Publ. Math. Debrecen
72/1-2 (2008), 81–94

Sufficient conditions for starlikeness of order α

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Abstract. In this paper we obtain some sufficient conditions for an analytic function to be starlike of order α, by using the differential operator recently introduced by F. Al-Oboudi. For such classes we also give some applications of a result due to M. Robertson.

1. Introduction and preliminaries

Let $H(\mathrm{U})$ be the space of all analytic functions in the unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $n \in \mathbb{N} = \{1, 2, \dots\}$ let define the class of functions

$$A_n = \left\{ f \in H(\mathrm{U}) : f(z) = z + a_{n+1}z^{n+1} + \dots, \quad z \in \mathrm{U} \right\}.$$

Let $A \equiv A_1$ and let S denotes the subclass of A consisting in those functions that are univalent in U.

A function $f \in A$ is called to be a *starlike function of order* α, if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathrm{U},$$

where $\alpha < 1$. The class of all starlike functions of order α is denoted by $S^*(\alpha)$; we write $S^* \equiv S^*(0)$ and, moreover $S^*(\alpha) \subset S$ for $0 \leq \alpha < 1$.

We mention that the class of all functions $f \in A_n$ that satisfy the above inequality is denoted by $S^*_n(\alpha)$, that is $S^*_n(\alpha) = S^*(\alpha) \cap A_n$.

Mathematics Subject Classification: 30C45.
Key words and phrases: analytic function, starlike and convex function, differential operator, differential subordination.

Recently, F. Al-Oboudi defined in [Ob04] the differential operator $D_\lambda^m : H(\mathrm{U}) \to H(\mathrm{U})$ by

$$D_\lambda^0 f(z) = f(z),$$

$$D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z), \tag{1.1}$$

$$D_\lambda^m f(z) = D_\lambda^1 \left(D_\lambda^{m-1} f(z) \right). \tag{1.2}$$

If $f \in A_n$, then from (1.1) and (1.2) we may easily deduce that

$$D_\lambda^m f(z) = z + \sum_{k=n+1}^{\infty} \left[1 + (k-1)\lambda\right]^m a_k z^k, \tag{1.3}$$

where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\lambda \geq 0$. Remark that, for $\lambda = 1$ we get the operator introduced by Gr. Şt. Sălăgean in [Sal83].

Definition 1.1. Let $S^m(n, \lambda, \alpha)$ denotes the class of functions $f \in A_n$ which satisfy the condition

$$\operatorname{Re} \frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} > \alpha, \quad z \in \mathrm{U}, \tag{1.4}$$

for some $\alpha < 1$, $\lambda \geq 0$ and $m \in \mathbb{N}$.

In order to prove that all the functions of $S^m(n, \lambda, \alpha)$ are univalent in U, first we will show an inclusion and a sharp inclusion relation between these classes.

To prove our main results we will need the next definition and lemmas.

If $f, g \in H(\mathrm{U})$ we say that the function f is *subordinate* to g, or g is *superordinate* to f, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathrm{U}$, such that $f(z) = g(w(z))$ for all $z \in \mathrm{U}$. In such a case we write $f(z) \prec g(z)$.

Remark that, if g is univalent in U, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathrm{U}) \subseteq g(\mathrm{U})$.

The next lemma represents a result concerning the generalized *Libera integral operator* introduced by S. D. Bernardi in [Ber69], which shows that this operator preserves the starlikeness, the convexity and the close-to-convexity. We will give now only a part of the original form.

Lemma 1.1 ([LewMiZl76], [Pa79])**.** *If $L_c : A \to A$ is the integral operator defined by $L_c(f) = F$, where*

$$F(z) = \frac{c+1}{z^c} \int_0^z f(t)\, t^{c-1}\, \mathrm{d}\, t,$$

and $\operatorname{Re} c \geq 0$, *then* $L_c(S^*) \subset S^*$.

The next two lemmas deal with the so called *Briot–Bouquet* differential subordinations:

Lemma 1.2 ([EeMiMoRe83, Theorem 1]). *Let $\beta, \gamma \in \mathbb{C}$ and let h be a convex function in* U *with* $\operatorname{Re}[\beta h(z)+\gamma]>0$, $z \in \mathrm{U}$. *If p is analytic in* U, *with* $p(0)=h(0)$, *then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Lemma 1.3 ([MiMo00, Theorem 3.3e]). *Let* $\beta>0$, $\beta+\gamma>0$ *and consider the integral operator* $\mathrm{I}_{\beta,\gamma}$ *defined by*

$$\mathrm{I}_{\beta,\gamma}(f)(z)=\left[\frac{\beta+\gamma}{z^{\gamma}}\int_0^z f^{\beta}(t)t^{\gamma-1}\,\mathrm{d}\,t\right]^{1/\beta}.$$

