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Geometrical structures on Lie algebroids

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Abstract. In this paper we study on the prolongation of Lie algebroid the notions such as: nonlinear connection, related connections, torsion and curvature, semispray and complex structures. The case of homogeneous connections and some examples are presented.

1. Introduction

The Lie algebroid is a generalization of the notions of Lie algebra and integrable distribution. A unitary study of Lie algebroids has realized by K. Mackenzie in [12]. A. Weinstein [15] developed a generalized theory of Lagrangian Mechanics on Lie algebroids and obtained the equations of motion, using the Poisson structure on the dual and Legendre transformation. Later, E. Martinez [11] developed the Klein's formalism on Lie algebroids using the notion of prolongation of Lie algebroid over a smooth map [7], and has proposed a modified version of symplectic formalism, in which the bundles tangent to E and E^* are replaced by the prolongations $\mathcal{T}E$ and $\mathcal{T}E^*$. The notion of nonlinear connection on Lie algebroid is a natural extension of the usual concept on the tangent bundle (see [4], [1]). In the last years diverse aspects of this topic have been studied in a number of works. In [2] a generalized notion of connection over a vector bundle map is presented. The nonlinear connection on the prolongation of Lie algebroid over the vector bundle projection of a dual bundle is investigated in [8]. In [13] a definition for torsion and curvature of a connection on affine Lie algebroid is given and the connection generated by a pseudo-SODE is pointed out.

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The purpose of the present paper is to study the nonlinear connection on the prolongation of Lie algebroid and its properties. The paper is organized as follows. In Section 2 the known results on Lie algebroid and its prolongation over the vector bundle projection are recalled. In section 3 we introduce the nonlinear connection on the prolongation Lie algebroid $\mathcal{T}E$ such that $\mathcal{T}E = V\mathcal{T}E \oplus H\mathcal{T}E$. We show that the vertical part of the bracket of horizontal sections from the basis represents the local coordinates of the curvature tensor of the connection. We study the related connections and show that a connection on TE generates a connection on $\mathcal{T}E$. We introduce an almost complex structure and prove that its integrability is characterized by a zero torsion and curvature property of the connection. The nonlinear connection generated by a semispray and its properties are studied, and in the homogeneous case a canonical connection associated to a Finsler function is determined.

2. Preliminaries on Lie algebroids

2.1. The notion of Lie algebroid. Let M be a differentiable, n-dimensional manifold and (TM, π_M, M) its tangent bundle. Let (E, π, M) be a vector bundle with the dimension of type fibre m. A Lie algebroid over a manifold M (see [12]) is a vector bundle (E, π, M) equipped with a Lie algebra structure $[\ ,\ ]$ on its space of sections, denoted $\Gamma(E)$, and a map $\sigma : E \to TM$ (called the anchor) which induces a Lie algebra homomorphism (also denoted σ) from sections of E to vector fields on M, satisfying the Leibniz rule

$$[s_1, fs_2] = f[s_1, s_2] + (\sigma(s_1)f)s_2, \tag{1}$$

where $f \in C^\infty(M)$ and $s_1,\ s_2 \in \Gamma(E)$. Therefore, we have

$$[\sigma(s_1), \sigma(s_2)] = \sigma[s_1, s_2], \quad [s_1, [s_2, s_3]] + [s_2, [s_3, s_1]] + [s_3, [s_1, s_2]] = 0$$

and the triple $(E, [\ ,], \sigma)$ is called a Lie algebroid over M. If ω is a k-form, $\omega \in \bigwedge^k(E) = \Gamma((E^*)^k \to M$, then the exterior derivative $d\omega \in \bigwedge^{k+1}(E)$ is given by the formula

$$\begin{aligned} d\omega(s_1, \ldots, s_{k+1}) = & \sum_{i=1}^{k+1} (-1)^{i+1} \sigma(s_i) \omega(s_1, \ldots, \hat{s}_i, \ldots, s_{k+1}) \\ & + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([s_{i,} s_j], s_1, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_{k+1}). \end{aligned} \tag{2}$$

For $\xi \in \Gamma(E)$ can be defined the *Lie derivative* with respect to ξ given by $\mathcal{L}_\xi = i_\xi \circ d + d \circ i_\xi$. If we take the local coordinates (x^i) on an open $U \subset M$ and a local basis $\{s_\alpha\}$ of sections of the bundle $\pi^{-1}(U) \to U$, then we have the local coordinates (x^i, y^α) on E. These coordinates determine the local functions $\sigma^i_\alpha(x)$, $L^\gamma_{\alpha\beta}(x)$ on M given by

$$\sigma(s_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta] = L^\gamma_{\alpha\beta} s_\gamma, \quad i = \overline{1,n}, \ \alpha, \beta, \gamma = \overline{1,m}, \tag{3}$$

and satisfying the relations

$$\sigma^j_\alpha \frac{\partial \sigma^i_\beta}{\partial x^j} - \sigma^j_\beta \frac{\partial \sigma^i_\alpha}{\partial x^j} = \sigma^i_\gamma L^\gamma_{\alpha\beta}, \qquad \sum_{(\alpha,\beta,\gamma)} \left(\sigma^i_\alpha \frac{\partial L^\delta_{\beta\gamma}}{\partial x^i} + L^\delta_{\alpha\eta} L^\eta_{\beta\gamma} \right) = 0, \tag{4}$$

which are called the *structure equations* of Lie algebroid. In local coordinates the differential d is determined by $dx^i = \sigma^i_\alpha s^\alpha$, $ds^\alpha = -\frac{1}{2} L^\alpha_{\beta\gamma} s^\beta \wedge s^\gamma$, where $\{s^\alpha\}$ is the dual basis of $\{s_\alpha\}$ and we have the relations $d^2 x^i = 0$ and $d^2 s^\alpha = 0$. The differential of a function f on M is given by $df = \frac{\partial f}{\partial x^i} \sigma^i_\alpha s^\alpha$ and in particular we have $\dot{x}^i = \sigma^i_\alpha y^\alpha$.

