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A Radon–Nikodym theorem for completely n-positive linear maps on pro-C^*-algebras and its applications

By MARIA JOIŢA (Bucharest)

Abstract. The order relation on the set of completely n-positive linear maps from a pro-C^*-algebra A to $L(H)$, the C^*-algebra of bounded linear operators on a Hilbert space H, is characterized in terms of the representation associated with each completely n-positive linear map. Also, the pure elements in the set of all completely n-positive linear maps from A to $L(H)$ and the extreme points in the set of completely n-positive linear maps from a unital C^*-algebra A to $L(H)$ are characterized in terms of the representation induced by each completely n-positive linear map.

1. Introduction and preliminaries

Stinespring [12] showed that any completely positive linear map from a C^*-algebra A to $L(H)$, the C^*-algebra of bounded linear operators on a Hilbert space H, induces a representation of A on another Hilbert space that generalizes the GNS construction for positive linear functionals on C^*-algebras. In [1], Arveson proved a Radon–Nikodym type theorem which gives a description of the order relation on the set $CP_\infty(A, L(H))$ of all completely positive linear maps from a C^*-algebra A to $L(H)$ in terms of the Stinespring representation associated with each completely positive linear map, and using this theorem, he established characterizations of pure elements in the set of completely positive linear maps from A to $L(H)$ and extreme elements in the set of completely n-positive linear

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maps $[p_{ij}]_{i,j=1}^n$ from a unital C^*-algebra A to $L(H)$ such that $p_{ij}(1) = I_H$ for all $i \in \{1, 2, \ldots, n\}$ and $p_{ij}(1) = 0$ for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ in terms of the Stinespring representation associated with each completely positive linear map.

Kaplan [10] introduced the notion of multi-positive linear functional on a C^*-algebra A (that is, an $n \times n$ matrix of linear functionals on A which verifies the positivity condition) and showed that any multi-positive linear functional on A induces a representation of A on a Hilbert space. Also, he proved a Radon–Nikodym type theorem for multi-positive linear functionals on a C^*-algebra A. In [13], [14] Suen considered the $n \times n$ matrices of continuous linear maps from a C^*-algebra A to $L(H)$ and showed that a unital completely n-positive linear map from a unital C^*-algebra A to $L(H)$ induces a representation of A on a Hilbert space in terms of the Stinespring construction.

A pro-C^*-algebra A is a complete Hausdorff topological $*$-algebra over $\mathbb{C}$ whose topology is determined by its continuous C^*-seminorms in the sense that the net $\{a_i\}_{i \in I}$ converges to 0 in A if and only if the net $\{p(a_i)\}_{i \in I}$ converges to 0 for any continuous C^*-seminorm p on A; equivalently, A is homeomorphically $*$-isomorphic to an inverse limit of C^*-algebras. So, pro-C^*-algebras are generalizations of C^*-algebras. Instead of being given by a single norm, the topology on a pro-C^*-algebra is defined by a directed family of C^*-seminorms. Besides an intrinsic interest in pro-C^*-algebras as topological algebras comes from the fact that they provide an important tool in investigation of certain aspects of C^*-algebras (like multipliers of the Pedersen ideal [2], [11]; tangent algebra of a C^*-algebra [11]; quantum field theory [3]). In the literature, pro-C^*-algebras have been given different name such as b^*-algebras (C. Apostol), LMC^*-algebras (G. Lassner, K. Schmüdgen) or locally C^*-algebras [4], [6]–[9].

A representation of a pro-C^*-algebra A on a Hilbert space H is a continuous $*$-morphism from A to $L(H)$. In [4] it is shown that a continuous positive linear functional on a pro-C^*-algebra A induces a representation of A on a Hilbert space in terms of the GNS construction. Bhatt and Karia [2] extended the Stinespring construction for completely positive linear maps from a pro-C^*-algebra A to $L(H)$. In [9], we extend the KSGNS (Kasparov, Stinespring, Gel'fend, Naimark, Segal) construction for strict completely n-positive linear maps from a pro-C^*-algebra A to another pro-C^*-algebra B.

In this paper we generalize various earlier results by Arveson and Kaplan. The paper is organized as follows. In Section 2, we establish a relationship between the comparability of nondegenerate representations of A and the matricial order structure of $\mathcal{B}(A, L(H))$, the vector space of continuous linear maps from A to

$L(H)$ (Proposition 2.6). This is a generalization of Proposition 2.2, [10]. Section 3 is devoted to a Radon–Nikodym type theorem for completely multi-positive linear maps from A to $L(H)$. As a consequence of this theorem, we obtain a criterion of pureness for elements in $CP_{\infty}^{n}(A, L(H))$ in terms of the representation associated with each completely n-positive linear map (Corollary 3.6). In Section 4 we prove a sufficient criterion for a completely n-positive linear map from A to $L(H)$ to be pure in terms of its components (Lemma 4.1) and using this result we determine a certain class of extreme points in the set of all unital completely positive linear maps from A to $L(H^n)$, where H^n denotes the direct sum of n copies of the Hilbert space H (Corollary 4.3). Finally, we give a characterization of the extreme points in the set of completely n-positive linear maps $[\rho_{ij}]_{i,j=1}^{n}$ from a unital C^*-algebra A to $L(H)$ such that $\rho_{ii}(1) = I_H$ for all $i \in \{1, 2, \ldots, n\}$ and $\rho_{ij}(1) = 0$ for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ (Theorem 4.4) that extends the Arveson characterization of the extreme points in the set of all unital completely positive linear maps from a C^*-algebra A to $L(H)$.