If $\alpha \in \left[-\frac{\gamma}{\beta},1\right)$ *then* the order of starlikeness of the class $\mathrm{I}_{\beta,\gamma}(S_n^*(\alpha))$, *i.e. the largest number* $\delta=\delta_n(\alpha;\beta,\gamma)$ *such that* $\mathrm{I}_{\beta,\gamma}(S_n^*(\alpha)) \subset S_n^*(\delta)$, *is given by the number* $\delta_n(\alpha;\beta,\gamma)=\inf\{\operatorname{Re} q_n(z) : z \in \mathrm{U}\}$, *where*

$$q_n(z)=\frac{1}{\beta Q_n(z)}-\frac{\gamma}{\beta} \quad \text{and} \quad Q_n(z)=\frac{1}{n}\int_0^1\left(\frac{1-z}{1-tz}\right)^{\frac{2\beta(1-\alpha)}{n}} t^{\frac{\beta+\gamma}{n}-1}\,\mathrm{d}\,t.$$

Moreover, if $\alpha \in [\alpha_0,1)$, *where* $\alpha_0=\max\left\{\frac{\beta-\gamma-n}{2\beta};-\frac{\gamma}{\beta}\right\}$ *and* $g=\mathrm{I}_{\beta,\gamma}(f)$ *with* $f \in S_n^*(\alpha)$, *then*

$$\operatorname{Re}\frac{zg'(z)}{g(z)}>\delta_n(\alpha;\beta,\gamma)=\frac{\beta+\gamma}{{}_2F_1\left(1,\frac{2\beta(1-\alpha)}{n},\frac{\beta+\gamma+n}{n};\frac{1}{2}\right)\cdot\beta}-\frac{\gamma}{\beta}, \quad z \in \mathrm{U},$$

where ${}_2F_1$ *represents the hypergeometric function.*

Lemma 1.4 ([MiMo87]). *Let* $\Omega \subset \mathbb{C}$, *and suppose that the mapping* $\psi : \mathbb{C}^2 \times \mathrm{U} \to \mathbb{C}$ *satisfies* $\psi(ix,y;z) \notin \Omega$ *for* $z \in \mathrm{U}$, *and for all* $x, y \in \mathbb{R}$ *such that* $y \leq -n\left(1+x^2\right)/2$. *If the function* $p(z)=1+c_nz^n+\dots$ *is analytic in* U *and* $\psi(p(z),zp'(z);z) \in \Omega$ *for all* $z \in \mathrm{U}$, *then* $\operatorname{Re} p(z)>0$, *for all* $z \in \mathrm{U}$.

For the result presented in the last section, we will need the next lemma of M. Robertson.

Lemma 1.5 ([Rob61]). *Let* $F : \mathrm{U} \times [0,+\infty) \to \mathbb{C}$ *be an analytic function in the unit disc* U *for all* $0 \leq t \leq 1$, *with* $F(0,t)=0$ *for all* $0 \leq t \leq 1$. *Suppose that* $F(\cdot,0)=f \in S$, *and let* $p>0$ *a such number for which it exists*

$$F(z)=\lim_{t\to 0}\frac{F(z,t)-F(z,0)}{zt^p}.$$

If $F(z,t) \prec f(z)$ *for all* $0 \le t \le 1$, *then*

$$\operatorname{Re}\frac{F(z)}{f'(z)} \le 0, \quad z \in \mathrm{U}.$$

If in addition, F *is also analytic in the unit disc* U *and* $\operatorname{Re} F(0) \neq 0$, *then*

$$\operatorname{Re}\frac{F(z)}{f'(z)} < 0, \quad z \in \mathrm{U}.$$

2. Inclusion relations between the $S^m(n,\lambda,\alpha)$ subclasses

Theorem 2.1. 1) *For all* $0 \le \lambda \le 1$ *and* $1-\lambda \le \alpha < 1$, *the inclusion*

$$S^{m+1}(n,\lambda,\alpha) \subseteq S^m(n,\lambda,\alpha). \tag{2.1}$$

holds for all $m \in \mathbb{N}$.