2.2. The prolongation of a Lie algebroid. Let (E, π, M) be a vector bundle. For the projection $\pi : E \to M$ we can construct the prolongation of E (see [7], [9], [11]). The associated vector bundle is $(\mathcal{T}E, \pi_2, E)$ where $\mathcal{T}E = \cup_{w \in E} \mathcal{T}_w E$ with

$$\mathcal{T}_w E = \{(u_x, v_w) \in E_x \times T_w E \mid \sigma(u_x) = T_w \pi(v_w), \quad \pi(w) = x \in M\},$$

and the projection $\pi_2(u_x, v_w) = \pi_E(v_w) = w$, where $\pi_E : TE \to E$ is the tangent projection. We have also the canonical projection $\pi_1 : \mathcal{T}E \to E$ given by $\pi_1(u, v) = u$. The projection onto the second factor $\sigma^1 : \mathcal{T}E \to TE$, $\sigma^1(u, v) = v$ will be the anchor of a new Lie algebroid over manifold E. An element of $\mathcal{T}E$ is said to be vertical if it is in the kernel of the projection π_1. We will denote $(V\mathcal{T}E, \pi_{2|_{V\mathcal{T}E}}, E)$ the vertical bundle of $(\mathcal{T}E, \pi_2, E)$.

If $f \in C^\infty(M)$ we will denote by f^c and f^v the *complete and vertical lift* to E of f defined by

$$f^c(u) = \sigma(u)(f), \quad f^v(u) = f(\pi(u)), \quad u \in E.$$

For $s \in \Gamma(E)$ we can consider the *vertical lift* of s given by

$$s^v(u) = s(\pi(u))^v_u, \quad u \in E,$$

where ${}^v_u : E_{\pi(u)} \to T_u(E_{\pi(u)})$ is the canonical isomorphism. There exists a unique vector field s^c on E, the *complete lift* of s satisfying the two following conditions:

i) s^c is π-projectable on $\sigma(s)$,

ii) $s^c(\hat{\alpha}) = \widehat{\mathcal{L}_s\alpha}$

for all $\alpha \in \Gamma(E^*)$, where $\hat{\alpha}(u) = \alpha(\pi(u))(u)$, $u \in E$ (see [5], [6]).

Considering the prolongation $\mathcal{T}E$ of E over the projection π, we may introduce the *vertical lift* $s^{\mathbf{v}}$ and the *complete lift* $s^{\mathbf{c}}$ of a section $s \in \Gamma(E)$ as the sections of $\mathcal{T}E \to E$ given by (see [11])

$$s^{\mathrm{v}}(u) = (0, s^v(u)), \quad s^{\mathrm{c}}(u) = (s(\pi(u)), s^c(u)), \quad u \in E.$$

Other two canonical objects on $\mathcal{T}E$ are the *Euler section* $\mathcal{C}$ and *vertical endomorphism* $\mathcal{J}$. $\mathcal{C}$ is the section of $\mathcal{T}E \to E$ defined by $\mathcal{C}(u) = (0, u_u^v)$ for all $u \in E$ and $\mathcal{J}$ is the section of the bundle $(\mathcal{T}E) \oplus (\mathcal{T}E)^* \to E$ characterized by

$$J(s^{\mathrm{v}}) = 0, \quad J\,(s^{\mathrm{c}}) = s^{\mathrm{v}}, \quad s \in \Gamma(E).$$

The vertical endomorphism satisfies $J^2 = 0$, $\operatorname{im} J = \ker J = V\mathcal{T}E$ and is homogeneous of degree 0, that is $[\mathcal{C}, \mathcal{J}] = -\mathcal{J}$. Moreover, the Nijenhuis tensor of the vertical endomorphism vanishes (see [11]). Finally, a section ξ of $\mathcal{T}E \to E$ is called *semispray* (or *second order differential equation -SODE*) on E if $\mathcal{J}(\xi) = \mathcal{C}$. The local basis of $\Gamma(\mathcal{T}E)$ is given by $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$, where

$$\mathcal{X}_\alpha(u) = \left(s_\alpha(\pi(u)), \sigma^i_\alpha \frac{\partial}{\partial x^i}\bigg|_u \right), \quad \mathcal{V}_\alpha(u) = \left(0, \frac{\partial}{\partial y^\alpha}\bigg|_u\right)$$

and $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha})$ is the local basis on TE. The structure functions of $\mathcal{T}E$ are given by the following formulas

$$\sigma^1(\mathcal{X}_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \quad \sigma^1(\mathcal{V}_\alpha) = \frac{\partial}{\partial y^\alpha}, \tag{5}$$

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = L^\gamma_{\alpha\beta}\mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{V}_\beta] = 0, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0. \tag{6}$$

If V is a section of $\mathcal{T}E$ then in terms of basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ it is $V = Z^\alpha \mathcal{X}_\alpha + V^\alpha \mathcal{V}_\alpha$, and the vector field $\sigma^1(V) \in \chi(E)$ has the expression $\sigma^1(V) = \sigma^i_\alpha Z^\alpha \frac{\partial}{\partial x^i} + V^\alpha \frac{\partial}{\partial y^\alpha}$. The vertical lift of a section $\rho = \rho^\alpha s_\alpha$ and the corresponding vector field are $\rho^{\mathrm{v}} = \rho^\alpha \mathcal{V}_\alpha$ and $\sigma^1(\rho^{\mathrm{v}}) = \rho^\alpha \frac{\partial}{\partial y^\alpha}$. The coordinate expressions of $\mathcal{C}$ and $\sigma^1(\mathcal{C})$ are

$$\mathcal{C} = y^\alpha \mathcal{V}_\alpha, \quad \sigma^1(\mathcal{C}) = y^\alpha \frac{\partial}{\partial y^\alpha}$$

and the local expression of $\mathcal{J}$ is

$$\mathcal{J} = \mathcal{X}^\alpha \otimes \mathcal{V}_\alpha$$

where$\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$ denotes the corresponding dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$. The local expression of the differential of a function L on $\mathcal{T}E$ is $dL = \sigma^i_\alpha \frac{\partial L}{\partial x^i}\mathcal{X}^\alpha + \frac{\partial L}{\partial y^\alpha}\mathcal{V}^\alpha$, and therefore, we have $dx^i = \sigma^i_\alpha \mathcal{X}^\alpha$ and $dy^\alpha = \mathcal{V}^\alpha$. The differential of sections of $(\mathcal{T}E)^*$ is determined by

$$d\mathcal{X}^\alpha = -\frac{1}{2}L^\alpha_{\beta\gamma}\mathcal{X}^\beta \wedge \mathcal{X}^\gamma, \quad d\mathcal{V}^\alpha = 0.$$

