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Weak compactness of vector measures on topological spaces

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Abstract. Let X be a completely regular Hausdorff space, E a quasi-complete locally convex space, $C_b(X)$ the space of all bounded, scalar-valued continuous functions on X, and $\mathcal{B}(X)$ and $\mathcal{B}_0(X)$ be the classes of Borel and Baire subsets of X. We study the subsets of the spaces $M_t(X,E)$, $M_\tau(X,E)$, $M_\sigma(X,E)$ of respectively tight, τ-smooth, and σ-smooth, E-valued measures on X for weak compactness.

1. Introduction and notations

In this paper K will always denote the field of real or complex numbers (we will call them scalars), X a completely regular Hausdorff space and E a quasi-complete locally convex space space over K with topology generated by an increasing family of semi-norms $\{\|.\|_p,\ p \in P\}$; E' will denote the topological dual of E. For a $p \in P$, $V_p = \{x \in E : \|x\|_p \le 1\}$; polars will be taken in the duality $\langle E, E'\rangle$ (for two vector spaces F and G in duality, $\langle .\,,.\rangle : F \times G \to K$ will denote be the bilinear mapping). We denote by $C(X)$ the space of all K-valued continuous functions on X, and by $C_b(X)$ the space of all bounded elements of $C(X)$. The zero-sets of X are the elements of the class $\{f^{-1}(0) : f \in C_b(X)\}$; the positive-sets of X are complements of zero-sets in X. For locally convex spaces, the notations and results of [18] will be used. For a vector space F, F^* will denote its algebraic dual. N will denote the set of natural numbers. For topological measure theory notations and results of [19], [20], [12] will be used. All locally convex spaces are assumed to be Hausdorff. The elements of the smallest σ-algebra, on X, relative

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to which all functions in $C_b(X)$ are measurable, are called Baire sets and the elements of the σ-algebra generated by open sets are called Borel sets. $\mathcal{B}(X)$and $\mathcal{B}_0(X)$ will be the classes of Borel and Baire subsets of X. $\tilde{X}$ will denote the Stone$-\check{C}ech$ compactification of X. $M_\sigma(X)$, $M_\tau(X)$, $M_t(X)$ denote the spaces of σ-additive, τ-smooth and tight Baire measures on X [21], [20], [19]. The elements of $M_\sigma(X)$ are scalar-valued, countably additive measures on $\mathcal{B}_0(X)$. An element $\mu \in M_\sigma(X)$ is called τ-smooth if for any decreasing net $\{f_\alpha\} \subset C_b(X)$, $f_\alpha \downarrow 0$, we have $\mu(f_\alpha) \to 0$. Every τ-smooth measure has a unique extension to a Borel measure which is inner regular by closed subsets and outer regular by open subsets of X; an element $\mu \in M_\sigma(X)$ is called tight if for any uniformly bounded net $\{f_\alpha\} \subset C_b(X)$, $f_\alpha \to 0$, uniformly on compact subsets of X, we have $\mu(f_\alpha) \to 0$. Every tight measure has a unique extension to a Borel measure which is inner regular by compact subsets and outer regular by open subsets of X [21], [20]. Also the so-called strict topologies β_z, $z = \sigma, \tau, t$ have been defined on $C_b(X)$, with the result that $(C_b(X), \beta_z)' = M_z(X)$ (see [20]) (notations like β_1, β, β_0 are also used for these topologies in [19]). The topology β_t is the finest locally convex on $C_b(X)$, agreeing with the topology of uniform convergence on the compact subsets of X, on the norm bounded subsets of $C_b(X)$. To define the topology β_σ, take a zero-set, in $\tilde{X}$, $Z \subset \tilde{X} \setminus X$. The topology β_t on $C_b(\tilde{X} \setminus Z)$, we denote by β_Z. Evidently $C_b(\tilde{X} \setminus Z)$ can be identified with $C_b(X)$ (there is a natural 1-1, onto, norm-preserving mapping) and so β_Z can be considered a locally convex topology on $C_b(X)$. The β_σ is defined as $\bigwedge\{\beta_Z : Z \text{ a zero-set in } \tilde{X},\ Z \subset \tilde{X} \setminus X\}$. Similary β_τ is defined as $\bigwedge\{\beta_C : C \text{ a compact set in } \tilde{X},\ C \subset \tilde{X} \setminus X\}$.