2. Representations associated with completely n-positive linear maps

Let A be a pro-C^*-algebra and let H be a Hilbert space. An $n \times n$ matrix $[\rho_{ij}]_{i,j=1}^{n}$ of continuous linear maps from A to $L(H)$ can be regarded as a linear map ρ from $M_n(A)$ to $M_n(L(H))$ defined by

$$\rho\left([a_{ij}]_{i,j=1}^{n}\right) = [\rho_{ij}(a_{ij})]_{i,j=1}^{n}.$$

It is not difficult to check that ρ is continuous. We say that $[\rho_{ij}]_{i,j=1}^{n}$ is an *n-positive* (respectively *completely n-positive*) linear map from A to $L(H)$ if the linear map ρ from $M_n(A)$ to $M_n(L(H))$ is positive (respectively completely positive). The set of all completely positive linear maps from A to $L(H)$ is denoted by $CP_{\infty}(A, L(H))$ and the set of all completely n-positive linear maps from A to $L(H)$ is denoted by $CP_{\infty}^{n}(A, L(H))$. If ρ and θ are two elements in $CP_{\infty}^{n}(A, L(H))$, we say that $\theta \leq \rho$ if $\rho - \theta$ is an element in $CP_{\infty}^{n}(A, L(H))$.

Remark 2.1. In the same manner as in the prof of Theorem 1.4 in [5], we can show that the map $\mathcal{S}$ from $CP_{\infty}^{n}(A, L(H))$ to $CP_{\infty}(A, M_n(L(H)))$ defined by

$$\mathcal{S}\left([\rho_{ij}]_{i,j=1}^{n}\right)(a) = [\rho_{ij}(a)]_{i,j=1}^{n}$$

is an affine order isomorphism.

The following theorem is a particular case of Theorem 3.4 in [9].

Theorem 2.2. *Let A be a pro-C^*-algebra, let H be a Hilbert space and let $\rho = [\rho_{ij}]_{i,j=1}^n$ be a completely n-positive linear map from A to $L(H)$. Then there is a representation Φ_ρ of A on a Hilbert space H_ρ and there are n elements $V_{\rho,1}, \dots, V_{\rho,n}$ in $L(H, H_\rho)$ such that*

(a) $\rho_{ij}(a) = V_{\rho,i}^* \Phi_\rho(a) V_{\rho,j}$ *for all* $a \in A$ *and for all* $i, j \in \{1, \dots, n\}$;

(b) $\{\Phi_\rho(a)V_{\rho,i}\xi;\ a \in A,\ \xi \in H,\ 1 \leq i \leq n\}$ *spans a dense subspace of* H_ρ.

The $n+2$ tuple $(\Phi_\rho, H_\rho, V_{\rho,1}, \dots, V_{\rho,n})$ constructed in Theorem 2.2 is said to be the Stinespring representation of A associated with the completely n-positive linear map ρ.

Remark 2.3. The representation associated with a completely n-positive linear map is unique up to unitary equivalence [9, Theorem 3.4].

Remark 2.4. Let $(\Phi_\rho, H_\rho, V_{\rho,1}, \dots, V_{\rho,n})$ be the Stinespring representation associated with a completely n-positive linear map $\rho = [\rho_{ij}]_{i,j=1}^n$. For each $i \in \{1, \dots, n\}$, we denote by H_i the Hilbert subspace of H_ρ generated by $\{\Phi_\rho(a)V_{\rho,i}\xi;\ a \in A,\ \xi \in H\}$ and by P_i the projection in $L(H_\rho)$ whose range is H_i. Then $P_i \in \Phi_\rho(A)'$, the commutant of $\Phi_\rho(A)$ in $L(H_\rho)$, and $a \mapsto \Phi_\rho(a)P_i$ is a representation of A on H_i which is unitarily equivalent with the Stinespring representation associated with ρ_{ii} for all $i \in \{1, \dots, n\}$. Since Φ_ρ is a nondegenerate representation of A,

$$\lim_\lambda \Phi_\rho(e_\lambda)\xi = \xi$$

for some approximate unit $\{e_\lambda\}_{\lambda\in\Lambda}$ for A and for all $\xi \in H$ [7, Proposition 4.2], and then

$$P_i V_{\rho,i}\xi = \lim_\lambda P_i \Phi_\rho(e_\lambda) V_{\rho,i}\xi = V_{\rho,i}\xi$$

for all $\xi \in H$ and for all $i \in \{1, \dots, n\}$. Therefore $P_i V_{\rho,i} = V_{\rho,i}$ for all $i \in \{1, \dots, n\}$.

Remark 2.5. If $\rho = [\rho_{ij}]_{i,j=1}^n$ is a diagonal completely n-positive linear map from A to $L(H)$ (that is, $\rho_{ij} = 0$ if $i \neq j$), then the Stinespring representation associated with ρ is unitarily equivalent with the direct sum of the Stinespring representations associated with ρ_{ii}, $i \in \{1, \dots, n\}$.

Two completely n-positive linear maps ρ and θ from A to $L(H)$ are called *disjoint* (respectively *unitarily equivalent*) if the representations of A induced by ρ respectively θ are disjoint (respectively unitarily equivalent).

The following proposition is an analogue of Proposition 2.2 in [10] for completely n-positive linear maps.