In the local coordinates a semispray has the expression

$$\xi(x,y) = y^\alpha \mathcal{X}_\alpha + \xi^\alpha(x,y)\mathcal{V}_\alpha$$

and the associated vector field is $\sigma^1(\xi) = \sigma^i_\alpha y^\alpha \frac{\partial}{\partial x^i} + \xi^\alpha \frac{\partial}{\partial y^\alpha}$. The integral curves of $\sigma^1(\xi)$ satisfy the differential equations

$$\frac{dx^i}{dt} = \sigma^i_\alpha(x)y^\alpha, \quad \frac{dy^\alpha}{dt} = \xi^\alpha(x,y). \tag{7}$$

If $[\mathcal{C}, \xi] = \xi$ then ξ is called spray and ξ^α are homogeneous of degree 2 in y^α.

3. Nonlinear connections

As in the classical case $E = TM$ we can define the nonlinear connection.

Definition 3.1. A nonlinear connection (or connection) on $\mathcal{T}E$ is an almost product structure $\mathcal{N}$ on $\pi_2 : \mathcal{T}E \to E$ (i.e. a bundle morphism $\mathcal{N} : \mathcal{T}E \to \mathcal{T}E$, such that $\mathcal{N}^2 = \mathrm{id}$) smooth on $\mathcal{T}E \backslash \{0\}$ such that $V\mathcal{T}E = \ker(\mathrm{id} + \mathcal{N})$.

If $\mathcal{N}$ is a connection on $\mathcal{T}E$ then $H\mathcal{T}E = \ker(\mathrm{id} - \mathcal{N})$ is the horizontal subbundle associated to $\mathcal{N}$ and $\mathcal{T}E = V\mathcal{T}E \oplus H\mathcal{T}E$. Each $\rho \in \Gamma(\mathcal{T}E)$ can be written as $\rho = \rho^{\mathrm{h}} + \rho^{\mathrm{v}}$ where ρ^{h}, ρ^{v} are sections in the horizontal and respective vertical subbundles. If $\rho^{\mathrm{h}} = 0$ then ρ is called *vertical* and if $\rho^{\mathrm{v}} = 0$ then ρ is called *horizontal*. A connection $\mathcal{N}$ on $\mathcal{T}E$ induces two projectors $\mathrm{h}, \mathrm{v} : \mathcal{T}E \to \mathcal{T}E$ such that $\mathrm{h}(\rho) = \rho^{\mathrm{h}}$ and $\mathrm{v}(\rho) = \rho^{\mathrm{v}}$ for every $\rho \in \Gamma(\mathcal{T}E)$. We have

$$\begin{gathered} \mathrm{h} = \frac{1}{2}(\mathrm{id} + \mathcal{N}), \quad \mathrm{v} = \frac{1}{2}(\mathrm{id} - \mathcal{N}), \\ \ker \mathrm{h} = \operatorname{im} \mathrm{v} = V\mathcal{T}E, \quad \operatorname{im} \mathrm{h} = \ker \mathrm{v} = H\mathcal{T}E. \end{gathered} \tag{8}$$

Locally, a connection can be expressed as

$$\mathcal{N}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha - 2\mathcal{N}^\beta_\alpha \mathcal{V}_\beta, \quad \mathcal{N}(\mathcal{V}_\beta) = -\mathcal{V}_\beta, \tag{9}$$

where $\mathcal{N}_\alpha^\beta = \mathcal{N}_\alpha^\beta(x,y)$ are the local coefficients of $\mathcal{N}$. The sections

$$\delta_\alpha = \mathrm{h}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha - \mathcal{N}_\alpha^\beta \mathcal{V}_\beta, \tag{10}$$

generate a basis of $H\mathcal{T}E$. The frame $\{\delta_\alpha, \mathcal{V}_\alpha\}$ is a local basis of $\mathcal{T}E$ called *adapted*. The dual adapted basis is $\{\mathcal{X}^\alpha, \delta\mathcal{V}^\alpha\}$ where $\delta\mathcal{V}^\alpha = \mathcal{V}^\alpha - \mathcal{N}_\beta^\alpha \mathcal{X}^\beta$.

Proposition 3.1. *The Lie brackets of the adapted basis* $\{\delta_\alpha, \mathcal{V}_\alpha\}$ *are*

$$[\delta_\alpha, \delta_\beta] = L_{\alpha\beta}^\gamma \delta_\gamma + \mathcal{R}_{\alpha\beta}^\gamma \mathcal{V}_\gamma, \quad [\delta_\alpha, \mathcal{V}_\beta] = \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\beta}\mathcal{V}_\gamma, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0, \tag{11}$$

where

$$\mathcal{R}_{\alpha\beta}^\gamma = \sigma_\beta^i \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial x^i} - \sigma_\alpha^i \frac{\partial \mathcal{N}_\beta^\gamma}{\partial x^i} - \mathcal{N}_\beta^\varepsilon \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\varepsilon} + \mathcal{N}_\alpha^\varepsilon \frac{\partial \mathcal{N}_\beta^\gamma}{\partial y^\varepsilon} + L_{\alpha\beta}^\varepsilon \mathcal{N}_\varepsilon^\gamma. \tag{12}$$

Proof. Using (5) and (6) we get

$$[\delta_\alpha, \delta_\beta] = \left(\sigma_\beta^i \frac{\partial \mathcal{N}_\alpha^\varepsilon}{\partial x^i} - \mathcal{N}_\beta^\gamma \frac{\partial \mathcal{N}_\alpha^\varepsilon}{\partial y^\gamma} - \sigma_\alpha^i \frac{\partial \mathcal{N}_\beta^\varepsilon}{\partial x^i} + \mathcal{N}_\alpha^\gamma \frac{\partial \mathcal{N}_\beta^\varepsilon}{\partial y^\gamma}\right)\mathcal{V}_\varepsilon + L_{\alpha\beta}^\gamma \mathcal{X}_\gamma.$$

If we replace $\mathcal{X}_\gamma = \delta_\gamma + \mathcal{N}_\gamma^\varepsilon \mathcal{V}_\varepsilon$ then the first relation from (11) is obtained. □

We recall that the Nijenhuis tensor of an endomorphism A is given by

$$\mathrm{N}_A(z,w) = [Az, Aw] - A[Az, w] - A[z, Aw] + A^2[z,w] \tag{13}$$

Definition 3.2. The curvature of a connection $\mathcal{N}$ on $\mathcal{T}E$ is given by $\Omega = -\mathrm{N}_\mathrm{h}$ where h is horizontal projector and N_h is the Nijenhuis tensor of h.