With norm topology on $C_b(X)$, the dual of $C_b(X)$ is denoted by $M(X)$; $M(X)$ can also be interpreted as the space of bounded finitely additive measures, on the algebra generated by zero-sets of X, which are inner regular by zero-sets and outer regular by positive-sets of X (Alexanderov's Theorem [21], [20]).

For a topological space Y and a set A, Y^A will denote the topological space of all functions from $A \to Y$ with the topology of pointwise convergence on A (this will be called the product topology on Y^A). For a function $f \in C_b(X)$, $\tilde{f}$ will denote its unique continuous extension to $\tilde{X}$. It is easily verified that $\mathcal{B}(\tilde{X}) \cap X \supset \mathcal{B}(X)$ and $\mathcal{B}_0(\tilde{X}) \cap X \supset \mathcal{B}_0(X)$.

Now we come to vector-valued measures. If $\mathcal{A}$ is a σ-algebra of subsets of a set Y, $\mu : \mathcal{A} \to E$ a countably additive vector measure and $p \in P$, we denote the p-semi-variation of μ by $\bar{\mu}_p$, $\bar{\mu}_p(A) = \sup\{|g \circ \mu|(A) : g \in V_p^0\}$ (here V_p^0 is the polar of V_p in the duality $\langle E, E' \rangle$) [15]; also we consider the submeasure $\dot{\mu}_p : \mathcal{A} \to R^+$, $\dot{\mu}_p(A) = \sup\{\|\mu(B)\|_p : B \in \mathcal{A},\ B \subset A\}$ [12], [3]. It is easily verified that $\dot{\mu}_p$ is countably sub-additive [3] and $\dot{\mu}_p \leq \bar{\mu}_p \leq 4\dot{\mu}_p$ ([4], p. 97, Lemma 5). Also there is a control measure for $\bar{\mu}_p$; this will be denoted by λ_p; this control measure can be taken in the closed convex hull of $\{|g \circ \mu| : g \in V_p^0\}$, with norm topology on measures ([15], p. 20, proof of Theorem 1). This control measure also has the properties: (i) $|f \circ \mu| \ll \lambda_p$ for every f in E' with $\|f\|_p \leq 1$ (note $\|f\|_p = \sup\{|f(x)| : x \in V_p\}$); (ii) if $\lambda_p(A) = 0$ then $\bar{\mu}_p(A) = 0$; (iii) $\lim_{\lambda_p(A)\to 0} \bar{\mu}_p(A) = 0$; (iv) $\lambda_p \leq \bar{\mu}_p$. We also have the result that if $f : Y \to K$ is a measurable function, $B \in \mathcal{A}$ and $|f| \leq c$ on B, then $\| \int_B f d\mu\|_p \leq c\bar{\mu}_p(B)$.

$L^1(\mu)$ will denote the space of μ-integrable functions [15]. For any $f \in L^1(\mu)$, we take $\bar{\mu}_p(f) = \sup\{|g \circ \mu|(|f|) : g \in V_p^0\}$ ([15], Lemma 2, p. 23).

If X is a compact Hausdorff space then there is 1-1 correspondence between regular Borel E-valued measures μ and linear weakly compact operators $T : C(X) \to E$ such that $T(f) = \int f d\mu$, $\forall f \in C(X)$ ([16], Therorem 3.1, p. 163); regularity means that for any Borel $B \subset X$, $p \in P$, and $c > 0$, there exists a compact C and an open V, $C \subset B \subset V$ such that $\bar{\mu}_p(V \setminus C) < c$. In this case, for $p \in P$, the control measure λ_p is positive regular Borel measure in X.

We start with a proposition whose proof is contained in ([14], Theorem 2, Theorem 3, Theorem 4).

Proposition 1. a) *Let* $M_\sigma(X, E) = \{(\mu : \mathcal{B}_0 \to E) : g \circ \mu \in M_\sigma(X)\ \forall g \in E'\}$. *Then:*

(i) *$\mu \in M_\sigma(X, E)$ is countably additive in the original topology of E and μ is inner regular by zero-sets and outer regular by positive sets (in the original topology of E);*

(ii) *A linear, weakly compact (that is bounded sets are mapped into relatively weakly compact sets) mapping $\mu : (C_b(X), \beta_\sigma) \to E$ is continuous if and only if $\mu \in M_\sigma(X, E)$.*

b) *A Baire measure $\mu : \mathcal{B}_0 \to E$ will be called τ-smooth if for every $g \in E'$, $g \circ \mu \in M_\tau(X)$. The set of all E-valued τ-smooth measures will be denoted by $M_\tau(X, E)$. Then:*