Proposition 2.6. *Let ρ_{11} and ρ_{22} be two completely positive linear maps from A to $L(H)$. Then ρ_{11} and ρ_{22} are disjoint if and only if there are no nonzero continuous linear maps ρ_{12} and ρ_{21} such that $\rho = [\rho_{ij}]_{i,j=1}^{2}$ is a completely 2-positive linear map from A to $L(H)$.*

Proof. Suppose that ρ_{11} and ρ_{22} are disjoint and $\rho = [\rho_{ij}]_{i,j=1}^{2}$ is a completely 2-positive linear map from A to $L(H)$. Let $(\Phi_\rho, H_\rho, V_{\rho,1}, V_{\rho,2})$ be the Stinespring representation associated with ρ. Since ρ_{11} and ρ_{22} are disjoint, the central carriers of projections P_1 and P_2 (see Remark 2.4) are orthogonal and so $P_1P_2 = 0$. Then

$$\rho_{12}(a) = V_{\rho,1}^{*}\Phi_\rho(a)V_{\rho,2} = V_{\rho,1}^{*}P_1\Phi_\rho(a)P_2V_{\rho,2} = V_{\rho,1}^{*}\Phi_\rho(a)P_1P_2V_{\rho,2} = 0$$

for all $a \in A$, and since $\rho_{21}(a) = (\rho_{12}(a^*))^*$ for all $a \in A$, we conclude that $\rho_{12} = \rho_{21} = 0$.

Conversely, suppose that there are no nonzero continuous linear maps ρ_{12} and ρ_{21} such that $\rho = [\rho_{ij}]_{i,j=1}^{2}$ is a completely 2-positive linear map from A to $L(H)$. Let $(\Phi_{\rho_{ii}}, H_{\rho_{ii}}, V_{\rho_{ii}})$ be the Stinespring representation associated with ρ_{ii}, $i \in \{1,2\}$. The direct sum $\Phi = \Phi_{\rho_{11}} \oplus \Phi_{\rho_{22}}$ of the representations $\Phi_{\rho_{11}}$ and $\Phi_{\rho_{22}}$ is a representation of A on the Hilbert space $H_0 = H_{\rho_{11}} \oplus H_{\rho_{22}}$. We consider the linear operators $V_1, V_2 \in L(H, H_0)$ defined by $V_1(\xi) = V_{\rho_{11}}\xi \oplus 0$ respectively $V_2(\xi) = 0 \oplus V_{\rho_{22}}\xi$. For each $i \in \{1,2\}$, the orthogonal projection of H_0 on $H_{\rho_{ii}}$ is denoted by E_i. It is not difficult to check that the range of E_i is generated by $\{\Phi(a)E_iV_i\xi;\ a \in A,\ \xi \in H\}$, $i \in \{1,2\}$. Moreover, $\rho_{ii}(a) = V_i^*\Phi(a)V_i$, $i \in \{1,2\}$.

Suppose that ρ_{11} and ρ_{22} are not disjoint. Then there are two nonzero projections F_1 and F_2 in $\Phi(A)'$ majorized by E_1 respectively E_2, and a partial isometry V in $\Phi(A)'$ such that $V^*V = F_1$ and $VV^* = F_2$. It is not difficult to check that the range of F_i is generated by $\{\Phi(a)F_iV_i\xi;\ a \in A,\ \xi \in H\}$, $i \in \{1,2\}$. We consider the linear map θ from A to $M_2(L(H))$ defined by

$$\theta(a) = \begin{bmatrix} V_1^*F_1\Phi(a)V_1 & V_1^*\Phi(a)V^*V_2 \\ V_2^*V\Phi(a)V_1 & V_2^*F_2\Phi(a)V_2 \end{bmatrix}.$$

It is not difficult to check that θ is continuous and

$$\theta\left(a^*b\right) = (M(a)W)^*M(b)W,$$

where

$$M(a) = \begin{bmatrix} \Phi(a) & \Phi(a) \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} F_1V_1 & 0 \\ 0 & V^*V_2 \end{bmatrix},$$

for all a and b in A. Then

$$\sum_{k.l=1}^{m} T_k^* \theta\left(a_k^* a_l\right) T_l = \left(\sum_{k=1}^{m} M(a_k) W T_k\right)^* \left(\sum_{l=1}^{m} M(a_l) W T_l\right) \geq 0$$

for all $T_1, \dots, T_m \in M_2(L(H))$ and for all $a_1, \dots, a_m \in A$. This implies that $\theta \in CP_\infty(A, M_2(L(H)))$ and so $\mathcal{S}^{-1}(\theta) \in CP_\infty^2(A, L(H))$ (see Remark 2.1). Let $\rho = [\rho_{ij}]_{i,j=1}^2$, where $\rho_{12} = \left(\mathcal{S}^{-1}(\theta)\right)_{12}$ and $\rho_{21} = \left(\mathcal{S}^{-1}(\theta)\right)_{21}$. Then

$$(\mathcal{S}(\rho) - \theta)(a) = \begin{bmatrix} V_1^*\left(E_1 - F_1\right)\Phi(a)V_1 & 0 \\ 0 & V_2^*\left(E_2 - F_2\right)\Phi(a)V_2 \end{bmatrix}$$

for all $a \in A$. Therefore $\mathcal{S}(\rho) - \theta \in CP_\infty(A, M_2(L(H)))$. From this fact and taking into account that $\theta \in CP_\infty(A, M_2(L(H)))$, we conclude that $\mathcal{S}(\rho) \in CP_\infty(A, M_2(L(H)))$ and then by Remark 2.1, $\rho \in CP_\infty^2(A, L(H))$. If $\rho_{12} = 0$, then $V_1^*\Phi(a)V^*V_2 = 0$ for all $a \in A$. This implies that