Proposition 3.2. *In the local coordinates we have*

$$\Omega = -\frac{1}{2}\mathcal{R}_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma,$$

where $\mathcal{R}_{\alpha\beta}^\gamma$ *are given by (12) and represent the local coordinate functions of the curvature tensor* Ω *in the frame* $\bigwedge^2 \mathcal{T}E^* \otimes \mathcal{T}E$ *induced by* $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$.

Proof. Since $\mathrm{h}^2 = \mathrm{h}$ we obtain

$$\Omega(z,w) = -[\mathrm{h}z, \mathrm{h}w] + \mathrm{h}[\mathrm{h}z, w] + \mathrm{h}[z, \mathrm{h}w] - \mathrm{h}[z,w],$$

$$\Omega(\mathrm{h}z, \mathrm{h}w) = -\mathrm{v}[\mathrm{h}z, \mathrm{h}w], \quad \Omega(\mathrm{h}z, \mathrm{v}w) = \Omega(\mathrm{v}z, \mathrm{v}w) = 0$$

and in local coordinates we get

$$\Omega(\delta_\alpha, \delta_\beta) = -\mathrm{v}[\delta_\alpha, \delta_\beta] = -\mathcal{R}_{\alpha\beta}^\gamma \mathcal{V}_\gamma$$

which ends the proof. □

The curvature of nonlinear connection is an obstruction to the integrability of $H\mathcal{T}E$, understanding that a vanishing curvature entails that horizontal sections are closed under the Lie algebroid bracket of $\mathcal{T}E$.

Remark 3.1. $H\mathcal{T}E$ *is integrable if and only if the curvature vanishes.*

In the following we will study the related connections which permit us to find the relations between the coefficients of related nonlinear connections.

Let Ψ a morphism of vector bundles E and $\overline{E}$. We recall that the connections N on E and $\overline{N}$ on $\overline{E}$ are Ψ-related if $\Psi \circ N = \overline{N} \circ \Psi$. We consider the connections $\mathcal{N}$ on $\mathcal{T}E$ and $\mathbb{N}$ on TE which are σ^1-related and a connection N on T^2M which is σ_* -related with $\mathbb{N}$ on TE and $\widetilde{\sigma}$-related with $\mathcal{N}$ on $\mathcal{T}E$, where $\widetilde{\sigma} : \mathcal{T}E \to T^2M$ is given by $\widetilde{\sigma} = \sigma_* \circ \sigma^1$ and $\sigma_* : TE \to T^2M$ is the tangent application of σ. It follows

$$\mathbb{N} \circ \sigma^1 = \sigma^1 \circ \mathcal{N}, \quad N \circ \sigma_* = \sigma_* \circ \mathbb{N}, \quad N \circ \widetilde{\sigma} = \widetilde{\sigma} \circ \mathcal{N}, \tag{14}$$

Let us consider the adapted basis $\left(\overset{E}{\delta}_i, \frac{\partial}{\partial y^\beta}\right)$ of $\mathbb{N}$ and $\left(\overset{TM}{\delta_i}, \frac{\partial}{\partial y^j}\right)$ of N given by $\overset{E}{\delta}_i = \frac{\partial}{\partial x^i} - \mathbb{N}_i^\beta \frac{\partial}{\partial y^\beta}$ and $\overset{TM}{\delta_i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}$. Therefore, we get

$$\sigma_*\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} + \frac{\partial \sigma^k}{\partial x^i}\frac{\partial}{\partial y^k}, \quad \sigma_*\left(\frac{\partial}{\partial y^\alpha}\right) = \sigma_\alpha^i \frac{\partial}{\partial y^i}.$$

Theorem 3.1. *The following relations hold*

$$\begin{aligned}
\sigma^1(\delta_\alpha) &= \sigma_\alpha^i \overset{E}{\delta}_i, & \mathcal{N}_\alpha^\beta &= \sigma_\alpha^i \mathbb{N}_i^\beta, \\
\sigma_*(\overset{E}{\delta}_i) &= \overset{TM}{\delta_i}, & \frac{\partial \sigma^j}{\partial x^i} + N_i^j &= \mathbb{N}_i^\beta \sigma_\beta^j \\
\widetilde{\sigma}(\delta_\alpha) &= \sigma_\alpha^i \overset{TM}{\delta_i}, & \sigma_\alpha^i \frac{\partial \sigma^j}{\partial x^i} + \sigma_\alpha^i N_i^j &= \mathcal{N}_\alpha^\beta \sigma_\beta^j.
\end{aligned} \tag{15}$$

Proof. The first relation from (14) leads to the relation $\mathbb{N}(\sigma^1(\delta_\alpha)) = \sigma^1(\delta_\alpha)$ from which we get $\sigma^1(\delta_\alpha) = \sigma_\alpha^i \overset{E}{\delta}_i$ and $\mathcal{N}_\alpha^\beta = \sigma_\alpha^i \mathbb{N}_i^\beta$. In the similar way the others relations are obtained. □

Proposition 3.3. *For the curvature tensors of σ^1-related connections $\mathcal{N}$ and $\mathbb{N}$ we have the relation*

$$\mathcal{R}_{\alpha\beta}^\gamma = \sigma_\alpha^i \sigma_\beta^j \mathbb{R}_{ij}^\gamma, \tag{16}$$

where $\mathbb{R}_{ij}^\gamma = \overset{E}{\delta}_i(\mathbb{N}_j^\gamma) - \overset{E}{\delta}_j(\mathbb{N}_i^\gamma)$ *is the curvature tensor of nonlinear connection on* TE.