(i) *$\mu \in M_\tau(X, E)$ can be uniquely extended to a Borel measure which is inner regular by closed sets and outer regular by open sets (in the original topology of E);*

(ii) *A linear, weakly compact mapping $\mu : (C_b(X), \beta_\tau) \to E$ is continuous if and only if $\mu \in M_\tau(X, E)$.*

c) *A countably additive Baire measure $\mu : \mathcal{B}_0 \to E$ will be called tight if for every $g \in E'$, $g \circ \mu \in M_t(X)$. The set of all E-valued tight measures will be denoted by $M_t(X, E)$. Then:*

(i) *$\mu \in M_t(X,E)$ can be uniquely extended to a Borel measure which is inner regular by compact sets and outer regular by open sets (in the original topology of E);*

(ii) *A linear, weakly compact mapping $\mu : (C_b(X), \beta_t) \to E$ is continous if and only if $\mu \in M_t(X,E)$.*

$H \subset M_t(X,E)$ is called uniformly tight if for a $p \in P$ and $c > 0$, there is a compact $C \subset X$ such that $\bar{\mu}_p(X \setminus C) < c$, $\forall \mu \in H$; H will be called weakly uniformly tight if $g \circ H$ is uniformly tight in $M_t(X)$, $\forall g \in E'$.

2. Weak compactness in $M_z(X,E)$

$M_z(X,E)$, $(z = \sigma, \tau, t)$, can be considered as subspaces of $E^{C_b(X)}$. If we take product topology on $E^{C_b(X)}$, with weak topology on E, then the topology induced on $M_z(X,E)$ is called the weak topology [2], [17], [9]. In [10], [11], [17], some results are proved about weak compactness of vector measures on topological spaces and mostly the measures in $M_t(X,E)$ are considered. In this paper we consider σ-smooth, τ-smooth and other measures also and extend some of their results to more general settings. Several of our proofs use quite different methods.

In the next theorem, we prove some general results about the relative compactness in the weak topology.

Theorem 2. *Suppose $H \subset M_z(X,E)$ $(z = \sigma, t, \tau)$, S is the closed unit ball of $C_b(X)$ and C is an absolutely convex, weakly compact subset of E such that $\mu(f) \in C$, for every $f \in S$ and for every $\mu \in H$.*

For z= σ, the following conditions are equivalent:

(i) *H is relatively weakly compact.*

(ii) *For any sequence $\{f_n\} \subset C_b(X)$, $f_n \downarrow 0$, the sequence $\mu(f_n) \to 0$, uniformly for $\mu \in H$, with weak topology on E.*

(iii) *For every $g \in E'$, $g \circ H$ is relatively weakly compact in $M_\sigma(X)$.*

For $z = t$: H is relatively weakly compact if and only if for every $g \in E'$, $g \circ H$ is relatively weakly in $M_t(X)$.

If in addition, X is a complete metric space or a paracompact locally compact space then H is relatively weakly compact if and only if for every $g \in E'$, $g \circ H$ is uniformly tight in $M_t(X)$.

For z= τ: The following conditions on H are equivalent:

(i) *H is relatively weakly compact.*

(ii) *for every* $g \in E'$, $g \circ H$ *is relatively weakly in* $M_\tau(X)$.

If X *is also paracompact then* (i) *and* (ii) *are also equivalent to:*

(iii) *For any net* $\{f_\alpha\} \subset C_b(X)$, $f_\alpha \downarrow 0$, *we have* $\mu(f_\alpha) \to 0$, *uniformly for* $\mu \in H$.

Proof. We first consider the case $z = \sigma$. $H \subset C^S$ and C^S, with product topology, is compact when C has the topology induced by $\sigma(E, E')$. Suppose (i) is satisfied. For a fixed $g \in E'$, the mapping $\mu \to g \circ \mu$, with weak topologies, is continuous. Thus (iii) is satisfied. Also by ([21], Theorem 28, p. 203), (ii) and (iii) are equivalent. Now suppose (ii) is satisfied. Take a net $\{\mu_\alpha\} \subset H$. There exists a subnet, which again we denote by $\{\mu_\alpha\}$, such that $\mu_\alpha \to \mu \in C^S$, in C^S. Now $\mu : C_b(X) \to E$ is a linear weakly compact operator and $g \circ \mu \in M_\sigma(X)$, for every $g \in E'$. By Proposition 1(a), $\mu \in M_\sigma(X, E)$. This proves (i).