2) *If* $0 < \lambda \le 1$ *and* $1-\lambda \le \alpha < 1$, *then the inclusion*

$$S^{m+1}(n,\lambda,\alpha) \subseteq S^m(n,\lambda,\beta(n,\lambda,\alpha)), \tag{2.2}$$

where

$$\beta(n,\lambda,\alpha) = \frac{1}{{}_2F_1\left(1, \frac{2(1-\alpha)}{n\lambda}, \frac{1+n\lambda}{n\lambda}; \frac{1}{2}\right)},$$

is sharp and holds for all $m \in \mathbb{N}$.

Proof. For the special case $\lambda = 0$, since $D_0^m f(z) = f(z)$, $z \in \mathrm{U}$, for all $m \in \mathbb{N}_0$, we have the equality $S^{m+1}(n,0,\alpha) = S^m(n,0,\alpha)$, $m \in \mathbb{N}$.

Let now consider the case $\lambda > 0$. If we let

$$p(z) = \frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)}, \tag{2.3}$$

then $p(0) = 1$, and the first step of our proof is to show that $p \in H(\mathrm{U})$.

According to the definition (1.4), if $f \in S^{m+1}(n,\lambda,\alpha)$ then $f \in A_n$ and

$$\operatorname{Re}\frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} > \alpha, \quad z \in \mathrm{U}. \tag{2.4}$$

If we denote by $H(z) = D_\lambda^m f(z)$ and using the definitions (1.1) and (1.2), the inequality (2.4) becomes

$$\operatorname{Re}\frac{zH'(z)}{H(z)} > \frac{\alpha+\lambda-1}{\lambda}, z \in \mathrm{U}. \tag{2.5}$$

From (1.3) we have $H(0) = H'(0) - 1 = 0$, and combining this with the inequality (2.5) we obtain that $H \in S^*$, whenever $1 - \lambda \leq \alpha < 1$.

Denoting by $h(z) = D_\lambda^{m-1} f(z)$, where $m \in \mathbb{N}$, then $h(0) = h'(0) - 1 = 0$, and from (1.1) and (1.2) we get

$$(1-\lambda)h(z) + \lambda z h'(z) = H(z). \tag{2.6}$$

For $\lambda = 1$ the above differential equation has the solution

$$h(z) = \int_0^z \frac{H(t)}{t} \, \mathrm{d}\, t,$$

where $H \in S^*$. From the well-known result concerning the *Alexander integral operator* we deduce that h is convex in U, so is a univalent function in U.

For $0 < \lambda < 1$, the relation (2.6) becomes

$$h(z) + \frac{\lambda}{1-\lambda} z h'(z) = \frac{H(z)}{1-\lambda}, \tag{2.7}$$

where $H \in S^*$. It is easy to see that the differential equation (2.7) has the solution

$$h(z) = \frac{1}{\lambda} \frac{1}{z^{\frac{1}{\lambda}-1}} \int_0^z H(t)\; t^{\frac{1}{\lambda}-2} \, \mathrm{d}\, t = L_c(H)(z),$$

where $c = \frac{1}{\lambda} - 1$. Since $0 < \lambda < 1$, then $\operatorname{Re} c \geq 0$, and from Lemma 1.1 it follows that $h \in S^*$, so h is a univalent function in U.

From the above results we conclude that h is a univalent function in U with the single zero $z_0 = 0$, i.e. $D_\lambda^{m-1} f(0) = 0$, $(D_\lambda^{m-1} f)'(0) = 1 \neq 0$ and $D_\lambda^{m-1} f(z) \neq 0$ for all $z \in \dot{\mathrm{U}} \equiv \mathrm{U} \setminus \{0\}$, hence we conclude that the function p defined by (2.3) is analytic in U.

The inequality (2.4) together with (1.1) and (1.2) shows that

$$p(z) + \lambda \frac{z p'(z)}{p(z)} = \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} \prec h(z) = \frac{1 + (1-2\alpha) z}{1+z}.$$

In the case $\lambda > 0$, according to Lemma 1.2 for $\beta := 1/\lambda$ and $\gamma := 0$, and using the fact that

$$\operatorname{Re}\left[\beta h(z) + \gamma\right] > \frac{\alpha}{\lambda} \geq 0, ; z \in \mathrm{U},$$

we deduce that $p(z) \prec h(z)$, i.e. $f \in S^m(n, \lambda, \alpha)$.