$$\langle V^*\Phi(a)V_2\xi, \Phi(a)F_1V_1\eta\rangle = \langle V_1^*\Phi\left(a^*a\right)V^*V_2\xi, \eta\rangle = 0$$

for all $a \in A$ and for all $\xi, \eta \in H$. Since the range of F_1 is generated by $\{\Phi(a)F_1V_1\xi;\ a \in A,\ \xi \in H\}$ and for all $\xi \in H$ and $a \in A$, $V^*\Phi(a)V_2\xi$ is an element in the range of F_1, the preceding relation yields that $V^*\Phi(a)V_2\xi = 0$ for all $a \in A$ and for all $\xi \in H$. Then $\Phi(a)F_2V_2\xi = VV^*\Phi(a)V_2\xi = 0$ for all $a \in A$ and for all $\xi \in H$ and so $F_2 = 0$. This is a contradiction, since F_2 is a nonzero projection in $\Phi(A)'$. Therefore $\rho_{12} \neq 0$. Thus we have found a completely 2-positive linear map $\rho = [\rho_{ij}]_{ij=1}^2$ such that $\rho_{12} \neq 0$, a contradiction. Therefore ρ_{11} and ρ_{22} are disjoint and the proposition is proved. □

3. The Radon–Nikodym theorem for completely n-positive linear maps

Let A be a pro-C^*-algebra and let H be a Hilbert space.

Lemma 3.1. *Let $\rho = [\rho_{ij}]_{ij=1}^n$ be an element in $CP_\infty^n(A, L(H))$. If T is a positive element in $\Phi_\rho(A)'$, the commutant of $\Phi_\rho(A)$ in $L\left(H_\rho\right)$, then the map ρ_T from $M_n(A)$ to $M_n\left(L(H)\right)$ defined by*

$$\rho_T\left([a_{ij}]_{i,j=1}^n\right) = \left[V_{\rho,i}^* T\Phi_\rho(a_{ij})V_{\rho,j}\right]_{i,j=1}^n$$

is a completely n-positive linear map from A to $L(H)$.

PROOF. It is not difficult to check that ρ_T is a matrix of continuous linear maps from A to $L(H)$, the (i,j)-entry of the matrix ρ_T being the continuous linear map $(\rho_T)_{ij}$ from A to $L(H)$ defined by

$$(\rho_T)_{ij}(a) = V_{\rho,i}^* T\Phi_\rho(a)V_{\rho,j}.$$

Also it is not difficult to check that

$$\mathcal{S}(\rho_T)(a^*b) = \left(M_{T^{\frac{1}{2}}}(a)V\right)^* \left(M_{T^{\frac{1}{2}}}(b)V\right),$$

for all $a, b \in A$, where $T^{\frac{1}{2}}$ is the square root of T,

$$M_{T^{\frac{1}{2}}}(a) = \begin{bmatrix} T^{\frac{1}{2}}\Phi_\rho(a) & \dots & T^{\frac{1}{2}}\Phi_\rho(a) \\ 0 & \dots & 0 \\ . & \dots & . \\ 0 & \dots & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} V_{\rho,1} & \dots & 0 \\ . & \dots & . \\ 0 & \dots & V_{\rho,n} \end{bmatrix}.$$

Then

$$\begin{aligned}\sum_{k,l=1}^m T_l^* \mathcal{S}(\rho_T)(a_l^* a_k) T_k &= \sum_{k,l=1}^m T_l^* \left(M_{T^{\frac{1}{2}}}(a_l)V\right)^* \left(M_{T^{\frac{1}{2}}}(a_k)V\right) T_k \\ &= \left(\sum_{l=1}^m M_{T^{\frac{1}{2}}}(a_l)VT_l\right)^* \left(\sum_{k=1}^m M_{T^{\frac{1}{2}}}(a_k)VT_k\right) \geq 0\end{aligned}$$

for all $T_1, \dots, T_m \in M_n(L(H))$ and for all $a_1, \dots, a_m \in A$. This shows that $\mathcal{S}(\rho_T) \in CP_\infty(A, M_n(L(H)))$ and by Remark 2.1, $\rho_T \in CP_\infty^n(A, L(H))$. □

Remark 3.2. Let $\rho = [\rho_{ij}]_{ij=1}^n \in CP_\infty^n(A, L(H))$.

1. If I_{H_ρ} is the identity map on H_ρ, then $\rho_{I_{H_\rho}} = \rho$.
2. If T_1 and T_2 are two positive elements in $\Phi_\rho(A)'$, then $\rho_{T_1+T_2} = \rho_{T_1} + \rho_{T_2}$.
3. If T is a positive element in $\Phi_\rho(A)'$ and α is a positive number, then $\rho_{\alpha T} = \alpha\rho_T$.

Lemma 3.3. *Let $\rho = [\rho_{ij}]_{ij=1}^n$ be an element in $CP_\infty^n(A, L(H))$ and let T_1 and T_2 be two positive elements in $\Phi_\rho(A)'$. Then $T_1 \leq T_2$ if and only if $\rho_{T_1} \leq \rho_{T_2}$.*

PROOF. First, we suppose that $T_1 \leq T_2$. Then $\rho_{T_2-T_1} \in CP_\infty^n(A, L(H))$, and since

$$\begin{aligned}(\rho_{T_2} - \rho_{T_1})\left([a_{ij}]_{i,j=1}^n\right) &= \left[V_{\rho,i}^*(T_2 - T_1)\Phi_\rho(a_{ij})V_{\rho,j}\right]_{i,j=1}^n \\ &= \rho_{T_2-T_1}\left([a_{ij}]_{i,j=1}^n\right)\end{aligned}$$

for all $[a_{ij}]_{i,j=1}^n \in M_n(A)$, we conclude that $\rho_{T_1} \leq \rho_{T_2}$.