Now consider the case $z = t$. If H is relatively weakly compact then trivially $g \circ H$ is relatively weakly compact in $M_t(X)$, for every $g \in E'$. Conversely suppose $g \circ H$ is relatively weakly compact in $M_t(X)$, for every $g \in E'$. Taking a net $\{\mu_\alpha\} \subset H$ and proceeding as in the case of $z = \sigma$, we get a cluster point μ such that $\mu : C_b(X) \to E$ is a linear weakly compact operator, and $g \circ \mu \in M_t(X)$ for every $g \in E'$. By Proposition 1(c), $\mu \in M_t(X, E)$. When X is a complete metric space or a paracompact locally compact space then $(C_b(X), \beta_t)$ is strongly Mackey [20] and so the weakly compact subsets of $M_t(X)$ are equicontinuous. Since the β_t-equicontinuous subsets of $M_t(X)$ are uniformly tight, the result follows.

Now the case $z = \tau$. This case is very similar to the case $z = t$ (note, if X is paracompact then the space $(C_b(X), \beta_\tau)$ is strongly Mackey [20]); details are omitted. □

Now we will establish some additional results with some additional assumptions.

Theorem 3. *Suppose* B *is an absolutely convex compact subset of* E, S *the closed unit ball of* $C_b(X)$ *and* $H \subset M_t(X, E)$ *such that* $\mu(S) \subset B$, $\forall \mu \in H$. *If* H *is weakly uniformly tight then it is uniformly tight.*

Proof. Fix a $p \in P$ and $c > 0$. Take $\{g_i\} \subset E'$ $(1 \le i \le n)$ and an $\eta > 0$ such that $x \in B$ with $|g_i(x)| \le \eta$, for $1 \le i \le n$, implies $\|x\|_p \le \frac{c}{4}$ (here we are very much using that the original topology of E and the weak topology of E coincide on B). Take a compact $K \subset X$ such that, for $1 \le i \le n$, $|g_i \circ \mu|(X \setminus K) \le \eta$, for every $\mu \in H$. For any Borel set $A \subset X \setminus K$, $|g_i \circ \mu(A)| \le \eta$ and so $\|\mu(A)\|_p \le \frac{c}{4}$. From this we get $\dot{\mu}_p(X \setminus K) \le \frac{c}{4}$ and so $\bar{\mu}_p(X \setminus K) \le c$. This proves the result. □

Remark 4. This theorem answers in the positive a conjecture raised in ([9], Remark 2).

A locally convex space is called semi-Montel if every bounded subset is relatively compact ([8], p. 229).

Corollary 5. *Suppose E is semi-Montel and H a uniformly bounded subset of $M_t(X, E)$. Assume further that $(C_b(X), \beta_t)$ is strongly Mackey ([20]; e.g. when X is a complete metric space or a paracompact locally compact space). Then H is relatively weakly compact if and only if H is uniformly tight.*

Proof. The result is well-known when E is scalars [20]. Using Theorem 3, we prove this corollary. □

In the next theorem we remove the condition that the range of H is contained in a weakly compact subset of E.

Theorem 6. *Suppose $H \subset M_t(X, E)$ has the properties:*

(i) *there is an absolutely convex, bounded and closed subset $B \subset E$ such that $\mu(S) \subset B$, $\forall \mu \in H$, S being the closed unit ball of $C_b(X)$;*

(ii) *for every compact $C \subset X$, there is an absolutely convex, weakly compact $K_C \subset E$ such that for every $f \in S$, $\int_C f d\mu \subset K_C$, $\forall \mu \in H$;*

(iii) *H is uniformly tight.*

Then H is relatively weakly compact.

