mathsf{E}\Big(|\varepsilon_1^{(n)} - \lambda_n|^2 \mathbb{1}_{\{|\varepsilon_1^{(n)} - \lambda_n| > \theta n^{(1+\varrho)/2}\}}\Big) = o(n^\varrho)$ *as* $n \to \infty$ *for all* $\theta > 0$.

Then

$$n^{-(1+\varrho)/2}\big(\mathcal{X}^{(n)} - \mathsf{E}\mathcal{X}^{(n)}\big) \xrightarrow{\mathcal{D}} \widetilde{\mathcal{X}} \quad \text{as } n \to \infty,$$

where $(\widetilde{\mathcal{X}}(t))_{t\in\mathbb{R}_+}$ *is an Ornstein–Uhlenbeck process defined by the SDE*

$$\mathrm{d}\widetilde{\mathcal{X}}(t) = \alpha\widetilde{\mathcal{X}}(t)\,\mathrm{d}t + \sqrt{\sigma^2\mu(t) + b^2}\,\mathrm{d}W(t), \quad \widetilde{\mathcal{X}}(0) = 0, \tag{9}$$

where $(W(t))_{t\in\mathbb{R}_+}$ *is a standard Wiener process.*

Remark 2.13. Note that in this case no Lindeberg condition is needed for the offspring distributions. Condition (iv) is the Lindeberg condition for the triangular system $\{\varepsilon_j^{(n)}/n^{(1+\varrho)/2} : n \in \mathbb{N},\ 1 \leq j \leq n\}$. If there exists $\gamma > 0$ such that $n^{-\varrho-(1+\varrho)\gamma/2}\mathsf{E}|\varepsilon_1^{(n)} - \lambda_n|^{2+\gamma} \to 0$ as $n \to \infty$ then condition (iv) is satisfied.

Example 2.14. If $\xi_{1,1}^{(n)}$ has a Bernoulli distribution with mean $1-\alpha n^{-1}$, where $\alpha \geq 0$, and $\mathsf{P}(\varepsilon_1^{(n)} = 0) = 1 - n^{-1}\ln^2 n$ and $\mathsf{P}(\varepsilon_1^{(n)} = \lfloor bn \ln^{-1} n \rfloor) = n^{-1}\ln^2 n$ for all $n \in \mathbb{N}$ with some $b > 0$, then conditions of Theorem 2.12 hold with $\varrho = 1$.

3. A general functional limit theorem

First we need the following definitions and notations. Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathsf{P})$ be a stochastic basis, i.e. $(\Omega, \mathcal{F}, \mathsf{P})$ is a probability space and $\mathbf{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is a filtration on it with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let $\mathcal{O}$ be the optional σ-field on $\Omega \times \mathbb{R}_+$ generated by all càdlàg adapted processes. Let $\mathcal{P} \subseteq \mathcal{O}$ be the predictable σ-field on $\Omega \times \mathbb{R}_+$ generated by the collection of sets $A \times (s, t]$, where $0 \leq s < t$ and $A \in \mathcal{F}_s$. Denote by $\mathbb{V}$ the set of all real-valued processes $(\mathcal{U}(t))_{t\in\mathbb{R}_+}$ on $(\Omega, \mathcal{F}, \mathbf{F}, \mathsf{P})$ that are càdlàg adapted, $\mathcal{U}(0) = 0$, and whose paths $t \mapsto \mathcal{U}(\omega, t)$ are of locally finite variation for all $\omega \in \Omega$. A random measure $\mu := \{\mu(\omega; \mathrm{d}t, \mathrm{d}x) : \omega \in \Omega\}$, where $\mu(\omega; \cdot)$ is a nonnegative Borel measure on $\mathbb{R}_+ \times \mathbb{R}$ satisfying $\mu(\omega; \{0\} \times \mathbb{R}) = 0$ for all $\omega \in \Omega$, is called optional (predictable) if the integral process $I_\mu^f(\omega; t) = \int_0^t \int_{\mathbb{R}} f(\omega; s, x)\mu(\omega; \mathrm{d}s, \mathrm{d}x)$ is $\mathcal{O}$-measurable ($\mathcal{P}$-measurable) for all integrable $\mathcal{O} \otimes \mathcal{B}(\mathbb{R})$-measurable ($\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable) function f. A random measure ν is called the compensator measure of μ if it is predictable and $I_\nu^f(\cdot; \infty) = I_\mu^f(\cdot; \infty)$ for every nonnegative $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable f. The compensator measure is unique up to a P-null set.

A function $h : \mathbb{R} \to \mathbb{R}$ is said to be a truncation function if it is bounded, continuous with compact support satisfying $h(x) = x$ in a neighbourhood of 0. Let $X = (X(t, \omega))_{t\in\mathbb{R}_+, \omega\in\Omega}$ be a semimartingale and define the process $X_h(t) := X(t) - \sum_{s\leq t}[\Delta X(s) - h(\Delta X(s))]$, $t \in \mathbb{R}_+$, where $\Delta X(t) := X(t) - X(t-)$ if $t > 0$ and $\Delta X(0) := 0$. Then X_h is a special semimartingale with canonical decomposition $X_h = X_0 + M_h + B_h$, where X_0 is $\mathcal{F}_0$-measurable, M_h is a local martingale with $M_h(0) = 0$ and $B_h \in \mathbb{V}$ is predictable, see Jacod and Shiryaev [10, Lemma I.4.24]. Let δ_a denote the Dirac measure at point $a \in \mathbb{R}_+ \times \mathbb{R}$.

Definition 3.1. A triplet (B, C, ν) is called *characteristics* of a semimartingale X with respect to h if (i) $B = B_h$ is a predictable process in $\mathbb{V}$, (ii) $C = \langle X^c, X^c\rangle$, where X^c is the continuous martingale part of X, is an increasing continuous process in $\mathbb{V}$, and (iii) ν is the compensator of the random measure $\mu(\omega;\cdot) := \sum_s \mathbb{1}_{\{\Delta X(s,\omega)\neq 0\}}\delta_{\{(s,\Delta X(s,\omega))\}}$ associated to the jumps of X. The process $\widetilde{C}(t) := C(t)+\int_0^t\int_{\mathbb{R}} h^2(x)\nu(\mathrm{d}s,\mathrm{d}x)-\sum_{s\le t}(B(s)-B(s-))^2$, $t\in\mathbb{R}_+$, is called the modified second characteristic of X.