To prove the second part of the theorem we will use Lemma 1.3 for the special case $\beta := 1/\lambda$ and $\gamma := 0$. We see that it is necessary to have $\alpha \in [\alpha_0, 1)$ and $1-\lambda \leq \alpha < 1$, where $\alpha_0 \equiv \max\left\{\frac{1-n\lambda}{2}; 0\right\}$, hence

$$1 > \alpha \geq \max\left\{\frac{1-n\lambda}{2}; 1-\lambda; 0\right\} = 1-\lambda.$$

Since the conditions of Lemma 1.3 are satisfied, we obtain the sharp bound

$$\operatorname{Re} p(z) > \delta_n(\alpha; \beta, \gamma) = \beta(n, \lambda, \alpha) = \frac{1}{{}_2F_1\left(1, \frac{2(1-\alpha)}{n\lambda}, \frac{1+n\lambda}{n\lambda}; \frac{1}{2}\right)}, \quad z \in \mathrm{U},$$

that is $f \in S^m(n, \lambda, \beta(n, \lambda, \alpha))$. □

Considering in the above theorem the special case $n = 1$, for $\lambda = 1$ we need to have that $0 \leq \alpha < 1$. For $\alpha = 0$, since

$$\delta_1\left(\frac{\beta-\gamma-1}{2\beta}; \beta, \gamma\right) = \frac{\beta-\gamma}{2\beta}$$

we get $\beta(1, 0, \lambda) = 1/2$. Taking in the relation (2.2) of Theorem 2.1 the special case $\alpha = 0$ we obtain the next result:

Corollary 2.1. *The inclusion*

$$S^{m+1}(1, 1, 0) \subseteq S^m\left(1, 1, \frac{1}{2}\right)$$

is sharp and holds for all $m \in \mathbb{N}$.

3. Sufficient conditions for starlikeness

Recently, Li and Owa [LiOw02] obtained the following result: if $f \in A$ satisfies

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right)\right] > \frac{-\alpha}{2}, \quad z \in \mathrm{U},$$

for some $\alpha \geq 0$, then $f \in S^*$.

In fact, Lewandowski, Miller and Złotkiewicz in [LewMiZl76] and Ramesha, Kumar and Padmanabhan in [RaKuPa95] have proved the next weaker from of this theorem: if $f \in A$ satisfies

$$\left|\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)} - 1\right)\right| < \rho, \quad z \in \mathrm{U},$$

where $\rho = 2.2443697$, then $f \in S^*$.

The above result with $\rho = 3/2$ and $\rho = 1/6$ were earlier proved by Li and Owa in [LiOw98] and Obradović and Ruscheweyh in [ObRu92] respectively. Also, Ravichandran, Selvaraj and Rajalaksmi in [RaSeRa02] obtained some sufficient condition for functions in A_n to be starlike of order β.

We will obtain some other sufficient condition for functions to be starlike of order α, by using the differential operator D_λ^m already defined by (1.1) and (1.2).

Theorem 3.1. *Let $\alpha \geq 0$, $\beta < 1$, $m \in \mathbb{N}$ and $\lambda \geq 0$. If the function $f \in A_n$ satisfies*

$$\operatorname{Re} \frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} \left[\alpha \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} + (1-\alpha)\right] > \alpha\beta\left(\beta + \frac{n\lambda}{2} - 1\right) + \left(\beta - \frac{\alpha\lambda n}{2}\right), \quad z \in \mathrm{U},$$

then $f \in S^m(n, \lambda, \beta)$.

Proof. Let define the function p by

$$p(z) = \frac{1}{1-\beta}\left(\frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} - \beta\right).$$

From the assumption it follows $p \in H(\mathrm{U})$ with $p(z) = 1 + c_n z^n + \dots$, and a simple computation shows that

$$\alpha \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} = \frac{\alpha\lambda(1-\beta) z p'(z)}{(1-\beta)p(z) + \beta} + \alpha\left[(1-\beta)p(z) + \beta\right].$$

Hence

$$\frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} \left[\alpha \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} - (1-\alpha)\right] = \alpha(1-\beta)\lambda z p'(z) + \alpha(1-\beta)^2 p^2(z) + (1-\beta)(2\alpha\beta + 1 - \alpha)p(z) + \beta(\alpha\beta + 1 - \alpha) = \psi(p(z), zp'(z); z),$$

where

$$\psi(r, s; t) = \alpha\lambda(1-\beta)s + \alpha(1-\beta)^2 r^2 + (1-\beta)(2\alpha\beta + 1 - \alpha)r^2 + \beta(\alpha\beta + 1 - \alpha).$$