Conversely, suppose that $\rho_{T_1} \leq \rho_{T_2}$. Let $\sum_{k=1}^{m} \sum_{i=1}^{n} \Phi_\rho(a_{ki})V_{\rho,i}\xi_{ki} \in H_\rho$. Then

$$\left\langle (T_2 - T_1)\left(\sum_{k=1}^{m}\sum_{i=1}^{n}\Phi_\rho(a_{ki})V_{\rho,i}\xi_{ki}\right), \sum_{k=1}^{m}\sum_{i=1}^{n}\Phi_\rho(a_{ki})V_{\rho,i}\xi_{ki}\right\rangle$$
$$= \sum_{k,l=1}^{m}\sum_{i,j=1}^{n}\left\langle V_{\rho,i}^{*}(T_2-T_1)\Phi_\rho\left(a_{ki}^{*}a_{lj}\right)V_{\rho,j}\xi_{lj}, \xi_{ki}\right\rangle$$
$$= \sum_{k,l=1}^{m}\sum_{i,j=1}^{n}\left\langle \left(\mathcal{S}(\rho_{T_2}) - \mathcal{S}(\rho_{T_1})\right)\left(a_{ki}^{*}a_{lj}\right)\left(\widetilde{\xi}_{ljp}\right)_{p=1}^{n}, \left(\widetilde{\xi}_{kip}\right)_{p=1}^{n}\right\rangle \geq 0,$$

where $\widetilde{\xi}_{kip} = \begin{cases} \xi_{ki} & \text{if } p = i \text{ and } 1 \leq k \leq m \\ 0 & \text{if } p \neq i \text{ and } 1 \leq k \leq m \end{cases}$. From this fact and taking into account that $\{\Phi_\rho(a)V_{\rho,i}\xi;\ a \in A,\ \xi \in H,\ 1 \leq i \leq n\}$ spans a dense subspace of H_ρ, we conclude that $T_2 - T_1 \geq 0$ and the lemma is proved. □

Lemma 3.4. *Let $\rho = [\rho_{ij}]_{ij=1}^{n}$ and $\theta = [\theta_{ij}]_{ij=1}^{n}$ be two elements in $CP_\infty^n(A, L(H))$. If $\theta \leq \rho$, then there is an element $W \in L(H_\rho, H_\theta)$ such that*

(a) $\|W\| \leq 1$;

(b) $WV_{\rho,i} = V_{\theta,i}$ *for all* $i \in \{1, \ldots, n\}$;

(c) $W\Phi_\rho(a) = \Phi_\theta(a)W$ *for all* $a \in A$.

Proof. Since

$$\left\langle \sum_{k=1}^{m}\sum_{i=1}^{n}\Phi_\theta(a_{ki})V_{\theta,i}\xi_{ki}, \sum_{k=1}^{m}\sum_{i=1}^{n}\Phi_\theta(a_{ki})V_{\theta,i}\xi_{ki}\right\rangle$$
$$= \sum_{k,l=1}^{m}\sum_{i,j=1}^{n}\left\langle V_{\theta,j}^{*}\Phi_\theta\left(a_{lj}^{*}a_{ki}\right)V_{\theta,i}\xi_{ki}, \xi_{lj}\right\rangle = \sum_{k,l=1}^{m}\sum_{i,j=1}^{n}\left\langle \theta_{ji}\left(a_{lj}^{*}a_{ki}\right)\xi_{ki}, \xi_{lj}\right\rangle$$
$$= \sum_{k,l=1}^{m}\sum_{i,j=1}^{n}\left\langle \mathcal{S}(\theta)\left(a_{lj}^{*}a_{ki}\right)\left(\widetilde{\xi}_{kip}\right)_{p=1}^{n}, \left(\widetilde{\xi}_{ljp}\right)_{p=1}^{n}\right\rangle$$
$$\leq \sum_{k,l=1}^{m}\sum_{i,j=1}^{n}\left\langle \mathcal{S}(\rho)\left(a_{lj}^{*}a_{ki}\right)\left(\widetilde{\xi}_{kip}\right)_{p=1}^{n}, \left(\widetilde{\xi}_{ljp}\right)_{p=1}^{n}\right\rangle$$
$$= \left\langle \sum_{k=1}^{m}\sum_{i=1}^{n}\Phi_\rho(a_{ki})V_{\rho,i}\xi_{ki}, \sum_{k=1}^{m}\sum_{i=1}^{n}\Phi_\rho(a_{ki})V_{\theta,i}\xi_{ki}\right\rangle$$

for all $a_{ki} \in A$ and $\xi_{ki} \in H$, $1 \leq i \leq n$, $1 \leq k \leq m$, and since $\{\Phi_\rho(a)V_{\rho,i}\xi;\ a \in A,\ \xi \in H,\ 1 \leq i \leq n\}$ spans a dense subspace of H_ρ, there is a unique bounded linear operator W from H_ρ to H_θ such that

$$W\left(\sum_{k=1}^{m}\sum_{i=1}^{n}\Phi_\rho(a_{ki})V_{\rho,i}\xi_{ki}\right)=\sum_{k=1}^{m}\sum_{i=1}^{n}\Phi_\theta(a_{ki})V_{\theta,i}\xi_{ki}.$$