We recall that each semimartingale defined on $(\Omega,\mathcal{F},\mathbf{F},\mathsf{P})$ has characteristics (B,C,ν) associated with a truncation function h. Moreover, C and ν are unique if h is fixed. (See Jacod and Shiryaev [10, Section II.2a].) We assume that each semimartingale $\mathcal{U}$ considered in this paper starts from zero, i.e. $\mathcal{U}(0)=0$. Denote by $\Pi(B,C,\nu)$ the martingale problem associated with characteristics (B,C,ν). A real-valued process $(\mathcal{U}(t))_{t\in\mathbb{R}_+}$ is a solution of this martingale problem if it is a semimartingale on the basis $(\Omega,\mathcal{F},\mathbf{F},\mathsf{P})$ with characteristics (B,C,ν) relative to the truncation function h.

The next theorem provides sufficient conditions in terms of characteristics for the weak convergence of semimartingales to a limiting semimartingale.

Theorem 3.2. *For each $n\in\mathbb{N}$, let $(\mathcal{U}^{(n)}(t))_{t\in\mathbb{R}_+}$ be a sequence of semimartingales with characteristics $(B^{(n)},C^{(n)},\nu^{(n)})$. Assume that the martingale problem $\Pi(0,C,0)$, where C is a càdlàg adapted and increasing process with $C(0)=0$, has a locally unique solution for all deterministic initial condition. Let $(\mathcal{U}(t))_{t\in\mathbb{R}_+}$ be a solution with $\mathcal{U}(0)=0$, and assume that there exists a function $\Phi:\mathbb{D}(\mathbb{R}_+,\mathbb{R})\to\mathbb{D}(\mathbb{R}_+,\mathbb{R})$ such that $C=\Phi(\mathcal{U})$. Suppose that for each $T>0$,*

(i) *there is an increasing continuous function $\varphi_{a,T}:\mathbb{R}_+\to\mathbb{R}$ for all $a>0$ such that $\Phi(x)(t)-\Phi(x)(s)\le\varphi_{a,T}(t)-\varphi_{a,T}(s)$ for all $0\le s\le t\le T$ if $\|x\|_\infty\le a$, $x\in\mathbb{D}(\mathbb{R}_+,\mathbb{R})$ (local strong majoration hypothesis),*

(ii) *the functions $x\mapsto\Phi(x)(t)$ are Skorokhod-continuous for all $0\le t\le T$ (local continuity condition),*

(iii) $\sup\limits_{t\in[0,T]}|B^{(n)}(t)|\stackrel{\mathsf{P}}{\longrightarrow}0$ *as $n\to\infty$,*

(iv) $\sup\limits_{t\in[0,T]}\left|\widetilde{C}^{(n)}(t)-\Phi(\mathcal{U}^{(n)})(t)\right|\stackrel{\mathsf{P}}{\longrightarrow}0$ *as $n\to\infty$,*

(v) $\nu^{(n)}([0,T]\times\{x:|x|>\theta\})\stackrel{\mathsf{P}}{\longrightarrow}0$ *as $n\to\infty$ for all $\theta>0$.*

Then
$$\mathcal{U}^{(n)}\stackrel{\mathcal{D}}{\longrightarrow}\mathcal{U}\quad\text{as } n\to\infty.$$

Proof. The proof is based on a general limit theorem of Jacod and Shiryaev [10, Theorem IX.3.39] and a standard localization procedure, see Ispány and Pap [9]. Note that the local condition on big jumps holds trivially since the third characteristic of the limiting semimartingale is zero. □

In the sequel, we suppose that $(\mathcal{U}(t))_{t\in\mathbb{R}_+}$ is a diffusion process with zero drift, i.e., it is a weak solution of a SDE

$$\mathrm{d}\mathcal{U}(t) = G(t,\mathcal{V}(t))dW(t), \quad t \in \mathbb{R}_+, \tag{10}$$

where $G : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a Borel function and $(W(t))_{t\in\mathbb{R}_+}$ is a standard Wiener process. If SDE (10) has a unique weak solution $(\mathcal{U}(t))_{t\in\mathbb{R}_+}$ with $\mathcal{U}(0) = 0$, then it is a semimartingale with characteristics

$$B(t) = 0, \quad C(t) = \int_0^t G^2(s,\mathcal{U}(s))\mathrm{d}s, \quad \nu([0,t]\times A) = 0,$$

where A is a Borel set and $t \in \mathbb{R}_+$. (See Jacod and Shiryaev [10, Section III.2c].) Thus $C = \Phi(\mathcal{U})$, where the function $\Phi : \mathbb{D}(\mathbb{R}_+,\mathbb{R}) \to \mathbb{D}(\mathbb{R}_+,\mathbb{R})$ is defined by $\Phi(x)(t) := \int_0^t G^2(s,x(s))\mathrm{d}s$, $x \in \mathbb{D}(\mathbb{R}_+,\mathbb{R})$. One can easily check that assumptions (i) and (ii) of Theorem 3.2 hold with $\varphi_{a,T} := \sup_{0\le t\le T}\sup_{\|x\|\le a} G^2(t,x)$. Clearly, there exists $A \ge 1$ such that $h(x) = x$ for $|x| \le 1/A$, $h(x) = 0$ for $|x| \ge A$, and $|h(x)| \le A$ for all $x \in \mathbb{R}$. Thus, $|h(x)| \le A^2|x|$ if $|x| > 1/A$. Hence we may throw off the truncation function, and for martingale differences Theorem 3.2 can be simplified in the following way, see [9, Corollary 2.2].