For all $x, y \in \mathbb{R}$ satisfing $y \leq -n\left(1 + x^2\right)/2$ we have the inequalities

$$\begin{aligned}\operatorname{Re} \psi(ix, y; z) &= \alpha\lambda(1-\beta)y - \alpha(1-\beta)^2 x^2 + \beta(\alpha\beta + 1 - \alpha) \\ &\leq \frac{-n\lambda\alpha}{2}(1-\beta) - \left[\frac{n\lambda\alpha}{2}(1-\beta) + \alpha(1-\beta)^2\right] x^2 + \beta(\alpha\beta + 1 - \alpha)\end{aligned}$$

$$\leq \beta(\alpha\beta+1-\alpha)-\frac{n\lambda\alpha}{2}(1-\beta)=\alpha\beta\left(\beta+\frac{n\lambda}{2}-1\right)+\left(\beta-\frac{n\lambda\alpha}{2}\right).$$

If we let

$$\Omega=\left\{\omega\in\mathbb{C}:\operatorname{Re}\omega>\alpha\beta\left(\beta+\frac{n\lambda}{2}-1\right)+\left(\beta-\frac{\lambda n\alpha}{2}\right)\right\},$$

then $\psi(p(z),zp'(z);z)\in\Omega$ and $\psi(ix,y;z)\notin\Omega$, for all $x,y\in\mathbb{R}$ with $y\leq -n\left(1+x^2\right)/2$ and for all $z\in \mathrm{U}$, hence by applying Lemma 1.4 we obtain the required result. □

Combining the above result together with the inclusion (2.1) of Theorem 2.1 we get the next corollary:

Corollary 3.1. *Let $0\leq\lambda\leq 1$, $1-\lambda\leq\alpha<1$, $\beta<1$ and $m\in\mathbb{N}$. If $f\in A_n$ satisfies*

$$\operatorname{Re}\left\{\lambda\frac{D_\lambda^m f(z)}{D_\lambda^{m-1}f(z)}\left[\alpha\frac{D_\lambda^{m+1}f(z)}{D_\lambda^m f(z)}+1-\alpha\right]+1-\lambda\right\}$$
$$>\lambda\alpha\beta\left(\beta+\frac{n\lambda}{2}-1\right)+\left(\lambda\beta-\frac{\lambda^2 n\alpha}{2}-\lambda+1\right),\quad z\in\mathrm{U},$$

then $f\in S^*(\rho)$, where $\rho=\frac{\beta-(1-\lambda)}{\lambda}$.

Taking $m=1$ in Theorem 3.1, we obtain the following implication:

Corollary 3.2 ([RaSeRa02])**.** *Let $\alpha\geq 0$ and $\beta<1$. If $f\in A_n$ satisfies*

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right]>\alpha\beta\left(\beta+\frac{n}{2}-1\right)+\left(\beta-\frac{\alpha n}{2}\right),\quad z\in\mathrm{U},$$

then $f\in S^*(\beta)$.

If we take in this corollary $\beta=\alpha/2$ and $n=1$, we deduce the next result:

Corollary 3.3 ([LiOw02])**.** *Let $0\leq\alpha<2$. If $f\in A$ satisfies*

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right]>-\frac{\alpha^2}{4}(1-\alpha),\quad z\in\mathrm{U},$$

then $f\in S^*\left(\frac{\alpha}{2}\right)$.

Now we shall prove another sufficient condition for a function $f\in A_n$ to be in the class $S^m(n,\lambda,\beta)$.

Theorem 3.2. *Let $\lambda \geq 0$, $0 \leq \beta < 1$ and suppose that the numbers*

$$a = \left(\frac{\lambda n}{2} + 1 - \beta\right)^2 \quad \text{and} \quad b = \left(\frac{\lambda n}{2} + \beta\right)^2 \tag{3.1}$$

satisfy the inequality

$$(a+b)\beta^2 < b(1-2\beta). \tag{3.2}$$

If t_0 is the positive root of the equation

$$2a(1-\beta)^2 t^2 + \left[3a\beta^2 + b(1-\beta)^2\right] t + \left[(\alpha + 2b)\beta^2 - (1-\beta)^2 b\right] = 0,$$

let denote

$$\rho = \sqrt{\frac{(1-\beta)^3 (1+t_0)^2 (at_0 + b)}{\beta^2 + (1-\beta)^2 t_0}}.$$

If $f \in A_n$ satisfies

$$\left| \left(\frac{D_\lambda^{m+1} f(z)}{D^m f(z)} - 1\right) \left(\frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} - 1\right) \right| < \rho, \quad z \in \mathrm{U}, \tag{3.3}$$

then $f \in S^m(n, \lambda, \beta)$, where $m \in \mathbb{N}$.