Moreover, $\|W\|\leq 1$. Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for A, $\xi\in H$ and $i\in\{1,\dots,n\}$. Then

$$WV_{\rho,i}\xi=\lim_\lambda W\Phi_\rho(e_\lambda)V_{\rho,i}\xi=\lim_\lambda\Phi_\theta(e_\lambda)V_{\theta,i}\xi=V_{\theta,i}\xi.$$

Therefore $WV_{\rho,i}=V_{\theta,i}$ for all $i\in\{1,\dots,n\}$. It is not difficult to check that $W\Phi_\rho(a)=\Phi_\theta(a)W$ for all $a\in A$. □

Let $\rho=[\rho_{ij}]_{ij=1}^n\in CP_\infty^n(A,L(H))$. We consider the following sets:

$$[0,\rho]=\{\theta\in CP_\infty^n(A,L(H));\ \theta\leq\rho\}$$

and

$$[0,I]_\rho=\{T\in\Phi_\rho(A)';\ 0\leq T\leq I_{H_\rho}\}.$$

The following theorem can be regarded as a Radon–Nikodym type theorem for completely multi-positive linear maps.

Theorem 3.5. *The map $T\mapsto\rho_T$ from $[0,I]_\rho$ to $[0,\rho]$ is an affine order isomorphism.*

Proof. According to Lemmas 3.1 and 3.3 and Remark 3.2, it remains to show that the map $T\mapsto\rho_T$ from $[0,I]_\rho$ to $[0,\rho]$ is bijective.

Let $T\in[0,I]_\rho$ such that $\rho_T=0$. Then $V_{\rho,i}^*T\Phi_\rho\left(a^*a\right)V_{\rho,i}=0$ and so $T^{\frac{1}{2}}\Phi_\rho(a)V_{\rho,i}=0$ for all $a\in A$ and for all $i\in\{1,\dots,n\}$. From this fact and taking into account that $\{\Phi_\rho(a)V_{\rho,i}\xi;\ a\in A,\ \xi\in H,\ 1\leq i\leq n\}$ spans a dense subspace of H_ρ, we conclude that $T=0$ and so the map $T\mapsto\rho_T$ from $[0,I]_\rho$ to $[0,\rho]$ is injective.

Let $\theta\in[0,\rho]$. If W is the bounded linear operator from H_ρ to H_θ constructed in Lemma 3.4, then $W^*W\in[0,I]_\rho$. Let $T=W^*W$. From

$$\begin{aligned}\theta\left([a_{ij}]_{i,j=1}^n\right)&=\left[V_{\theta,i}^*\Phi_\theta(a_{ij})V_{\theta,j}\right]_{i,j=1}^n=\left[(WV_{\rho,i})^*\Phi_\theta(a_{ij})WV_{\rho,j}\right]_{i,j=1}^n\\&=\left[V_{\rho,i}^*W^*W\Phi_\rho(a_{ij})V_{\rho,j}\right]_{i,j=1}^n=\rho_T\left([a_{ij}]_{i,j=1}^n\right)\end{aligned}$$

for all $[a_{ij}]_{i,j=1}^n\in M_n(A)$, we deduce that $\theta=\rho_T$ and the theorem is proved. □

We say that a completely n-positive linear map ρ from a pro-C^*-algebra A to $L(H)$ is *pure* if for every $\theta \in CP^n_\infty(A, L(H))$ with $\theta \leq \rho$ there is $\alpha \in [0,1]$ such that $\rho = \alpha\theta$.

A representation Φ of a pro-C^*-algebra A on a Hilbert space H is *irreducible* if and only if the commutant $\Phi(A)'$ of $\Phi(A)$ in $L(H)$ consists of the scalar multiplies of the identity map on H [4].

Corollary 3.6. *Let $\rho \in CP^n_\infty(A, L(H))$. Then the following statements are equivalent:*

1. *ρ is pure;*
2. *The representation Φ_ρ of A associated with ρ is irreducible.*

Remark 3.7. If ρ is a continuous positive functional on A, then $\rho \in CP^1_\infty(A, L(\mathbb{C}))$. Moreover, Φ_ρ is the GNS representation of A associated with ρ, and Corollary 3.6 states that ρ is pure if and only if the representation Φ_ρ is irreducible. This is a particular case of the known result which states that a continuous positive linear functional ρ on a lmc*-algebra A with bounded approximate unit is pure if and only if the GNS representation of A associated with ρ is topological irreducible (see, for example, Corollary 3.7 of [4]).

4. Applications of the Radon–Nikodym theorem

Let A be a unital pro-C^*-algebra and let H be a Hilbert space.