Corollary 3.3. *Let $G : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that the SDE (10) has a unique weak solution with $\mathcal{U}(0) = u_0$ for all $u_0 \in \mathbb{R}$. Let $(\mathcal{U}(t))_{t\in\mathbb{R}_+}$ be a solution with $\mathcal{U}(0) = 0$. For each $n \in \mathbb{N}$, let $(U_k^{(n)})_{k\in\mathbb{N}}$ be a sequence of martingale differences with respect to the natural filtration $(\mathcal{F}_k^{(n)})_{k\in\mathbb{Z}_+}$, i.e. $\mathcal{F}_k^{(n)} := \sigma\{U_1^{(n)},\dots,U_k^{(n)}\}$, $\mathsf{E}(U_k^{(n)} \mid \mathcal{F}_{k-1}^{(n)}) = 0$ and $\mathsf{E}(|U_k^{(n)}|^2 \mid \mathcal{F}_{k-1}^{(n)}) < \infty$ for all $k \in \mathbb{N}$. Let*

$$\mathcal{U}^{(n)}(t) := \sum_{k=1}^{\lfloor nt\rfloor} U_k^{(n)}, \quad t \in \mathbb{R}_+,\ n \in \mathbb{N}.$$

Suppose that, for each $T > 0$,

$$\text{(i)}\ \sup_{t\in[0,T]}\left|\sum_{k=1}^{\lfloor nt\rfloor}\mathsf{E}((U_k^{(n)})^2 \mid \mathcal{F}_{k-1}^{(n)}) - \int_0^t G^2(s,\mathcal{U}^{(n)}(s))\mathrm{d}s\right| \xrightarrow{\mathsf{P}} 0,$$

$$\text{(ii)}\ \sum_{k=1}^{\lfloor nT\rfloor}\mathsf{E}\left((U_k^{(n)})^2\mathbb{1}_{\{|U_k^{(n)}|>\theta\}} \mid \mathcal{F}_{k-1}^{(n)}\right) \xrightarrow{\mathsf{P}} 0 \text{ for all } \theta > 0$$

as $n \to \infty$. Then

$$\mathcal{U}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{U} \quad \text{as } n \to \infty.$$

Condition (ii) is the conditional Lindeberg condition for the triangular system $\{U_k^{(n)} : k,n \in \mathbb{N}\}$.

4. Proof of the main theorems

In the proofs we apply the following simple formulas and recursions for moments and covariances of a branching process with immigration.

Lemma 4.1. *Let* $(X_k)_{k\in\mathbb{Z}_+}$ *be a branching process with immigration defined by recursion* (1) *with moments* m, σ^2, λ *and* b^2. *Then, for all* $k\in\mathbb{N}$,

$$\mathsf{E}X_k=\lambda\sum_{\ell=0}^{k-1}m^\ell,\qquad \mathsf{Var}X_k=b^2\sum_{\ell=0}^{k-1}m^{2\ell}+\frac{\lambda\sigma^2}{m+1}\sum_{\ell=0}^{k-1}m^\ell\sum_{\ell=0}^{k-2}m^\ell.$$

Moreover, for all $k,\ell\in\mathbb{Z}_+$,

$$\mathsf{Cov}(X_k,\,X_\ell)=m^{|k-\ell|}\mathsf{Var}X_{k\wedge\ell}.$$

Furthermore, for all $k\in\mathbb{N}$,

$$\mathsf{E}\big(M_k^2\mid\mathcal{F}_{k-1}\big)=\sigma^2X_{k-1}+b^2.$$

Remark 4.2. The expectation $\mathsf{E}X_k$ and the variance $\mathsf{Var}X_k$ are monoton increasing in k.

Remark 4.3. In order to prove (3) we note that

$$\sup_{t\in[0,T]}\left|\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor}m_n^{\varkappa k}-\int_0^t \mathrm{e}^{\varkappa\alpha s}\,\mathrm{d}s\right|\to 0\quad\text{as } n\to\infty$$

for each fixed $\varkappa\in\mathbb{R}$. Thus, by Definition 1.1, condition (ii), and Lemma 4.1 we have (3). Moreover, the convergence in (3) is uniform on each finite interval $[0,T]$, $T>0$.

The technique of the proof of the main theorems is the so-called "martingale method" initiated by Stroock and Varadhan, see Jacod and Shiryaev [10] for details. Namely, let $\mathcal{F}_k^{(n)}$ denote the σ-algebra generated by the random variables $X_0^{(n)},X_1^{(n)},\ldots,X_k^{(n)}$ for $n\in\mathbb{N}$ and $k\in\mathbb{Z}_+$. Plainly,

$$M_k^{(n)}:=X_k^{(n)}-\mathsf{E}(X_k^{(n)}\mid\mathcal{F}_{k-1}^{(n)})=X_k^{(n)}-m_nX_{k-1}^{(n)}-\lambda_n,\quad k,\ n\in\mathbb{N},$$

defines a martingale difference sequence $(M_k^{(n)})_{k\in\mathbb{N}}$ with respect to the filtration $(\mathcal{F}_k^{(n)})_{k\in\mathbb{Z}_+}$ for all $n\in\mathbb{N}$. Note that $M_k^{(n)}$ can be decomposed into a random sum and a centered random variable, which are independent, as follows

$$M_k^{(n)}=\sum_{j=1}^{X_{k-1}^{(n)}}\xi_{k,j}^{(n)}+\varepsilon_k^{(n)}-m_nX_{k-1}^{(n)}-\lambda_n=N_k^{(n)}+\delta_k^{(n)},\tag{11}$$

where

$$N_k^{(n)} := \sum_{j=1}^{X_{k-1}^{(n)}} (\xi_{k,j}^{(n)} - m_n), \qquad \delta_k^{(n)} := \varepsilon_k^{(n)} - \lambda_n.$$

Then we define a suitable sequence of random step functions and we prove weak convergence of this sequence to a continuous process applying a general functional limit theorem of Section 3. Finally, by recursion

$$X_k^{(n)} - \mathsf{E} X_k^{(n)} = m_n\big(X_{k-1}^{(n)} - \mathsf{E} X_{k-1}^{(n)}\big) + M_k^{(n)} = \sum_{j=1}^{k} m_n^{k-j} M_j^{(n)} \tag{12}$$

we show that $\mathcal{X}^{(n)}$ is a function of the introduced random step function for all $n \in \mathbb{N}$, and a continuous mapping type argument yields the desired convergence.