Proof. Let define the function $p \in H(\mathrm{U})$ by

$$p(z) = \frac{1}{1-\beta}\left(\frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} - \beta\right).$$

With this notation it follows that

$$\frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} - 1 = \frac{(1-\beta)\lambda z p'(z) + [(1-\beta)p(z) + \beta]^2 - [(1-\beta)p(z) + \beta]}{(1-\beta)p(z) + \beta},$$

hence

$$\left(\frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} - 1\right)\left(\frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} - 1\right) = \frac{(1-\beta)(p(z)-1)}{(1-\beta)p(z) + \beta}$$

$$\cdot \left\{(1-\beta)\lambda z p'(z) + [(1-\beta)p(z) + \beta]^2 - [(1-\beta)p(z) + \beta]\right\} = \psi(p(z), zp'(z); z).$$

Now, for all $x, y \in \mathbb{R}$ satisfying $y \leq -n(1+x^2)/2$, we have

$$|\psi(ix, y; z)|^2 = \frac{(1-\beta)^2(1+t)}{\beta^2 + (1-\beta)^2 t} \cdot \left\{[(1-\beta)\lambda y - \beta + \beta^2 - (1-\beta)^2 t]^2\right.$$

$$\left. + [2\beta(1-\beta) - (1-\beta)]^2 t\right\} = g(t, y),$$

where $t = x^2$ and $y \le -n(1+t)/2$. If $\lambda \ge 0$ and $0 \le \beta < 1$, since

$$\begin{aligned}\frac{\partial g(t,y)}{\partial y} &= \frac{2(1-\beta)^3(1+t)}{\beta^2+(1-\beta)^2 t}\left[(1-\beta)\lambda y - \beta + \beta^2 - (1-\beta)^2 t\right]\\ &= \frac{2(1-\beta)^4(1+t)\lambda}{\beta^2+(1-\beta)^2 t}\left[\lambda y - \beta - (1-\beta)t\right] < 0, \quad t \ge 0,\end{aligned}$$

then for all $y \le -n(1+t)/2$ we have

$$g(t,y) \ge g\left(t, \frac{-n(1+t)}{2}\right) = h(t), \quad t \ge 0.$$

According to the above results, we need to determine the minimum of the function $h : [0,+\infty) \to \mathbb{R}$,

$$h(t) = \frac{(1-\beta)^3(1+t)^2}{\beta^2+(1-\beta)^2 t}(at+b),$$

where a and b are defined by (3.1).

With these notations, the derivative $h'(t) = \frac{(\beta-1)^2(1+t)}{[\beta^2+(\beta-1)^2 t]^2}\, H(t)$, where

$$H(t) = 2a(1-\beta)^2 t^2 + \left[3a\beta^2 + b(1-\beta)^2\right] t + \left[(a+2b)\beta^2 - t(1-\beta)^2 b\right].$$

We have that $h'(-1) = 0$ and the other two roots of $h'(t) = 0$ are given by $H(t) = 0$, i.e.

$$2a(1-\beta)^2 t^2 + \left[3a\beta^2 + b(1-\beta)^2\right] t + \left[(a+2b)\beta^2 - (1-\beta)^2 b\right] = 0.$$

If we denote the discriminant of H by $D(\beta, \lambda n)$, then

$$D(\beta,\lambda n) = \lambda n\left(\beta - \frac{1}{2}\right)^2\left[\left(\beta - \frac{1}{2}\right)^2 - \frac{1+\lambda n}{4}\right] R(\beta,\lambda n), \tag{3.4}$$

where

$$R(\beta,\lambda n) = 4(\lambda n - 8)\beta^2 + 4(7\lambda n + 16)\beta - \left(9\lambda^2 n^2 + 32\lambda n + 32\right).$$

First we see that

$$\left(\beta - \frac{1}{2}\right)^2\left[\left(\beta - \frac{1}{2}\right)^2 - \frac{1+\lambda n}{4}\right] \le 0, \text{ for } \beta \in [0,1), \quad \lambda \ge 0,\ n \in \mathbb{N}. \tag{3.5}$$

Since for all $\beta \in [0,1)$ and $\lambda n \ge 0$ we have

$$R(\beta,\lambda n) = -9\lambda^2 n^2 + 4(\beta+8)(\beta-1)\lambda n - 32(\beta-1)^2 \le 0,$$

if we combine this inequality together with (3.5), from (3.4) it follows that $D(\beta, \lambda n) \geq 0$ for all $\lambda n \geq 0$ and $\beta \in [0,1)$, so the roots of H are real. If the roots of H are denoted by t_0 and t_1, then from the assumption (3.2) we have $t_0 t_1 > 0$, hence the equation $h'(t) = 0$ has one positive root t_0.