Proof. As stated in the introduction, for a topological space Y and a set A, Y^A will denote the topological space of all functions from $A \to Y$ with the topology of pointwise convergence on A (this will be called the product topology on Y^A). B_0, the closure of B in $\sigma(E'', E')$, is compact. With product topology, B_0^S is compact and with weak topology, $H \subset B_0^S$. Take a net $\{\mu_\alpha\} \subset H$ and suppose $\mu_\alpha \to \mu \in B_0^S$. Now for any $g \in E'$, $g \circ H$ is uniformly tight in $M_t(X)$ (note H is uniformly tight). Let $\tilde{E}$ be the completion of E. We first prove that μ is $\tilde{E}$-valued. To prove this we will use Grothendieck completeness theorem [18]. Fix a $p \in P$. Take a p-equicontinuous net $\{g_\gamma\} \subset E'$ such that $|g_\gamma(V_p)| \leq 1$, $\forall \gamma$ and $g_\gamma \to 0$, pointwise on $\tilde{E}$. Assume that, for some $h \in S$, there is a $c > 0$ such that $|\langle g_\gamma, \mu(h)\rangle| > c$, $\forall \gamma$. By uniform tightness of H, there is a compact set $C \subset X$ such that $\bar{\nu}_p(X \setminus C) \leq \frac{c}{4}$, $\forall \nu \in H$. Define, for $f \in C_b(X)$, $\nu_\alpha(f) = \int_C f d\mu_\alpha$. It is a simple verification that $\nu_\alpha \in M_t(X, E)$. By hypothesis, $\nu_\alpha(f) \subset K_C, \forall f \in S$ and so by Theorem 2, the net $\{\nu_\alpha\}$ is relatively weakly compact. By taking subnets, if necessary, we assume that $\int_C f d\mu_\alpha \to \int f d\nu$, $\forall f \in S$ for some $\nu \in M_t(X, E)$. Now, from $\int f d\mu_\alpha = \int_C f d\mu_\alpha + \int_{X \setminus C} f d\mu_\alpha$, we get $\| \int f d\mu_\alpha - \int_C f d\mu_\alpha \|_p \leq \frac{c}{4}$. This means, for every $g \in E'$, with $|g(V_p)| \leq 1$, $| \int f d(g \circ \mu_\alpha) - \int_C f d(g \circ \mu_\alpha)| \leq \frac{c}{4}$. Taking limits over α, we get $|\langle g, \mu(f)\rangle - \int f d(g \circ \nu)| \leq \frac{c}{4}$. Since ν is E-valued,

$\int f d(g_\gamma \circ \nu) \to 0$. So $| < g_\gamma, \mu(f))| \leq c$, for some γ, a contradiction. This means μ is $\tilde{E}$-valued. Fix an $f \in S$. Thus $x = \mu(f) \in B_0 \cap \tilde{E}$. Take a net $\{x_\alpha\} \subset B$ such that $x_\alpha \to x$ weakly. This means there is a net $\{y_\alpha\} \subset B$ such that $y_\alpha \to x$ in $\tilde{E}$ (note B is convex). Since E is quasi-complete and B is closed, $x \in B$.

Now we will prove $\mu : C_b(X) \to E$ is weakly compact. Take a sequence $\{f_n\} \subset S$ such that $f_n \cdot f_m = 0$ for every n and for every m with $n \neq m$ ([1], Cor. 17, p. 160 remains valid for quasi-complete locally convex spaces); we have to prove that $\mu(f_n) \to 0$. Suppose this is not true. There are $p \in P$, a sequence $\{g_n\} \subset E'$, with $|g_n(V_p)| \leq 1$, $\forall n$, and a $c > 0$ such that $|\langle g_n, \mu(f_n) rangle| > c$, $\forall n$. Proceeding as in the proof above that μ is E-valued, we get a $\nu \in M_t(X, E)$ such that $|\langle g, \mu(f)\rangle - \int f d(g \circ \nu)| \leq \frac{c}{4}$, for all $f \in S$ and for all $g \in E'$ with $|g(V_p)| \leq 1$. Thus $|\langle g_n, \mu(f_n)\rangle - \int f_n d(g_n \circ \nu)| \leq \frac{c}{4}$, for all n. Since ν is a weakly compact operator, $\int f_n d(g_n \circ \nu) \to 0$ and so we get a contradiction. Now H being weakly uniformly tight, $g \circ H$ is relatively weakly compact in $M_t(X)$, $\forall g \in E'$. Since $g \circ \mu_\alpha \to g \circ \mu$, $\forall g \in E'$, by Proposition 1(c) $\mu \in M_t(X, E)$. This completes the proof. □

3. Sequential weak compactness in $M_t(X, E)$

In this section we prove some results about the sequential compactness of $H \subset M_t(X, E)$. We will use the fact that, in a Fréchet space E, for a subset $B \subset E$, weak compactness and sequential weak compactness are equivalent.