Proof of Theorem 2.1. Introduce the random step functions

$$\mathcal{N}^{(n)}(t) := \sum_{k=1}^{\lfloor nt \rfloor} m_n^{-k} M_k^{(n)} \quad \text{for } t \in \mathbb{R}_+, \ n \in \mathbb{N},$$

and let $\widetilde{\mathcal{N}}^{(n)}(t) := (n\sigma_n^2)^{-1} \mathrm{e}^{\alpha t} \mathcal{N}^{(n)}(t) + \mu(t)$, $t \in \mathbb{R}_+$, $n \in \mathbb{N}$. Finally, let us introduce the stochastic process $\mathcal{N}(t) := \mathrm{e}^{-\alpha t}(\mathcal{X}(t) - \mu(t))$, $t \in \mathbb{R}_+$. By Itô's formula we have that $(\mathcal{N}(t))_{t\in\mathbb{R}_+}$ satisfies the SDE

$$\mathrm{d}\mathcal{N}(t) = \mathrm{e}^{-\alpha t} \sqrt{(\mathrm{e}^{\alpha t}\mathcal{N}(t) + \mu(t))_+}\, \mathrm{d}W(t), \quad t \in \mathbb{R}_+, \quad \mathcal{N}(0) = 0. \tag{13}$$

Note that, since $\mathrm{e}^{\alpha t}\mathcal{N}(t) + \mu(t) = \mathcal{X}(t) \geq 0$ almost surely for all $t \in \mathbb{R}_+$, we can replace the non-negative part by the original value under the square root. We will prove

$$(n\sigma_n^2)^{-1}\mathcal{N}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{N} \quad \text{as } n \to \infty \tag{14}$$

applying Corollary 3.3 for $\mathcal{U} = \mathcal{N}$ and $U_k^{(n)} = (n\sigma_n^2)^{-1} m_n^{-k} M_k^{(n)}$. If $(\mathcal{N}(t))_{t\in\mathbb{R}_+}$ satisfies the SDE (13) then $\mathcal{X}(t) := \mathrm{e}^{\alpha t}\mathcal{N}(t) + \mu(t)$, $t \in \mathbb{R}_+$, satisfies the SDE (4). Thus, the SDE (13) has a unique strong solution with $\mathcal{N}(0) = x$ for all $x \in \mathbb{R}$. The coefficient function $G(t, x) := \mathrm{e}^{-\alpha t}\sqrt{(\mathrm{e}^{\alpha t}x + \mu(t))_+}$, $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$, is continuous. It suffices to show that

$$\sup_{t\in[0,T]} \left| \frac{1}{\sigma_n^4 n^2} \sum_{k=1}^{\lfloor nt \rfloor} m_n^{-2k} \mathsf{E}\Big(\big(M_k^{(n)}\big)^2 \mid \mathcal{F}_{k-1}^{(n)}\Big) - \int_0^t \mathrm{e}^{-2\alpha s} \widetilde{\mathcal{N}}_+^{(n)}(s)\,\mathrm{d}s \right| \xrightarrow{\mathsf{P}} 0, \tag{15}$$

$$\frac{1}{\sigma_n^4 n^2} \sum_{k=1}^{\lfloor nT \rfloor} m_n^{-2k} \mathsf{E}\Big(\big(M_k^{(n)}\big)^2 \mathbb{1}_{\{|M_k^{(n)}| > \theta n \sigma_n^2 m_n^k\}} \mid \mathcal{F}_{k-1}^{(n)}\Big) \xrightarrow{\mathsf{P}} 0 \tag{16}$$

as $n \to \infty$ for all $T > 0$ and $\theta > 0$.

If $t \in [\ell/n, (\ell+1)/n)$, $\ell \in \mathbb{Z}_+$, then by (12) and Lemma 4.1 we have

$$\widetilde{\mathcal{N}}^{(n)}(t) = (n\sigma_n^2)^{-1} \mathrm{e}^{\alpha t} m_n^{-\ell} X_\ell^{(n)} - (n\sigma_n^2)^{-1} \lambda_n \mathrm{e}^{\alpha t} \sum_{k=1}^{\ell} m_n^{-k} + \mu(t).$$

Since $|(a+b)_+ - a| \le |b|$ for any $a \in \mathbb{R}_+$ and $b \in \mathbb{R}$, we have

$$\left|\widetilde{\mathcal{N}}_+^{(n)}(t) - (n\sigma_n^2)^{-1} \mathrm{e}^{\alpha t} m_n^{-\lfloor nt \rfloor} X_{\lfloor nt \rfloor}^{(n)}\right| \le \left|(n\sigma_n^2)^{-1} \lambda_n \mathrm{e}^{\alpha t} \sum_{k=1}^{\lfloor nt \rfloor} m_n^{-k} - \mu(t)\right|$$

for all $t \in \mathbb{R}_+$. On the other hand, by Lemma 4.1, we obtain

$$\frac{1}{\sigma_n^4 n^2} \sum_{k=1}^{\lfloor nt \rfloor} m_n^{-2k} \mathsf{E}\Big(\big(M_k^{(n)}\big)^2 \mid \mathcal{F}_{k-1}^{(n)}\Big) = \frac{1}{\sigma_n^2 n^2} \sum_{k=1}^{\lfloor nt \rfloor} m_n^{-2k} X_{k-1}^{(n)} + \frac{b_n^2}{\sigma_n^4 n^2} \sum_{k=1}^{\lfloor nt \rfloor} m_n^{-2k}.$$