PROOF. Using the relation $\mathcal{N}_\alpha^\varepsilon = \sigma_\alpha^i \mathbb{N}_i^\varepsilon$ we obtain

$$\mathcal{R}_{\alpha\beta}^\gamma = \sigma_\beta^j \sigma_\alpha^i \left(\overset{E}{\delta}_j \left(\mathbb{N}_i^\gamma\right) - \overset{E}{\delta}_i \left(\mathbb{N}_j^\gamma\right) \right) + \mathbb{N}_j^\gamma \left(\sigma_\beta^i \frac{\partial \sigma_\alpha^j}{\partial x^i} - \sigma_\alpha^i \frac{\partial \sigma_\beta^j}{\partial x^i} \right) + L_{\alpha\beta}^\varepsilon \mathcal{N}_\varepsilon^\gamma$$

and from structure equations of Lie algebroid (4), the second term is given by $\mathbb{N}_j^\gamma \sigma_\varepsilon^j L_{\beta\alpha}^\varepsilon = -\mathcal{N}_\varepsilon^\gamma L_{\alpha\beta}^\varepsilon$ which concludes the proof. □

Remark 3.2. A σ^1-related connection $\mathbb{N}$ on TE determines a connection $\mathcal{N}$ on $\mathcal{T}E$ with the coefficients $\mathcal{N}_\alpha^\beta = \sigma_\alpha^i \mathbb{N}_i^\beta$ and curvature $\mathcal{R}_{\alpha\beta}^\gamma = \sigma_\alpha^i \sigma_\beta^j \mathbb{R}_{ij}^\gamma$. Converse, is not true, because σ is only injective.

Let $\mathcal{J}$ be the vertical endomorphism. We have the following result

Remark 3.3. Let $\mathcal{N}$ be a bundle morphism of $\pi_2 : \mathcal{T}E \to E$, smooth on $\mathcal{T}E \backslash \{0\}$. Then $\mathcal{N}$ is a connection on $\mathcal{T}E$ if and only if

$$\mathcal{J}\mathcal{N} = \mathcal{J}, \quad \mathcal{N}\mathcal{J} = -\mathcal{J}.$$

The proof proceeds as in the case $E = TM$ and will be omitted (see [10]).

Definition 3.3. The torsion of nonlinear connection $\mathcal{N}$ is the vector valued two form $t = [\mathcal{J}, \mathrm{h}]$ where h is the horizontal projector and $[.,.]$ is the Frolicher–Nijenhuis bracket.

Proposition 3.4. *t is a semibasic vector-valued form. Its local expression is*

$$t = \frac{1}{2} t_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma, \tag{17}$$

where

$$t_{\alpha\beta}^\gamma = \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\beta} - \frac{\partial \mathcal{N}_\beta^\gamma}{\partial y^\alpha} - L_{\alpha\beta}^\gamma. \tag{18}$$

PROOF. We have

$$\begin{aligned}[\mathcal{J}, \mathrm{h}](z, w) = {} & [\mathcal{J}z, \mathrm{h}w] + [\mathrm{h}z, \mathcal{J}w] + \mathcal{J}[z, w] - \mathcal{J}[z, \mathrm{h}w] - \mathcal{J}[\mathrm{h}z, w] \\ & - \mathrm{h}[z, \mathcal{J}w] - \mathrm{h}[\mathcal{J}z, w]\end{aligned}$$

and in local coordinates we get

$$t(\mathcal{X}_\alpha, \mathcal{X}_\beta) = \left(\frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\beta} - \frac{\partial \mathcal{N}_\beta^\gamma}{\partial y^\alpha} - L_{\alpha\beta}^\gamma \right) \mathcal{V}_\gamma, \quad t(\mathcal{X}_\alpha, \mathcal{V}_\beta) = t(\mathcal{V}_\alpha, \mathcal{V}_\beta) = 0. \quad \square$$

Now, let us consider the linear mapping $\mathbb{F} : \mathcal{T}E \to \mathcal{T}E$, defined by

$$\mathbb{F}(\mathrm{h}z) = -\mathrm{v}z, \quad \mathbb{F}(\mathrm{v}z) = \mathrm{h}z,$$

for $z \in \Gamma(\mathcal{T}E)$ and h, v are the horizontal and vertical projectors of nonlinear connection on $\mathcal{T}E$.

Proposition 3.5. *The mapping $\mathbb{F}$ has the properties:*

i) $\mathbb{F}$ *is globally defined on* $\mathcal{T}E$

ii) *Locally it is given by*

$$\mathbb{F} = -\mathcal{V}_\alpha \otimes \mathcal{X}^\alpha + \delta_\alpha \otimes \delta\mathcal{V}^\alpha. \tag{19}$$

iii) $\mathbb{F}$ *is an almost complex structure* $\mathbb{F} \circ \mathbb{F} = -\,\mathrm{id}$.

Proof. It results by definition that $\mathbb{F}$ is globally defined and

$$(\mathbb{F} \circ \mathbb{F})(\mathrm{h}z) = \mathbb{F}(-\mathrm{v}z) = -\mathrm{h}z, \quad (\mathbb{F} \circ \mathbb{F})(\mathrm{v}z) = \mathbb{F}(\mathrm{h}z) = -\mathrm{v}z.$$

In local coordinates we get $\mathbb{F}(\delta_\alpha) = -\mathcal{V}_\alpha$ and $\mathbb{F}(\mathcal{V}_\alpha) = \delta_\alpha$ which ends the proof. □

Proposition 3.6. *The almost complex structure is integrable if and only if the nonlinear connection is locally flat, that is the curvature and torsion vanish.*