Theorem 7. *Suppose S is the closed unit ball of $C_b(X)$, E a Fréchet locally convex space and H a weakly compact subset of $M_z(X, E)$ ($z = \sigma, \tau, t$) such that $\mu(S) \subset B$, $\forall \mu \in H$, for some absolutely convex, weakly compact $B \subset E$. Assume $(C_b(X), \beta_z)$ is separable (this will be the case when X is a separable metric space and $z = t$ or τ [20]) or there is a countable subset of $C_b(X)$ which separates the points of H. Then from every net in H converging weakly to a $\mu \in H$, one can extract a sequence which weakly converges to μ (this implies that H is weakly sequential compact).*

Proof. Choose a sequence $\{f_n\} \subset S$ such that $\{f_n\}$ is total in $(C_b(X), \beta_z)$. Put $S_0 = \{f_n : n \in N\}$. Now H can be considered as a compact subset of B^{S_0}, a weakly compact subset of the Fréchet space E^N. Since B^{S_0} is angelic ([5], p. 32), the result follows. □

Theorem 8. *Suppose X is a metric space, S the closed unit ball of $C_b(X)$, E a Fréchet locally convex space and $H \subset M_t(X, E)$ having the properties:*

(i) *there is an absolutely convex, bounded and closed subset* $B \subset E$ *such that* $\mu(S) \subset B$, $\forall \mu \in H$;

(ii) *for every compact* $C \subset X$, *there is an absolutely convex, weakly compact* $K_C \subset E$ *such that for every* $f \in S$, $\int_C f d\mu \subset K_C$, $\forall \mu \in H$;

(iii) H *is uniformly tight.*

Then H *is relatively weakly compact and for every net in* H *converging to a* $\mu \in M_t(X, E)$, *one can extract a sequence, from the net, which converges to* μ *(this implies that* H *is relatively sequential compact).*

Proof. Relative weak compactness of H follows from Theorem 6. Let $\{p_n; n \in N\}$ be an increasing sequence of semi-norms on E such that $\{p_n^{-1}[0,1]\}$ is a 0-nbd base for E. Take an increasing sequence $\{C_n\}$ of compact subsets of X such that $\bar{\mu}_{p_n}(X \setminus C_n) \leq \frac{1}{n}$, $\forall n$ and $\forall \mu \in H$. Taking $H_0 = \{\chi_{C_n} H : n \in N\}$, H_0 can be considered a subset of $M_t(X, E^N)$, with product topology on E^N (it is easily verified that the elements of H_0 are weakly regular and so regular [16]). By Theorem 2, H_0 is relatively weakly compact. Take a net $\{\mu_\alpha\} \subset H$. There is a subnet, which again we denote by $\{\mu_\alpha\}$, such that $\{\chi_{C_n}\mu_\alpha : n \in N\} \subset H_0$ converges to $\{\nu_n : n \in N\} \in M_t(X, E^N)$ and $\mu_\alpha \to \mu \in M_t(X, E)$ (Theorem 7). Since C_n is a compact metric space, there exists a sequence $\{f_n\} \subset C_b(X)$ such that $\{(f_n)_{|C_k}\}$ is dense in $C(C_k)$, $\forall k$. This means the weak closure of H_0 satisfies the conditions of the Theorem 7. By Theorem 7, there is a sequence $\{\mu_k\} \subset \{\mu_\alpha\}$ such that $\{\chi_{C_n}\mu_k : n \in N\}$ converges to $\{\nu_n : n \in N\}$. We claim that $\mu_k \to \mu$. Fix an $f \in C_b(X)$, $0 \leq f \leq 1$, a $g \in E'$ and a $c > 0$. Fix an n, big enough that $|g(p_n^{-1}[0,1])| \leq 1$ and $cn > 4$. Choose a $k_0 \in N$ big enough such that $|\int f d(g \circ \nu_n) - \int_{C_n} f d(g \circ \mu_k)| < \frac{1}{n}$. $\forall k \geq k_0$. From $|\int f d(g \circ \mu_\alpha) - \int_{C_n} f d(g \circ \mu_\alpha| \leq \frac{1}{n}$, $\forall \alpha$, by taking limit over α, we get $|\int f d(g \circ \mu) - \int f d(g \circ \nu_n)| \leq \frac{1}{n}$. Now $|\int f d(g \circ \mu) - \int f d(g \circ \mu_k)| \leq |\int f d(g \circ \mu) - \int f d(g \circ \nu_n)| + |\int f d(g \circ \nu_n) - \int_{C_n} f d(g \circ \mu_k)| + |\int_{C_n} f d(g \circ \mu_k) - \int f d(g \circ \mu_k)| \leq \frac{1}{n} + \frac{1}{n} + \frac{1}{n} < c$, $\forall k \geq k_0$. This proves the result. □