Lemma 4.1. *Let $\rho = [\rho_{ij}]^n_{i,j=1} \in CP^n_\infty(A, L(H))$. If ρ_{ii}, $i \in \{1, 2, \ldots, n\}$ are unitarily equivalent pure unital completely positive linear maps from A to $L(H)$ and for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ there is a unitary element u_{ij} in A such that $\rho_{ij}(u_{ij}) = I_H$, then ρ is pure.*

Proof. Let $(\Phi_\rho, H_\rho, V_{\rho,1}, \ldots, V_{\rho,n})$ be the Stinespring representation associated with ρ. Let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Since

$$\begin{aligned}\|\Phi_\rho(u_{ij})V_{\rho,j} - V_{\rho,i}\|^2 &= \left\|\left(V^*_{\rho,j}\Phi_\rho\left(u^*_{ij}\right) - V^*_{\rho,i}\right)\left(\Phi_\rho(u_{ij})V_{\rho,j} - V_{\rho,i}\right)\right\| \\ &= \left\|\rho_{jj}(1) - \rho_{ij}(u_{ij}) - (\rho_{ij}(u_{ij}))^* + \rho_{ii}(1)\right\| = 0\end{aligned}$$

we conclude that $\Phi_\rho(u_{ij})V_{\rho,j} = V_{\rho,i}$. This implies that the vector subspaces H_i and H_j of H_ρ coincide. Therefore $H_\rho = H_i$ for all $i \in \{1, \ldots, n\}$, and since $\Phi_\rho(\cdot)P_i$ acts irreducibly on H_i, $i \in \{1, \ldots, n\}$, the representation Φ_ρ of A is irreducible and, by Corollary 3.6, ρ is pure. □

The following proposition is a generalization of Propositions 2.5 in [10] and 4.5 in [8].

Proposition 4.2. *Let* $\rho = [\rho_{ij}]_{i,j=1}^{n} \in CP_{\infty}^{n}(A, L(H))$ *such that* ρ_{ii}, $i \in \{1, 2, \ldots, n\}$ *are pure unital completely positive linear maps from* A *to* $L(H)$. *If whenever* ρ_{ii} *is unitarily equivalent with* ρ_{jj} *for some* $i, j \in \{1, \ldots, n\}$ *with* $i \neq j$, *there is a unitary element* u_{ij} *in* A *such that* $\rho_{ij}(u_{ij}) = I_H$, *then* ρ *is an extreme point in the set of all* $\theta = [\theta_{ij}]_{i,j=1}^{n} \in CP_{\infty}^{n}(A, L(H))$ *such that* $\theta_{ii}(1) = I_H$ *for all* $i \in \{1, \ldots, n\}$.

Proof. By Proposition 2.6 and Lemma 4.1, ρ is a block-diagonal sum of disjoint pure completely n_r-positive linear maps $\rho_r = [\rho_{i(k)j(k)}]_{k=1}^{n_r}$, $1 \leq r \leq m$, where m is the number of equivalence classes of the pure completely positive linear maps ρ_{ii} and $n_1 + \cdots + n_m = n$. By Corollary 3.6, the representation Φ_{ρ_r} of A induced by ρ_r is irreducible and, moreover, it is unitarily equivalent to a subrepresentation Φ_r of Φ_ρ. Therefore Φ_ρ is a direct sum of disjoint irreducible representations Φ_r, $r \in \{1, \ldots, m\}$. For each $r \in \{1, \ldots, m\}$, we denote by E_r the central support of Φ_r (this is, $\Phi_r(a) = \Phi_\rho(a)E_r$ for all $a \in A$).

Let $\theta = [\theta_{ij}]_{i,j=1}^{n}, \sigma = [\sigma_{ij}]_{i,j=1}^{n} \in CP_{\infty}^{n}(A, L(H))$ with $\theta_{ii}(1) = \sigma_{ii}(1) = I_H$ for all $i \in \{1, \ldots, n\}$ and $\lambda \in (0, 1)$ such that

$$\lambda\theta + (1 - \lambda)\sigma = \rho.$$

Then, by Theorem 3.5, there is T in $\Phi_\rho(A)'$, $0 \leq T \leq I_{H_\rho}$ such that $\lambda\theta = \rho_T$. From this fact and taking into account that Φ_r is irreducible and E_rTE_r is an element in $\Phi_r(A)'$ for all $r \in \{1, \ldots, m\}$ and $\theta_{ii}(1) = I_H$ for all $i \in \{1, \ldots, n\}$, we conclude that $E_rTE_r = \lambda E_r$ for all $r \in \{1, \ldots, m\}$. Consequently, $T = \lambda I_{H_\rho}$ and so $\theta = \rho$. In the same manner we obtain $\sigma = \rho$ and so ρ is an extreme point in the set of all $\theta = [\theta_{ij}]_{i,j=1}^{n} \in CP_{\infty}^{n}(A, L(H))$ such that $\theta_{ii}(1) = I_H$ for all $i \in \{1, \ldots, n\}$. □

Let A be a unital pro-C^*-algebra, H a Hilbert space and $CP_{\infty}^{n}(A, L(H), I) = \{\theta \in CP_{\infty}^{n}(A, L(H));\ \theta_{ii}(1) = I_H,\ 1 \leq i \leq n \text{ and } \theta_{ij}(1) = 0,\ 1 \leq i < j \leq n\}$. From Remark 2.1 and Proposition 4.2 we obtain the following corollary that generalizes Corollaries 2.7 in [10] and 4.7 in [8].

Corollary 4.3. *Let* $\rho = [\rho_{ij}]_{i,j=1}^{n} \in CP_{\infty}^{n}(A, L(H), I)$ *such that* ρ_{ii}, $i \in \{1, 2, \ldots, n\}$ *are pure completely positive linear maps from* A *to* $L(H)$. *If whenever* ρ_{ii} *is unitarily equivalent with* ρ_{jj} *for some* $i, j \in \{1, \ldots, n\}$ *with* $i \neq j$, *there is a unitary element* u_{ij} *in* A *such that* $\rho_{ij}(u_{ij}) = I_H$, *then the map* φ *from* A *to* $M_n(L(H))$ *defined by* $\varphi(a) = [\rho_{ij}(a)]_{i,j=1}^{n}$ *is an extreme point in the set of all unital completely positive linear maps from* A *to* $M_n(L(H))$.