Thus, in order to prove (15) it is enough to show that

$$D_n^{(1)} := \sup_{t \in [0,T]} \left| \frac{1}{\sigma_n^2 n^2} \sum_{k=1}^{\lfloor nt \rfloor} m_n^{-2k} X_{k-1}^{(n)} + \frac{b_n^2}{\sigma_n^4 n^2} \sum_{k=1}^{\lfloor nt \rfloor} m_n^{-2k} - \frac{1}{\sigma_n^2 n} \int_0^t \mathrm{e}^{-\alpha s} m_n^{-\lfloor ns \rfloor} X_{\lfloor ns \rfloor}^{(n)} \mathrm{d}s \right| \xrightarrow{\mathsf{P}} 0, \tag{17}$$

$$D_n^{(2)} := \int_0^T \mathrm{e}^{-2\alpha t} \left|(n\sigma_n^2)^{-1} \lambda_n \mathrm{e}^{\alpha t} \sum_{k=1}^{\lfloor nt \rfloor} m_n^{-k} - \mu(t)\right| \mathrm{d}t \to 0 \tag{18}$$

as $n \to \infty$. To prove (17) we note that

$$\int_0^t \mathrm{e}^{-\alpha s} m_n^{-\lfloor ns \rfloor} X_{\lfloor ns \rfloor}^{(n)} \mathrm{d}s = \sum_{k=1}^{\lfloor nt \rfloor} m_n^{-k+1} X_{k-1}^{(n)} \mathrm{e}^{-\alpha(k-1)/n} \int_0^{1/n} \mathrm{e}^{-\alpha s}\, \mathrm{d}s + m_n^{-\lfloor nt \rfloor} X_{\lfloor nt \rfloor}^{(n)} \int_{\lfloor nt \rfloor/n}^t \mathrm{e}^{-\alpha s}\, \mathrm{d}s.$$

Thus, we have

$$D_n^{(1)} \le \frac{1}{\sigma_n^2 n^2} \sum_{k=1}^{\lfloor nT \rfloor} d_k^{(n)} X_{k-1}^{(n)} + \frac{1}{\sigma_n^2 n^2} \mathrm{e}^{|\alpha| T} \max_{1 \le k \le \lfloor nT \rfloor} m_n^{-k} X_k^{(n)} + \frac{b_n^2}{\sigma_n^4 n^2} \sum_{k=1}^{\lfloor nT \rfloor} m_n^{-2k},$$

where the third term on the right hand side tends to 0 by condition (iii) and Remark 4.3, and

$$d_k^{(n)} := \left| m_n^{-2k} - m_n^{-k+1} \mathrm{e}^{-\alpha(k-1)/n} n \int_0^{1/n} \mathrm{e}^{-\alpha s} \, \mathrm{d}s \right|.$$

One can easily see that

$$\lim_{n\to\infty} \max_{1\leq k\leq \lfloor nT \rfloor} d_k^{(n)} = 0.$$

Thus, in order to prove (17) it is enough to see that $(\sigma_n^2 n^2)^{-1} \sum_{k=1}^{\lfloor nT \rfloor} X_{k-1}^{(n)}$ is stochastically bounded and $(\sigma_n^2 n^2)^{-1} \max_{1\leq k\leq \lfloor nT \rfloor} m_n^{-k} X_k^{(n)} \stackrel{\mathsf{P}}{\longrightarrow} 0$ as $n \to \infty$. The first statement follows by the Markov inequality since

$$\frac{1}{\sigma_n^2 n^2} \sum_{k=1}^{\lfloor nT \rfloor} \mathsf{E} X_{k-1}^{(n)} \leq \frac{T}{\sigma_n^2 n} \mathsf{E} X_{\lfloor nT \rfloor}^{(n)},$$

where the right hand side is bounded by (3). To prove the second statement we note that, by (12),

$$m_n^{-k} X_k^{(n)} = \sum_{j=1}^{k} m_n^{-j} M_j^{(n)} + m_n^{-k} \mathsf{E} X_k^{(n)}$$

for all $k, n \in \mathbb{N}$. By Definition 1.1 we obtain that $n \ln m_n \to \alpha$ as $n \to \infty$, hence $C := \sup_{n\in\mathbb{N}} |n \ln m_n - \alpha| < \infty$. Thus, we have

$$\max_{1\leq k\leq \lfloor nT \rfloor} m_n^{-k} X_k^{(n)} \leq \sum_{j=1}^{\lfloor nT \rfloor} m_n^{-j} |M_j^{(n)}| + \mathrm{e}^{(|\alpha|+C)T} \mathsf{E} X_{\lfloor nT \rfloor}^{(n)}.$$

By the Lyapunov and the Cauchy–Schwarz inequalities we obtain

$$\mathsf{E}\left(\frac{1}{\sigma_n^2 n^2} \sum_{j=1}^{\lfloor nT \rfloor} m_n^{-j} \big| M_j^{(n)} \big| \right) \leq \left(\frac{1}{n} \sum_{j=1}^{\lfloor nT \rfloor} m_n^{-2j} \right)^{1/2} \left(\frac{1}{\sigma_n^4 n^3} \sum_{j=1}^{\lfloor nT \rfloor} \mathsf{E}\big(M_j^{(n)}\big)^2 \right)^{1/2}.$$

The sequence $(n^{-1} \sum_{j=1}^{\lfloor nT \rfloor} m_n^{-2j})_{n\in\mathbb{N}}$ is bounded by Remark 4.3. Moreover, by Lemma 4.1 and Remark 4.2, we have

$$\frac{1}{\sigma_n^4 n^3} \sum_{j=1}^{\lfloor nT \rfloor} \mathsf{E}\big(M_j^{(n)}\big)^2 = \frac{1}{\sigma_n^4 n^3} \sum_{j=1}^{\lfloor nT \rfloor} (\sigma_n^2 \mathsf{E} X_{j-1}^{(n)} + b_n^2) \leq \frac{T \mathsf{E} X_{\lfloor nT \rfloor}^{(n)}}{\sigma_n^2 n^2} + \frac{T b_n^2}{\sigma_n^4 n^2},$$

where the second term on the right hand side tends to 0 by condition (iii). Thus, by Markov's inequality, in order to prove the second statement it is enough to see that $(\sigma_n^2 n^2)^{-1}\mathsf{E}X^{(n)}_{\lfloor nT\rfloor} \to 0$ as $n\to\infty$, which follows from (3).