Proof. Let $\mathrm{N}_\mathbb{F}$ be the Nijenhuis tensor of the almost complex structure. We find

$$\begin{aligned} \mathrm{N}_\mathbb{F}(\delta_\alpha, \delta_\beta) &= t^\gamma_{\alpha\beta}\delta_\gamma - \mathcal{R}^\gamma_{\alpha\beta}\mathcal{V}_\gamma \\ \mathrm{N}_\mathbb{F}(\delta_\alpha, \mathcal{V}_\beta) &= -\mathcal{R}^\gamma_{\alpha\beta}\delta_\gamma - t^\gamma_{\alpha\beta}\mathcal{V}_\gamma \\ \mathrm{N}_\mathbb{F}(\mathcal{V}_\alpha, \mathcal{V}_\beta) &= -\mathrm{N}_\mathbb{F}(\delta_\alpha, \delta_\beta). \end{aligned} \tag{20}$$

From (20) one reads immediately that $\mathrm{N}_\mathbb{F} = 0$ if and only if $t = 0$ and $\Omega = 0$. □

A curve $u : [t_0, t_1] \to E$ is called *admissible* if $\sigma(u(t)) = \dot{c}(t)$ where $c(t) = \pi(u(t))$ is the base curve. A nonlinear connection on $\mathcal{T}E$ induces a covariant derivative of the sections defined locally as follows

$$\mathcal{D}_\rho \eta = \rho^\alpha \left(\sigma^i_\alpha \frac{\partial \eta^\beta}{\partial x^i} + \mathcal{N}^\beta_\alpha \right) s_\beta,$$

where $\rho = \rho^\alpha s_\alpha$ and $\eta = \eta^\alpha s_\alpha$. The derivative is linear in the first argument and respect multiplication of second argument by real numbers, but not necessarily sums, except the case when the coefficients $\mathcal{N}^\beta_\alpha$ are the local coefficients of a linear connection. The linearity in the first argument permit us to define the derivative of a section $\eta \in \Gamma(E)$ with respect to $a \in E_u$ by setting

$$\mathcal{D}_a \eta = (\mathcal{D}_\rho \eta)(u),$$

where $\rho \in \Gamma(E)$ satisfying $\rho(u) = a$. Also, the covariant derivative allows us to take the derivative of sections along curves (see [3]). If we have a morphism of Lie algebroids ([12]) $\Phi : F \to E$ over the map $\varphi : N \to M$ and a section $\eta : N \to E$ along φ, i.e. $\eta(n) \in E_{\varphi(n)}$, $n \in N$, then η can be written in the form

$$\eta = \sum_{l=1}^{p} F_l(\xi_l \circ \varphi),$$

for some sections $\{\xi_1, \ldots, \xi_p\}$ of E and some functions $\{F_1, \ldots, F_p) \in C^\infty(N)$ and the derivative of η along φ is given by

$$\mathcal{D}_b \eta = \sum_{l=1}^{p} [(\sigma_F(b) F_l) \xi_l(\varphi(n)) + F_l(n) \mathcal{D}_{\Phi(b)} \xi_l], \quad b \in F_n$$

where σ_F is the anchor map of the Lie algebroid $F \to N$ (see [3]).

Let $a : I \to E$ be an admissible curve and let $b : I \to E$ be a curve in E, both of them projecting by π onto the same curve γ in M, $\pi(a(t)) = \pi(b(t)) = \gamma(t)$. Take the particular case of Lie algebroid structure $TI \to I$ and the morphism $\Phi : TI \to E$, $\Phi(t, \dot{t}) = \dot{t}\gamma(t)$ over $\gamma : I \to M$. Then one can define the derivative of $b(t)$ along $a(t)$ as $\mathcal{D}_{d/dt} b(t)$. In local coordinates, we obtain

$$\mathcal{D}_{a(t)} b(t) = \left(\frac{db^\beta}{dt} + \mathcal{N}_\alpha^\beta a^\alpha \right) s_\beta(\gamma(t)).$$

Definition 3.4. An admissible curve $c(t)$ is a path (autoparallel) for nonlinear connection $\mathcal{N}$ if and only if $\mathcal{D}_{c(t)} c(t) = 0$.

In local coordinates we get

$$\frac{dc^\beta}{dt} + \mathcal{N}_\alpha^\beta(x, y) c^\alpha = 0.$$

From the previous considerations we have

Proposition 3.7. *An admissible curve $c(t)$ in E is autoparallel for the nonlinear connection if and only if*

$$\frac{dx^i}{dt} = \sigma_\alpha^i y^\alpha, \quad \frac{dy^\beta}{dt} + \mathcal{N}_\alpha^\beta y^\alpha = 0, \tag{21}$$

where $x^i = x^i(t) = x^i(c(t))$, $y^\alpha = y^\alpha(t) = y^\alpha(c(t))$, $\sigma_\alpha^i = \sigma_\alpha^i(t) = \sigma_\alpha^i(c(t))$.

Let $\mathcal{N}$ be a nonlinear connection on $\mathcal{T}E$, ξ' an arbitrary semispray on $\mathcal{T}E$ and h the horizontal projector of $\mathcal{N}$. We consider $\xi = \mathrm{h}\xi'$ and for any other semispray ξ'' on $\mathcal{T}E$ we have $\mathrm{h}(\xi' - \xi'') = \mathrm{h}((\xi'^{\alpha} - \xi''^{\alpha})\mathcal{V}_{\alpha}) = 0$ and it results that ξ doesn't depends on the choose of ξ'. We have $\mathcal{J}\xi = \mathcal{J}\mathrm{h}\xi' = \mathcal{J}\xi' = \mathcal{C}$ so ξ is a semispray, which is called the *associated semispray* to $\mathcal{N}$.

Proposition 3.8. *A nonlinear connection $\mathcal{N}$ and its associated semispray have the same paths.*

Proof. For the arbitrary semispray $\xi' = y^{\alpha}\mathcal{X}_{\alpha} + \xi'^{\alpha}\mathcal{V}_{\alpha}$, the associated semispray of $\mathcal{N}$ is $\xi = \mathrm{h}\xi' = y^{\alpha}\mathcal{X}_{\alpha} - \mathcal{N}_{\alpha}^{\beta}y^{\alpha}\mathcal{V}_{\beta}$ so $\xi^{\beta} = -\mathcal{N}_{\alpha}^{\beta}y^{\alpha}$. From (7) and (21) it results the conclusion. □

Remark 3.4. If ξ is a semispray on $\mathcal{T}E$, then we have

$$\mathcal{J}[\xi, \mathcal{J}z] = -\mathcal{J}z, \quad z \in \Gamma(\mathcal{T}E). \tag{22}$$

The proof follows the classical case $E = TM$ (see [10]).

