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On Hölder continuous solutions of functional equations

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Abstract. In this work it is proved that the real solutions f of the functional equation

$$f(t) = h(t, y, f(y), f(g_1(t, y)), \dots, f(g_n(t, y))),$$

that are locally Hölder continuous with some exponent $0 < \alpha < 1$, are locally Hölder continuous with all exponent $\alpha,\ 0 < \alpha < 1$.

As it is treated in ACZÉL's classical book [1961], regularity is very important in the theory and practice of functional equations. The regularity problem of functional equations with two variables can be formulated as follows (see ACZÉL [1984] and JÁRAI [1986]):

Problem. Let T and Z be open subsets of $\mathbb{R}^s$ and $\mathbb{R}^m$, respectively, and let D be an open subset of $T \times T$. Let $f : T \to Z$, $g_i : D \to T$ $(i = 1, 2, \dots, n)$ and $h : D \times Z^{n+1} \to Z$ be functions. Suppose that

(1) $\quad f(t) = h(t, y, f(y), f(g_1(t, y)), \dots, f(g_n(t, y)))$ whenever $(t, y) \in D$;

(2) h is analytic;

(3) g_i is analytic and for each $t \in T$ there exists a y for which $(t, y) \in D$ and $\dfrac{\partial g_i}{\partial y}(t, y)$ has rank s $(i = 1, 2, \dots, n)$.

Is it true that every f, which is measurable or has the Baire property is analytic?

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The following steps may be used:

(I) Measurability implies continuity.
(II) Almost open solutions are continuous.
(III) Continuous solutions are locally Lipschitz.
(IV) Locally Lipschitz solutions are continuously differentiable.
(V) All p times continuously differentiable solutions are $p+1$ times continuously differentiable.
(VI) Infinitely many times differentiable solutions are analytic.

The complete answer to this problem is unknown. The problems corresponding to (I), (II), (IV) and (V) are solved in Járai [1986]. In the same paper, partial results in connection with (III) are treated. A partial result in connection with (VI) is treated in Járai [1988] (in Hungarian).

In this paper we deal with locally Hölder continuous real solution. The result is a new step in (III). The main tool is the fundamental lemma of the theory of Campanato spaces (Lemma 1), which is a generalization of the famous classical Morrey lemma from the regularity theory of partial differential equations. For further references about this lemma see Zeidler's book [1990], II/A pp. 90–93.

A real function f is called locally Hölder continuous with exponent $0<\alpha\le 1$, if each point of its domain has a neighbourhood V such that

$$\sup_{x,y\in V} |f(x)-f(y)| \,/\, |x-y|^{\alpha} < \infty.$$

Any constant not less then this supremum is called a (local) Hölder-constant for f. In the case $\alpha=1$ Hölder continuous functions and Hölder constants are also called Lipschitz functions and Lipschitz constants, respectively. It is well-known, that continuously differentiable functions are locally Lipschitz.

Lemma 1. *Let G be a nonempty open set in $\mathbb{R}^n$. Let $\mathbf{B}_r(y)$ denote the closed ball with center y and radius r, and define the mean value $\overline{f}_{y,r}$ of the real valued function f by*

$$\bar{f}_{y,r} = \frac{1}{\operatorname{meas}\mathbf{B}_r(y)} \int_{\mathbf{B}_r(y)} f(x)\,dx.$$

Let $0<\alpha\le 1$, $1\le p<\infty$, and $r_0>0$ be given. Then the inequality

$$\int_{\mathbf{B}_r(y)} |f(x)-\bar{f}_{y,r}|^p\,dx \le \operatorname{const} r^{n+p\alpha}$$

for all $r<\min(r_0, \operatorname{dist}(y,\partial G))$ and all $y\in G$ implies that f is locally Hölder continuous with exponent α on G.

Lemma 2. *Let V, W and U be open real intervals, $r, R>0$, $[t_0 - r, t_0 + r] \subset V$, $[y_0 - R, y_0 + R] \subset W$, $g : V \times W \to U$ a continuously differentiable function, and $f : U \to \mathbb{R}$ a continuous function. Suppose that all partial functions $y \mapsto g(t, y)$ are monotonic with inverse denoted by $x \mapsto G_t(x)$. If there exist constants B, B', L and L' such that $|f(x)| \leq B, |G'_t(x)| \leq B'$, $|g(t, y) - g(t', y')| \leq L(|t - t'| + |y - y'|)$ and $|G'_t(x) - G'_{t'}(x)| \leq L'|t - t'|$ whenever $|t - t_0| \leq r$, $|t' - t_0| \leq r$ and the left hand sides are defined, then the absolute value of the integral*

$$\int_{t_0-r}^{t_0+r} \int_{y_0-R}^{y_0+R} f(g(t, y)) - f(g(t', y))\, dy\, dt'$$

is bounded by $8LBB'r^2 + 8LBL'r^2(r + R)$ whenever $|t - t_0| \leq r$.