To prove (18) consider the estimation

$$D_n^{(2)} \le \frac{\lambda_n}{\sigma_n^2}\sup_{t\in[0,T]}\left|\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} m_n^{-k} - \int_0^t \mathrm{e}^{-\alpha s}\,\mathrm{d}s\right|\int_0^T \mathrm{e}^{-\alpha t}\,\mathrm{d}t \\ + \left|\frac{\lambda_n}{\sigma_n^2}-\lambda\right|\int_0^T \mathrm{e}^{-\alpha t}\int_0^t \mathrm{e}^{-\alpha s}\,\mathrm{d}s\mathrm{d}t.$$

Hence, condition (ii) and Remark 4.3 imply (18).

To prove (16) we apply inequality $\mathbb{1}_{\{|Y+Z|>\theta\}} \le \mathbb{1}_{\{|Y|>\theta/2\}} + \mathbb{1}_{\{|Z|>\theta/2\}}$, where Y and Z are random variables, and decomposition (11). Thus, it suffices to show that

$$L_n^{i,j} := \frac{1}{\sigma_n^4 n^2}\sum_{k=1}^{\lfloor nT\rfloor} m_n^{-2k}\mathsf{E}\Big(\big(\zeta_{n,k}^{(i)}\big)^2 \mathbb{1}_{\{|\zeta_{n,k}^{(j)}|>\theta n\sigma_n^2 m_n^k\}} \mid \mathcal{F}_{k-1}^{(n)}\Big) \xrightarrow{\mathsf{P}} 0 \qquad (19)$$

as $n\to\infty$ for all $T,\theta>0$ and $i,j=1,2$, where $\zeta_{n,k}^{(1)} = N_k^{(n)}$ and $\zeta_{n,k}^{(2)} = \delta_k^{(n)}$. Introduce the random variable $S_k^{(n)} := \sum_{j=1}^{X_{k-1}^{(n)}}(\xi_{1,j}^{(n)} - m_n)$ for all $k,n\in\mathbb{N}$. In case of $i=j=1$ we have to prove that $A_n\to 0$ and $B_n\to 0$ as $n\to\infty$, where

$$A_n := \frac{1}{\sigma_n^4 n^2}\sum_{k=1}^{\lfloor nT\rfloor} m_n^{-2k}\mathsf{E}\left(\sum_{j=1}^{X_{k-1}^{(n)}} |\xi_{1,j}^{(n)} - m_n|^2 \mathbb{1}_{\{|S_k^{(n)}|>\theta n\sigma_n^2 m_n^k\}}\right),$$

$$B_n := \frac{2}{\sigma_n^4 n^2}\sum_{k=1}^{\lfloor nT\rfloor} m_n^{-2k}\mathsf{E}\left(\sum_{i=2}^{X_{k-1}^{(n)}}\sum_{j=1}^{i-1}(\xi_{1,i}^{(n)} - m_n)(\xi_{1,j}^{(n)} - m_n) \mathbb{1}_{\{|S_k^{(n)}|>\theta n\sigma_n^2 m_n^k\}}\right).$$

Applying again the above mentioned inequality for the indicator function of random variables $\xi_{1,j}^{(n)} - m_n$ and $S_{k,j}^{(n)} := \sum_{\ell\ne j}^{X_{k-1}^{(n)}}(\xi_{1,\ell}^{(n)} - m_n)$ we obtain

$$A_n \le \frac{1}{\sigma_n^4 n^2}\sum_{k=1}^{\lfloor nT\rfloor} m_n^{-2k}\mathsf{E}X_{k-1}^{(n)}\cdot\mathsf{E}\left(|\xi_{1,1}^{(n)} - m_n|^2 \mathbb{1}_{\{|\xi_{1,1}^{(n)}-m_n|>\theta n\sigma_n^2 m_n^k/2\}}\right) \\ + \frac{1}{\sigma_n^4 n^2}\sum_{k=1}^{\lfloor nT\rfloor} m_n^{-2k}\mathsf{E}\left(\sum_{j=1}^{X_{k-1}^{(n)}} |\xi_{1,j}^{(n)} - m_n|^2 \mathbb{1}_{\{|S_{k,j}^{(n)}|>\theta n\sigma_n^2 m_n^k/2\}}\right). \qquad (20)$$

Since $m_n^k \geq \exp\{-T(|\alpha|+C)\}$ for all $1 \leq k \leq \lfloor nT \rfloor$, $n \in \mathbb{N}$, and, by Remark 4.2,

$$\frac{1}{\sigma_n^2 n^2} \sum_{k=1}^{\lfloor nT \rfloor} m_n^{-2k} \mathsf{E} X_{k-1}^{(n)} \leq \frac{1}{\sigma_n^2 n^2} \mathsf{E} X_{\lfloor nT \rfloor}^{(n)} \sum_{k=1}^{\lfloor nT \rfloor} m_n^{-2k},$$

where the right hand side is bounded by Remark 4.3, the first term in (20) tends to zero by condition (i). The second term in (20), by the Markov inequality, can be majorized by

$$\frac{4}{\theta^2 \sigma_n^4 n^4} \sum_{k=1}^{\lfloor nT \rfloor} m_n^{-4k} \mathsf{E}\big(X_{k-1}^{(n)}\big)^2 \leq \frac{4}{\theta^2 \sigma_n^4 n^4} \Big((\mathsf{E} X_{\lfloor nT \rfloor}^{(n)})^2 + \mathsf{Var} X_{\lfloor nT \rfloor}^{(n)} \Big) \sum_{k=1}^{\lfloor nT \rfloor} m_n^{-4k}$$