The following theorem gives a characterization of the extreme points in $CP_{\infty}^{n}(A,\, L(H), I)$.

Theorem 4.4. *Let $\rho \in CP_{\infty}^{n}(A, L(H), I)$ and P_{H_0} the projection of H_ρ on the Hilbert subspace H_0 generated by $\{V_{\rho,i}\xi;\ \xi \in H,\ 1 \leq i \leq n\}$. Then ρ is an extreme point in $CP_{\infty}^{n}(A, L(H), I)$ if and only if the map $T \longmapsto P_{H_0}TP_{H_0}$ from $\Phi_\rho(A)'$ to $L(H_\rho)$ is injective.*

Proof. First we suppose that ρ is an extreme point in $CP_{\infty}^{n}(A, L(H), I)$. Let $T \in \Phi_\rho(A)'$ such that $P_{H_0}TP_{H_0} = 0$. Since $P_{H_0}T^*P_{H_0} = (P_{H_0}TP_{H_0})^* = 0$, we can suppose that $T = T^*$. From

$$\left\langle V_{\rho,i}^{*}TV_{\rho,j}\xi, \eta\right\rangle = \langle TV_{\rho,j}\xi, V_{\rho,i}\eta\rangle = \langle P_{H_0}TP_{H_0}V_{\rho,j}\xi, V_{\rho,i}\eta\rangle = 0$$

for all $i, j \in \{1, \ldots, n\}$ and for all $\xi, \eta \in H$, it follows that $V_{\rho,i}^{*}TV_{\rho,j} = 0$ for all $i, j \in \{1, \ldots, n\}$. It is not difficult to check that there are two positive numbers α and β such that $\frac{1}{4}I_{H_\rho} \leq \alpha T + \beta I_{H_\rho} \leq \frac{3}{4}I_{H_\rho}$. Moreover, $\beta \in (0,1)$. Let $T_1 = \frac{\alpha}{\beta}T + I_{H_\rho}$ and $T_2 = I_{H_\rho} - \frac{\alpha}{1-\beta}T$. Clearly, T_1 and T_2 are positive elements in $\Phi_\rho(A)'$. Therefore ρ_{T_1} and ρ_{T_2} are completely n-positive linear maps from A to $L(H)$. Moreover, since

$$\left(\rho_{T_1}\right)_{ij}(1) = V_{\rho,i}^{*}\left(\frac{\alpha}{\beta}T + I_{H_\rho}\right)V_{\rho,j} = V_{\rho,i}^{*}V_{\rho,j} = \rho_{ij}(1)$$

and

$$\left(\rho_{T_2}\right)_{ij}(1) = V_{\rho,i}^{*}\left(I_{H_\rho} - \frac{\alpha}{1-\beta}T\right)V_{\rho,j} = V_{\rho,i}^{*}V_{\rho,j} = \rho_{ij}(1)$$

for all $i, j \in \{1, \ldots, n\}$, $\rho_{T_1},\, \rho_{T_2} \in CP_{\infty}^{n}(A, L(H), I)$. A simple calculation shows that

$$\beta\rho_{T_1} + (1-\beta)\rho_{T_2} = \rho.$$

From this fact, since ρ is an extreme point, we conclude that $\rho_{T_1} = \rho_{T_2} = \rho$ and then by Theorem 3.5, $T_1 = T_2 = I_{H_\rho}$, whence $T = 0$.

Conversely, we suppose that the map $T \longmapsto P_{H_0}TP_{H_0}$ from $\Phi_\rho(A)'$ to $L(H_\rho)$ is injective. Let $\theta, \sigma \in CP_{\infty}^{n}(A, L(H), I)$ and $\alpha \in (0,1)$ such that $\alpha\theta + (1-\alpha)\sigma = \rho$. Then by Theorem 3.5, there is $T \in \Phi_\rho(A)'$, $0 \leq T \leq I_{H_\rho}$ such that $\alpha\theta = \rho_T$ and so

$$V_{\rho,i}^{*}TV_{\rho,j} = \begin{cases} \alpha I_H & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

From

$$\left\langle P_{H_0}\left(T - \alpha I_{H_\rho}\right)P_{H_0}V_{\rho,j}\xi, V_{\rho,i}\eta\right\rangle = \left\langle V_{\rho,i}^{*}TV_{\rho,j}\xi, \eta\right\rangle - \alpha\left\langle V_{\rho,i}^{*}V_{\rho,j}\xi, \eta\right\rangle = 0$$

for all $i, j \in \{1, \dots, n\}$ and for all $\xi, \eta \in H$, we conclude that $P_{H_0}(T - \alpha I_{H_\rho})P_{H_0} = 0$ and so $T = \alpha I_{H_\rho}$. Consequently $\theta = \rho$. In the same way we obtain $\sigma = \rho$. Therefore ρ is an extreme point in $CP^n_\infty(A, L(H), I)$. $\square$

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MARIA JOIŢA
UNIVERSITY OF BUCHAREST
DEPARTMENT OF MATHEMATICS
FACULTY OF CHEMISTRY
BD. REGINA ELISABETA NR. 4-12
BUCHAREST
ROMANIA

E-mail: mjoita@fmi.unibuc.ro

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