From the fact that $h'(t) \leq 0$ for $t \in [0, t_0]$ and $h'(t) \geq 0$ for $t \geq t_0$, we get that $h(t) \geq h(t_0)$ for all $t \geq 0$, and it follows that

$$|\psi(ix, y; z)|^2 \geq h(t_0),$$

for all $x, y \in \mathbb{R}$ such that $y \leq -n\left(1 + x^2\right)/2$ and $z \in \mathrm{U}$.

If we define the set $\Omega = \{\omega \in \mathbb{C} : |\omega| < \rho\}$, then $\psi(p(z), zp'(z); z) \in \Omega$ and $\psi(ix, y; z) \notin \Omega$ for all $x, y \in \mathbb{R}$ with $y \leq -n\left(1 + x^2\right)/2$ and for all $z \in \mathrm{U}$, hence by applying Lemma 1.4 we obtain our result. □

Remarks 3.1. 1. For the special $m = 1$ and $\lambda = 1$, the result was studied in [RaSeRa02].

2. For the special case $n = 1$, $\beta = 0$, $m = 1$ and $\lambda = 1$, we may easily obtain $t_0 = (\sqrt{73} - 1)/36$ and therefore we have the following result from [LiOw02]: if $f \in A$ satisfies

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| < \rho, \quad z \in \mathrm{U},$$

where $\rho = \sqrt{\frac{827 + 73\sqrt{73}}{288}}$, then $f \in S^*$.

4. Some applications of a result of M. Robertson

Now, by using Lemma 1.5, we will obtain a sufficient condition such that a function $f \in A$ belongs to $S^m(1, \lambda, \rho)$.

Theorem 4.1. *Let $\alpha < 1$, $\lambda \geq 0$ and $m \in \mathbb{N}$. Let $f \in A$, and suppose that the next two relations hold for all $0 \leq t \leq 1$:*

$$g(z) = \frac{1}{1 - \alpha} \left[D_\lambda^m f(z) - \alpha D_\lambda^{m-1} f(z) \right] \in S,$$

and

$$G(z, t) = \frac{1}{1 - \alpha} \left[(1 - t) D_\lambda^m f(z) - \alpha \left(1 - t^2\right) D_\lambda^{m-1} f(z) \right] \prec g(z).$$

Then $f \in S^m(1, \lambda, \rho(\lambda, \alpha, m))$, where $\rho(\lambda, \alpha, m) = \alpha + 1 - \lambda + \mu(\lambda, \alpha, m)$ and

$$\mu(\lambda, \alpha, m) = \inf \left\{ \alpha(\lambda - 1) \operatorname{Re} \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} : z \in \mathrm{U} \right\}. \tag{4.1}$$

Proof. It is easy to see that

$$G(z)=\lim_{t\to 0^+}\frac{G(z,t)-G(z,0)}{zt}=\frac{-D_\lambda^m f(z)}{(1-\alpha)z}$$

and

$$g'(z)=\frac{1}{1-\alpha}\left[\left(D_\lambda^m f(z)\right)'-\alpha\left(D_\lambda^{m-1} f(z)\right)'\right].$$

Furthermore, it follows that $G\in H(\mathrm{U})$ and $\operatorname{Re}G(0)=-1/(1-\alpha)\neq 0$.

Consequently, by using Lemma 1.5 for the special case $p=1$, together with the definitions (1.1) and (1.2), we obtain

$$\operatorname{Re}\frac{g'(z)}{G(z)}=\operatorname{Re}\left[\alpha+1-\lambda+\alpha(\lambda-1)\frac{D_\lambda^{m-1}f(z)}{D_\lambda^m f(z)}-\frac{D_\lambda^{m+1}f(z)}{D_\lambda^m f(z)}\right]<0,\quad z\in\mathrm{U},$$

and multiplying by $\lambda\geq 0$ we get

$$\operatorname{Re}\frac{D_\lambda^{m+1}f(z)}{D_\lambda^m f(z)}\geq\alpha+1-\lambda+\alpha(\lambda-1)\operatorname{Re}\frac{D_\lambda^{m-1}f(z)}{D_\lambda^m f(z)},\quad z\in\mathrm{U}.$$

If $\mu(\lambda,\alpha,m)$ is given by (4.1), the above inequality shows that $f\in S^m(1,\lambda,\rho(\lambda,\alpha,m))$, which completes the proof of the theorem. □

Remark 4.1. If we take in the above theorem $\lambda=1$ we have the result of Owa, Obradović and Lee from [OwObLe86], while for $\lambda=1$ and $m=0$ we have the result of Obradović obtained in [Ob83].