which tends to 0 as $n \to \infty$. Let $V_k^{(n)} := \sum_{i=2}^{X_{k-1}^{(n)}} \sum_{j=1}^{i-1} (\xi_{1,i}^{(n)} - m_n)(\xi_{1,j}^{(n)} - m_n)$, $k, n \in \mathbb{N}$. Then $\mathsf{E}(V_k^{(n)})^2 = (\sigma_n^4/2) \mathsf{E}(X_{k-1}^{(n)}(X_{k-1}^{(n)} - 1))$ for all $k, n \in \mathbb{N}$. By the Cauchy–Schwarz inequality and the Markov inequality we have

$$\begin{aligned}
|B_n| &\leq \frac{2}{\sigma_n^4 n^2} \sum_{k=1}^{\lfloor nT \rfloor} m_n^{-2k} \Big(\mathsf{E}(V_k^{(n)})^2 \mathsf{P}(|S_k^{(n)}| > \theta n \sigma_n^2 m_n^k) \Big)^{1/2} \\
&\leq \frac{2^{1/2}}{\theta \sigma_n^3 n^3} \sum_{k=1}^{\lfloor nT \rfloor} m_n^{-3k} \Big(\mathsf{E}\big(X_k^{(n)}\big)^2 \mathsf{E} X_k^{(n)} \Big)^{1/2} \\
&\leq \frac{2^{1/2}}{\theta \sigma_n^3 n^3} \Big(\big(\mathsf{Var}(X_{\lfloor nT \rfloor}^{(n)}) + (\mathsf{E} X_{\lfloor nT \rfloor}^{(n)})^2\big) \mathsf{E} X_{\lfloor nT \rfloor}^{(n)} \Big)^{1/2} \sum_{k=1}^{\lfloor nT \rfloor} m_n^{-3k} \to 0
\end{aligned}$$

as $n \to \infty$ by Lemma 4.1, Remark 4.2 and 4.3. To prove (19) in case of $i = 1$ and $j = 2$ let us use the inequality

$$\mathsf{E}\Big(\big(N_k^{(n)}\big)^2 \mathbb{1}_{\{|\delta_k^{(n)}| > \theta n \sigma_n^2 m_n^k\}} \mid \mathcal{F}_{k-1}^{(n)} \Big) \leq \theta^{-2} n^{-2} m_n^{-2k} b_n^2 \sigma_n^{-2} X_{k-1}^{(n)}.$$

Thus, by Remark 4.2, we have

$$\mathsf{E}(L_n^{1,2}) \leq \frac{b_n^2}{\theta^2 \sigma_n^6 n^4} \sum_{k=1}^{\lfloor nT \rfloor} m_n^{-4k} \mathsf{E} X_{k-1}^{(n)} \leq \frac{b_n^2}{\theta^2 \sigma_n^6 n^4} \mathsf{E} X_{\lfloor nT \rfloor}^{(n)} \sum_{k=1}^{\lfloor nT \rfloor} m_n^{-4k} \to 0$$

as $n \to \infty$ by condition (iii) and Remark 4.3. To prove the cases $i = 2$ and $j = 1, 2$ we note that $L_n^{2,1}$ and $L_n^{2,2}$ can be majorized by $\sigma_n^{-4} n^{-2} b_n^2 \sum_{k=1}^{\lfloor nT \rfloor} m_n^{-2k}$, which tends to 0 by condition (iii) and Remark 4.3. Consequently, we proved (14).

By recursion (12) it is easy to see that

$$\mathcal{X}^{(n)}(t) - \mathsf{E}\mathcal{X}^{(n)}(t) = m_n^{\lfloor nt\rfloor}\mathcal{N}^{(n)}\left(\tfrac{\lfloor nt\rfloor}{n}\right).$$

Thus, $(n\sigma_n^2)^{-1}\mathcal{X}^{(n)}$ tends to the process $\mathcal{X}(t) := \mathrm{e}^{\alpha t}\mathcal{N}(t) + \mu(t)$, $t \in \mathbb{R}_+$, as $n \to \infty$ in the Skorokhod space. Since μ satisfies the ordinary differential equation $\mathrm{d}\mu(t) = (\lambda + \alpha\mu(t))\mathrm{d}t$ with initial condition $\mu(0) = 0$, Itô's formula yields

$$\mathrm{d}(\mathrm{e}^{\alpha t}\mathcal{N}(t) + \mu(t)) = (\lambda + \alpha(\mathrm{e}^{\alpha t}\mathcal{N}(t) + \mu(t)))\,\mathrm{d}t + \mathrm{e}^{\alpha t}\,\mathrm{d}\mathcal{N}(t),$$

which agrees with the SDE (4). Finally, $(n\sigma_n^2)^{-1}(\mathcal{X}^{(n)} - \mathsf{E}\mathcal{X}^{(n)})$ tends to the process $\widetilde{\mathcal{X}}(t) := \mathrm{e}^{\alpha t}\mathcal{N}(t)$, $t \in \mathbb{R}_+$, as $n \to \infty$ which satisfies SDE (5). □

Proof of Theorem 2.5. By Theorem 2.1 $(n\sigma_n^2)^{-1}\mathcal{X}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{X}'$ as $n \to \infty$, where $(\mathcal{X}'(t))_{t\in\mathbb{R}_+}$ is the unique solution of a SDE

$$\mathrm{d}\mathcal{X}'(t) = (\lambda' + \alpha\mathcal{X}'(t))\,\mathrm{d}t + \sqrt{\mathcal{X}'_+(t)}\,\mathrm{d}W(t), \quad t \in \mathbb{R}_+,$$

with initial condition $\mathcal{X}'(0) = 0$, where $\lambda' := \lim_{n\to\infty}\lambda_n/\sigma_n^2 = \lambda/\sigma^2$ by conditions (i) and (iii). Thus, $n^{-(1+\varrho)}\mathcal{X}^{(n)} \xrightarrow{\mathcal{D}} \sigma^2\mathcal{X}'$ as $n \to \infty$. Introduce the process $\mathcal{X} := \sigma^2\mathcal{X}'$. Then, by Itô's formula, we have that the process $(\mathcal{X}(t))_{t\in\mathbb{R}_+}$ satisfies SDE (6). □