Example 3.1. Let $\mathcal{L}$ be a regular Lagrangian on E. One can associate to $\mathcal{L}$ a remarkable semispray locally given by

$$\xi^{\varepsilon} = g^{\varepsilon\beta}\left(\sigma_{\beta}^{i}\frac{\partial\mathcal{L}}{\partial x^{i}} - \sigma_{\alpha}^{i}\frac{\partial^{2}\mathcal{L}}{\partial x^{i}\partial y^{\beta}}y^{\alpha} - L_{\beta\alpha}^{\theta}y^{\alpha}\frac{\partial\mathcal{L}}{\partial y^{\theta}}\right), \tag{23}$$

where $g_{\alpha\beta} = \frac{\partial^{2}\mathcal{L}}{\partial y^{\alpha}\partial y^{\beta}}$ and $g_{\alpha\beta}g^{\varepsilon\beta} = \delta_{\alpha}^{\varepsilon}$ (see [11]).

Theorem 3.2. *Let $\mathcal{J}$ be the vertical endomorphism on $\mathcal{T}E$. If ξ is a semispray then*

$$\mathcal{N} = -\mathcal{L}_{\xi}\mathcal{J}, \tag{24}$$

is a connection on $\mathcal{T}E$.

Proof. Since $\mathcal{N}(v) = -\mathcal{L}_{\xi}\mathcal{J}(v) = -[\xi, \mathcal{J}v] + \mathcal{J}[\xi, v]$ then using the Remark 3.4 we get $\mathcal{J}\mathcal{N}(v) = -\mathcal{J}[\xi, \mathcal{J}v] + \mathcal{J}^{2}[\xi, v] = \mathcal{J}v$ and $\mathcal{N}\mathcal{J}(v) = -[\xi, \mathcal{J}^{2}v] + \mathcal{J}[\xi, \mathcal{J}v] = -\mathcal{J}v$. By using the Remark 3.3 we get the proof of the theorem. □

Remark 3.5. The connection $\mathcal{N} = -\mathcal{L}_{\xi}\mathcal{J}$ is induced by $\mathcal{J}$ and ξ. Its local coefficients are given by

$$\mathcal{N}_{\alpha}^{\beta} = \frac{1}{2}(-\frac{\partial\xi^{\beta}}{\partial y^{\alpha}} + y^{\varepsilon}L_{\alpha\varepsilon}^{\beta}) \tag{25}$$

Proof. $\mathcal{N}(\mathcal{X}_\alpha) = -[\xi, \mathcal{J}(\mathcal{X}_\alpha)] + \mathcal{J}[\xi, \mathcal{X}_\alpha] = \mathcal{X}_\alpha + \frac{\partial \xi^\beta}{\partial y^\alpha}\mathcal{V}_\beta + \mathcal{J}(y^\beta L^\gamma_{\beta\alpha}\mathcal{X}_\gamma - \sigma^i_\alpha \frac{\partial \xi^\beta}{\partial x^i}\mathcal{V}_\beta) = \mathcal{X}_\alpha + \left(\frac{\partial \xi^\beta}{\partial y^\alpha} + y^\gamma L^\beta_{\gamma\alpha}\right)\mathcal{V}_\beta$ and using the relation (9) we obtain (25). □

Proposition 3.9. *The torsion of the connection $\mathcal{N} = -\mathcal{L}_\xi \mathcal{J}$ vanishes.*

Proof. We have $t = [\mathcal{J}, h] = \frac{1}{2}\left([\mathcal{J}, \mathrm{id}] + \left[\mathcal{J}, -[\xi, \mathcal{J}]\right]\right) = \frac{1}{2}\left[\mathcal{J}, [\mathcal{J}, \xi]\right]$. Using Jacobi identity we obtain that $t = 0$. Also, if we replace (25) into the relation (18), by direct computation, the same result is obtained. □

Proposition 3.10. *The associated semispray of $\mathcal{N} = -\mathcal{L}_\xi \mathcal{J}$ is given by $\frac{1}{2}(\xi - [\xi, \mathcal{C}])$.*

Proof. The associated semispray is

$$\mathrm{h}\xi = \frac{1}{2}\xi + \frac{1}{2}\mathcal{N}(\xi) = \frac{1}{2}(\xi - [\xi, \mathcal{J}\xi] + \mathcal{J}[\xi, \xi]) = \frac{1}{2}(\xi - [\xi, \mathcal{C}]).$$ □

3.1. Homogeneous connections. The morphism $\mathcal{T} = \frac{1}{2}\mathcal{L}_\mathcal{C}\mathcal{N}$ is called the *tension* of the nonlinear connection, where $\mathcal{L}$ is the Lie derivative $\mathcal{L}_\mathcal{C}\mathcal{N}(Z) = [\mathcal{C}, \mathcal{N}Z] - \mathcal{N}[\mathcal{C}, Z]$. In the local coordinates we get

$$\mathcal{T}(\mathcal{X}_\alpha) = \left(\mathcal{N}^\beta_\alpha - \frac{\partial \mathcal{N}^\beta_\alpha}{\partial y^\gamma}y^\gamma\right)\mathcal{V}_\beta, \quad \mathcal{T}(\mathcal{V}_\alpha) = 0 \Rightarrow \mathcal{T} = \left(\mathcal{N}^\beta_\alpha - \frac{\partial \mathcal{N}^\beta_\alpha}{\partial y^\gamma}y^\gamma\right)\mathcal{X}^\alpha \otimes \mathcal{V}_\beta.$$

It is obvious that $\mathcal{T}$ is vanishing if and only if the nonlinear connection is homogeneous of degree 1 with respect to y^α.