Theorem 4.2. *Let $\lambda>0$, $\alpha<1$ and $m\in\mathbb{N}_0$. If the function $f\in S^{m+1}(n,\lambda,\alpha)$, then*

$$\operatorname{Re}\left[\frac{D_\lambda^m f(z)}{z}\right]^\beta>\frac{n\lambda}{2\beta(1-\alpha)+n\lambda},\quad z\in\mathrm{U},\tag{4.2}$$

whenever $0<2\beta(1-\alpha)\leq\lambda n$. (The power in (4.2) is the principal one)

Proof. If $f\in S^{m+1}(n,\lambda,\alpha)$, according to the definition (1.4) and using (1.1) and (1.2), we have

$$1-\lambda+\lambda\operatorname{Re}\frac{z\left(D_\lambda^m f(z)\right)'}{D_\lambda^m f(z)}>\alpha,\quad z\in\mathrm{U}.$$

It follows that $D_\lambda^m f(z)\neq 0$ for all $z\in\dot{\mathrm{U}}\equiv\mathrm{U}\setminus\{0\}$, and combining this together with (1.3) we deduce that

$$\frac{D_\lambda^m f(z)}{z}\neq 0,\quad z\in\mathrm{U}.$$

Let now define the function p by

$$\left[\frac{D_\lambda^m f(z)}{z}\right]^\beta = (1-\mu)p(z) + \mu, \tag{4.3}$$

where

$$\frac{1}{2} \le \mu = \frac{n\lambda}{2\beta(1-\alpha)+n\lambda} < 1, \tag{4.4}$$

whenever $0 < 2\beta(1-\alpha) \le \lambda n$, $\lambda > 0$ and $\alpha < 1$. Then $p \in H(\mathrm{U})$ with $p(0) = 1$, and differentiating logarithmically both sides of (4.3) we obtain

$$\frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} - \alpha = \frac{\lambda z p'(z)}{\beta\big[p(z) + \frac{\mu}{1-\mu}\big]} + 1 - \alpha.$$

Using the fact $f \in S^{m+1}(n, \lambda, \alpha)$, this above relation shows that

$$\operatorname{Re} \frac{\lambda z p'(z)}{\beta\big[p(z) + \frac{\mu}{1-\mu}\big]} + 1 - \alpha > 0, \quad z \in \mathrm{U}, \tag{4.5}$$

and if define the function $\psi : \mathbb{C}^2 \times \mathrm{U} \to \mathbb{C}$ by

$$\psi(u, v; z) = \frac{\lambda v}{\beta\big(u + \frac{\mu}{1-\mu}\big)} + 1 - \alpha, \tag{4.6}$$

then (4.5) may be rewritten as $\operatorname{Re}\psi(p(z), zp'(z); z) > 0$, $z \in \mathrm{U}$.

From (4.6) it follows that ψ is continuous on the domain $D = \big(\mathbb{C} \setminus \big(-\frac{\mu}{1-\mu}\big)\big) \times \mathbb{C} \times \mathrm{U}$, $(1, 0; z) \in D$ and $\operatorname{Re}\psi(1, 0; z) = 1 - \alpha > 0$, for all $z \in \mathrm{U}$. Moreover, for all $(ix, y; z) \in D$ such that $x, y \in \mathbb{R}$ and $y \le -n\left(1 + x^2\right)/2$, a simple calculus combined with (4.4) shows that

$$\operatorname{Re}\psi(ix, y; z) \le -\frac{\lambda n}{2\beta} \cdot \frac{\mu}{1-\mu} \cdot \frac{x^2+1}{x^2 + \big(\frac{\mu}{1-\mu}\big)^2} + 1 - \alpha \le 0, \quad z \in \mathrm{U},$$

provided $0 < 2\beta(1-\alpha) \le \lambda n$, $\lambda > 0$ and $\alpha < 1$.

Consequently, the function ψ satisfies the conditions of Lemma 1.4 with $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$, and thus we deduce

$$\operatorname{Re} p(z) > 0, \quad z \in \mathrm{U}.$$

This inequality together with the relation (4.3) implies (4.2), and the proof is complete. $\square$

Remark 4.2. Taking in this theorem $\lambda = 1$ and $n = 1$ we obtain the result of Owa, Obradović and Lee from [OwObLe86], and letting $\lambda = 1$, $m = 0$ and $n = 1$ we obtain the result of Obradović [Ob83].

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(Received March 8, 2006; revised February 22, 2007)