4. Weak compactness in $M_c(X, E)$

In this section we assume that $K = R$. A subset $B \subset C(X)$ will be called order-bounded if there are elements f and g in $C(X)$ such that $f \leq b \leq g$, $\forall b \in B$. It is well-known that a linear map $\mu : C(X) \to R$, which maps order-bounded sets into bounded sets, gives a unique $\nu \in M_\sigma(X)$ such that $C(X) \subset L^1(\nu)$ and $\mu(f) = \int f d\nu$ ([21], Theorem 23; [7]). Also it is well-known that support of

$\tilde{\mu} : C(\tilde{X}) \to R$, $\tilde{\mu}(f) = \mu(f_{|X})$ is contained in υX [7], the real-compactfication of X. This can be extended to the vector valued case:

Let $\mu : C(X) \to E$ be a linear map such that order-bounded subsets are mapped into relatively weakly compact subsets of E. It is proved in ([14], Theorem 7) that:

(i) μ can be considered an element of $M_\sigma(X, E)$ with $C(X) \subset L^1(\nu)$ and $\mu(f) = \int f d\mu$, $\forall f \in C(X)$;

(ii) for every $p \in P$, there is a compact $C \subset \upsilon X$ (the real-compactification of X), depending on p, such that $\bar{\tilde{\mu}}_p(\tilde{X} \setminus C) = 0$.

The collection of all μ's satisfying the above conditions will be denoted by $M_c(X, E)$. Considering $M_c(X, E) \subset E^{C(X)}$, the weak topology on $M_c(X, E)$ is the one induced by the product topology on $E^{C(X)}$ when in E is given the weak topology.

We prove the following theorem:

Theorem 9. *Suppose X is a completely regular Hausdorff space, E a quasi-complete locally convex space and $H \subset M_c(X, E)$. Assume that for every $f \in C(X)$, $f \geq 0$, there exists a weakly compact set $W_f \subset E$ such that $\mu(h) \in W_f$, for every $h \in C(X)$, with $|h| \leq f$, and for every $\mu \in H$. Then H is relatively weakly compact.*

Proof. Every $\mu \in H$ gives a mapping $\phi_\mu : C(X) \times E' \to R$, $\phi_\mu(f, g) = \langle \mu(f), g \rangle$. For an $f \in C(X)$ and $g \in E'$, let $C_{(f,g)} = \sup |\langle W_{|f|}, g \rangle|$; by the given hypothesis, $|\phi_\mu(f, g)| \leq C_{(f,g)}$, $\forall \mu \in H$. This means the collection of mappings $\{\phi_\mu : \mu \in H\}$ is pointwise bounded. Now we will prove that H is relatively weakly compact.

Take a net $\{\mu_\alpha\} \subset H$; there is a subset, which again we denote by $\{\mu_\alpha\}$, such that ϕ_{μ_α} is pointwise convergent on $C(X) \times E'$. This means the net of the mappings $\mu_\alpha : C(X) \to R^{E'}$, with product topology on $R^{E'}$, is convergent, pointwise on $C(X)$. Let μ be its limit. E, with weak topology, is a subspace of $R^{E'}$ and so we have to prove that μ is E-valued.

Fix an $f \in C(X)$. By hypothesis, there is a weakly compact $W_{|f|} \subset E$ such that $\mu_\alpha(f) \in W_{|f|}$ $\forall \alpha$. It follows that $\mu(f) \in W_{|f|} \subset E$. This proves μ is E-valued. Now we want to prove that $\mu \in M_c(X, E)$.

Take $f \in C(X)$, $f \geq 0$ and an $h \in C(X)$ with $|h| \leq f$. We claim that $\mu(h) \in W_{|f|}$:
Now $\mu_\alpha(h) \in W_{|f|}$ $\forall \alpha$ and so $\mu(h) \in W_{|f|}$. This proves that $\mu \in M_c(X, E)$. So the result is proved. $\square$

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