Proposition 3.11. *If ξ is a spray then $\mathcal{N} = -\mathcal{L}_\xi \mathcal{J}$ is a homogeneous nonlinear connection.*

Proof. Using (25) we get $\mathcal{T} = \left(-\frac{\partial \xi^\gamma}{\partial y^\alpha} + y^\beta \frac{\partial^2 \xi^\gamma}{\partial y^\alpha \partial y^\beta}\right)\mathcal{X}^\alpha \otimes \mathcal{V}_\gamma$. But ξ is a spray and it results that ξ^γ is homogeneous of degree 2, so $2\xi^\gamma = y^\beta \frac{\partial \xi^\gamma}{\partial y^\beta}$ and $\frac{\partial \xi^\gamma}{\partial y^\alpha} = y^\beta \frac{\partial^2 \xi^\gamma}{\partial y^\alpha \partial y^\beta}$ and therefore the tension $\mathcal{T} = 0$. □

Definition 3.5. A function $\mathcal{F} : E \to [0, \infty]$ which satisfies the following properties:

1) $\mathcal{F}$ is C^∞ on $E\backslash\{0\}$
2) $\mathcal{F}(\lambda u) = \lambda \mathcal{F}(u)$ for $\lambda > 0$ and $u \in E_x$, $x \in M$
3) For each $y \in E_x\backslash\{0\}$ the quadratic form

$$g_{\alpha\beta}(x, y) = \frac{1}{2}\frac{\partial^2 \mathcal{F}^2}{\partial y^\alpha \partial y^\beta} \tag{26}$$

is positive definite, will be called the Finsler function on Lie algebroid.

If we replace $\mathcal{L} = \frac{1}{2}\mathcal{F}^2 = \frac{1}{2}g_{\alpha\beta}y^\alpha y^\beta$ into the expression of semispray (23) we obtain:

Remark 3.6. A homogeneous nonlinear connections has the coefficients given by (25) with

$$\xi^\delta = \frac{1}{2}g^{\delta\beta}\left(\sigma_\alpha^i\frac{\partial g_{\beta\gamma}}{\partial x^i} + \sigma_\gamma^i\frac{\partial g_{\alpha\beta}}{\partial x^i} - \sigma_\beta^i\frac{\partial g_{\alpha\gamma}}{\partial x^i} + g_{\varepsilon\alpha}L_{\beta\gamma}^\varepsilon + g_{\varepsilon\gamma}L_{\beta\alpha}^\varepsilon - g_{\varepsilon\beta}L_{\gamma\alpha}^\varepsilon\right)y^\alpha y^\gamma$$

and is called the canonical nonlinear connection associated to a Finsler function.

Remark 3.7. In the particular case of standard Lie algebroid $E = TM$ and $\sigma = \mathrm{id}$ the Cartan nonlinear connection (see [14]) is obtained.

Let us consider the canonical nonlinear connection and $\|y\|^2 = g_{\alpha\beta}y^\alpha y^\beta = \mathcal{F}^2$ is the square of the norm of the *Euler* section. The almost complex structure, characterized by (19), does not preserve the property of homogeneity of the sections. Indeed, it applies the 1-homogeneous section δ_α onto the 0-homogeneous section $\mathcal{V}_\alpha$, $\alpha \in \overline{1,m}$. We can solve this problem by defining a new kind of almost complex structure $\mathbb{F}_0 : \mathcal{T}E \to \mathcal{T}E$ given by

$$\mathbb{F}_0(\delta_\alpha) = -\frac{\mathcal{F}}{a}\mathcal{V}_\alpha, \quad \mathbb{F}_0(\mathcal{V}_\alpha) = \frac{a}{\mathcal{F}}\delta_\alpha, \quad a > 0.$$

It is not difficult to prove that $\mathbb{F}_0^2 = -\mathrm{id}$ and $\mathbb{F}_0$ preserves the property of the homogeneity of the sections.

Theorem 3.3. *The almost complex structure $\mathbb{F}_0$ is integrable if and only if the following relations hold*

$$\begin{aligned} \mathcal{R}_{\alpha\beta}^\gamma &= \frac{1}{a^2}\left(y_\alpha\delta_\beta^\gamma - y_\beta\delta_\alpha^\gamma\right), \\ \delta_\alpha(\mathcal{F}^2)\delta_\beta^\gamma &= \delta_\beta(\mathcal{F}^2)\delta_\alpha^\gamma, \end{aligned} \tag{27}$$

where $y_\alpha = g_{\alpha\beta}y^\beta$, $\alpha,\beta,\gamma = \overline{1,m}$.

Proof. For the Nijenjuis tensor $\mathrm{N}_{\mathbb{F}_0}$ we have

$$\begin{aligned} \mathrm{N}_{\mathbb{F}_0}(\delta_\alpha,\delta_\beta) = &\left(t_{\alpha\beta}^\gamma + \frac{1}{2\mathcal{F}^2}\left(\delta_\beta(\mathcal{F}^2)\delta_\alpha^\gamma - \delta_\alpha(\mathcal{F}^2)\delta_\beta^\gamma\right)\right)\delta_\gamma \\ &+ \left(\frac{1}{a^2}(y_\alpha\delta_\beta^\gamma - y_\beta\delta_\alpha^\gamma) - \mathcal{R}_{\alpha\beta}^\gamma\right)\mathcal{V}_\gamma, \end{aligned}$$

$$\begin{aligned}
\mathrm{N}_{\mathbb{F}_0}(\delta_\alpha, \mathcal{V}_\beta) &= \left(\frac{1}{\mathcal{F}^2}(y_\alpha \delta_\beta^\gamma - y_\beta \delta_\alpha^\gamma) - \frac{a^2}{\mathcal{F}^2}\mathcal{R}_{\alpha\beta}^\gamma\right)\delta_\gamma \\
&\quad - \left(t_{\alpha\beta}^\gamma + \frac{1}{2\mathcal{F}^2}\left(\delta_\beta(\mathcal{F}^2)\delta_\alpha^\gamma - \delta_\alpha(\mathcal{F}^2)\delta_\beta^\gamma\right)\right)\mathcal{V}_\gamma, \\
\mathrm{N}_{\mathbb{F}_0}(\delta_\alpha, \delta_\beta) &= -\frac{\mathcal{F}^2}{a^2}\mathrm{N}_{\mathbb{F}_0}(\mathcal{V}_\alpha, \mathcal{V}_\beta).
\end{aligned}$$

It follows that $\mathrm{N}_{\mathbb{F}_0} = 0$ if and only if the relations (27) are satisfied. □

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