Proof of Theorem 2.9. The proof is based on the martingale central limit theorem similarly to the proof of [8, Theorem 2.2.8]. Define the random step functions

$$\mathcal{M}^{(n)}(t) := \sum_{k=1}^{\lfloor nt\rfloor} M_k^{(n)} \quad \text{for } t \in \mathbb{R}_+,\ n \in \mathbb{N}. \tag{21}$$

We prove that $(nb_n^2)^{-1/2}\mathcal{M}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{M}$ as $n \to \infty$, where $(\mathcal{M}(t))_{t\in\mathbb{R}_+}$ is a Wiener process $\mathcal{M}(t) = W(T(t))$, $t \in \mathbb{R}_+$, with $T(t) := \sigma^2\int_0^t \mu(s)\mathrm{d}s + t$, $t \in \mathbb{R}_+$, $(W(t))_{t\in\mathbb{R}_+}$ is a standard Wiener process. By the martingale central limit theorem we have to prove that

$$\frac{1}{nb_n^2}\sum_{k=1}^{\lfloor nt\rfloor}\mathsf{E}\Big(\big(M_k^{(n)}\big)^2 \mid \mathcal{F}_{k-1}^{(n)}\Big) \xrightarrow{\mathsf{P}} T(t), \tag{22}$$

$$\frac{1}{nb_n^2}\sum_{k=1}^{\lfloor nt\rfloor}\mathsf{E}\Big(\big(M_k^{(n)}\big)^2\mathbb{1}_{\{|M_k^{(n)}|>\theta\sqrt{nb_n^2}\}} \mid \mathcal{F}_{k-1}^{(n)}\Big) \xrightarrow{\mathsf{P}} 0 \tag{23}$$

as $n \to \infty$ for all $t \in \mathbb{R}_+$ and $\theta > 0$.

By Lemma 4.1, in order to prove (22) we have to show that

$$\frac{\sigma_n^2}{nb_n^2} \sum_{k=1}^{\lfloor nt \rfloor} \mathsf{E} X_{k-1}^{(n)} \to \sigma^2 \int_0^t \mu(s) \mathrm{d}s \quad \text{as } n \to \infty, \tag{24}$$

$$\mathsf{Var}\left(\frac{\sigma_n^2}{nb_n^2} \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1}^{(n)} \right) \to 0 \qquad \text{as } n \to \infty. \tag{25}$$

Since $(nb_n^2)^{-1} \mathsf{E} X_{\lfloor nt \rfloor}^{(n)} \to \mu(t)$ as $n \to \infty$ uniformly on each finite interval $[0, T]$, $T > 0$, by condition (i) we have (24). To prove (25) we apply (12)

$$\sum_{k=1}^{\lfloor nt \rfloor} \left(X_{k-1}^{(n)} - \mathsf{E} X_{k-1}^{(n)} \right) = \sum_{j=1}^{\lfloor nt \rfloor - 1} \sum_{k=j}^{\lfloor nt \rfloor - 1} m_n^{k-j} M_j^{(n)}.$$

Thus, by Lemma 4.1 and Remark 4.2, we have

$$\mathsf{Var}\left(\frac{\sigma_n^2}{nb_n^2} \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1}^{(n)} \right) \le t \left(\frac{1}{n} \sum_{k=0}^{\lfloor nt \rfloor} m_n^k \right)^2 \left(\frac{(n\sigma_n^2)^3}{(nb_n^2)^2} \mathsf{E} X_{\lfloor nt \rfloor}^{(n)} + \frac{(n\sigma_n^2)^2}{nb_n^2} \right),$$

where the right hand side tends to 0 by Remark 4.3 and conditions of the theorem.

To prove the Lindeberg condition (23) it suffices to check that, for all $\theta > 0$ and $t \in \mathbb{R}_+$,

$$L_n^{i,j} := \frac{1}{nb_n^2} \sum_{k=1}^{\lfloor nt \rfloor} \mathsf{E}\Big(\big(\zeta_{n,k}^{(i)}\big)^2 \mathbb{1}_{\{|\zeta_{n,k}^{(j)}| > \theta \sqrt{nb_n^2}\}} \mid \mathcal{F}_{k-1}^{(n)} \Big) \xrightarrow{\mathsf{P}} 0 \quad \text{as } n \to \infty \tag{26}$$

for $i, j = 1, 2$, where $\zeta_{n,k}^{(1)} = N_k^{(n)}$ and $\zeta_{n,k}^{(2)} = \delta_k^{(n)}$. The proof of (26) in cases of $i = j = 1$ and $i \neq j$ is similar to one of (19) in these cases. Finally, in case of $i = j = 2$ (26) agrees exactly with Lindeberg condition (iv). □

Proof of Theorem 2.12. By Theorem 2.9 $(nb_n^2)^{-1/2}(\mathcal{X}^{(n)} - \mathsf{E}\mathcal{X}^{(n)}) \xrightarrow{\mathcal{D}} \widetilde{\mathcal{X}}'$ as $n \to \infty$, where $(\widetilde{\mathcal{X}}'(t))_{t \in \mathbb{R}_+}$ is an Ornstein–Uhlenbeck type process defined by the SDE

$$\mathrm{d}\widetilde{\mathcal{X}}'(t) = \alpha \widetilde{\mathcal{X}}'(t)\, \mathrm{d}t + \sqrt{\sigma^2 \mu'(t) + 1}\, \mathrm{d}W(t), \qquad \widetilde{\mathcal{X}}'(0) = 0,$$

with $\mu'(t) := (\lambda / b^2) \int_0^t \mathrm{e}^{\alpha s}\, \mathrm{d}s$, $t \in \mathbb{R}_+$. Thus, $n^{-(1+\varrho)/2}(\mathcal{X}^{(n)} - \mathsf{E}\mathcal{X}^{(n)}) \xrightarrow{\mathcal{D}} b\widetilde{\mathcal{X}}'$ as $n \to \infty$. The process $\widetilde{\mathcal{X}} := b\widetilde{\mathcal{X}}'$ clearly satisfies SDE (9). □

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