Proof. In the integral above the inner integral can be written as the difference of two integrals. Using the substitution $x = g(t, y)$ in the first, and the substitution $x = g(t', y)$ in the second integral respectively, we get

$$\int_{t_0-r}^{t_0+r} \left(\int_{g(t,y_0-R)}^{g(t,y_0+R)} f(x)G'_t(x)\, dx - \int_{g(t',y_0-R)}^{g(t',y_0+R)} f(x)G'_{t'}(x) dx \right) dt'.$$

The integrand of the outer integral can be rewritten as

$$\int_{g(t,y_0-R)}^{g(t',y_0-R)} f(x)G'_t(x)\, dx + \int_{g(t',y_0-R)}^{g(t,y_0+R)} f(x)(G'_t(x) - G'_{t'}(x))\, dx$$
$$+ \int_{g(t',y_0+R)}^{g(t,y_0+R)} f(x)G'_{t'}(x)\, dx.$$

The first and the last term can be estimated by $L|t - t'|BB'$, and the middle term by $L(2r + 2R)BL'|t - t'|$. Using that $|t - t'| \leq 2r$, we get the stated result.

Theorem. *Let $0 < \alpha < 1$. Let $T, Y, X_1, \ldots, X_n$ and $Z_1, Z_2, \ldots, Z_n$ be open subsets of $\mathbb{R}$, D an open subset of $T \times Y$. Consider the functions $f : T \to \mathbb{R}$, $f_i : X_i \to Z_i$ $(i = 1, \ldots, n)$, $g_i : D \to X_i$ $(i = 1, \ldots, n)$, $h : D \times Z_1 \times Z_2 \times \cdots \times Z_n \to \mathbb{R}$. Suppose, that*

(1) *for each $(t, y) \in D$,*

$$f(t) = h(t, y, f_1(g_1(t, y)), \ldots, f_n(g_n(t, y)));$$

(2) *h is twice continuously differentiable;*

(3) *g_i is twice continuously differentiable on D and for each $t \in T$ there exists a y such that $(t, y) \in D$ and $\dfrac{\partial g_i}{\partial y}(t, y) \neq 0$ for $i = 1, \ldots, n$;*

(4) *the functions $f_i, i = 1, \ldots, n$ are locally Hölder continuous with exponent α.*

Then f is locally Hölder continuous with exponent $2\alpha/(\alpha+1)$.

Proof. We have to prove that for each point $t_0 \in T$ the function f is Hölder continuous on a neighbourhood of t_0 with exponent $2\alpha/(1+\alpha)$. Let us choose y_0 by (3) for t_0. For an arbitrary set $V \subset \mathbb{R}$ let V_ε denote the ε-neighbourhood

$$V_\varepsilon = \{x : |x-y| < \varepsilon \text{ for some } y \in V\}$$

of V. Let V and W be open intervals containing t_0 and y_0 respectively, and $0 < \varepsilon \le 1$ such that $V_\varepsilon \times W_\varepsilon \subset D$ and $\dfrac{\partial g_i}{\partial y}$ does not vanish on $V_\varepsilon \times W_\varepsilon$. Hence the partial functions $y \mapsto g_i(t,y)$ have inverse on W_ε for all $t \in V_\varepsilon$ and $i = 1,2,\dots,n$. Decreasing V, W and ε if necessary we may suppose that these inverses have derivatives bounded (in absolute value) by B' and are Lipschitz continuous with Lipschitz constant L' for $i = 1,2,\dots,n$. Similarly, we may suppose that g_i is a Lipschitz function with Lipschitz constant L on $V_\varepsilon \times W_\varepsilon$, that f_i is Hölder continuous with exponent α and Hölder constant H and $|f_i|$ bounded by B on $g_i(V_\varepsilon \times W_\varepsilon)$ $(i = 1,2,\dots,n)$, moreover on

$$V_\varepsilon \times W_\varepsilon \times f_1(g_1(V_\varepsilon \times W_\varepsilon)) \times \cdots \times f_n(g_n(V_\varepsilon \times W_\varepsilon))$$

the functions $\dfrac{\partial h}{\partial z_i}$ are Lipschitz continuous with Lipschitz constant L_i', and the functions $\left|\dfrac{\partial h}{\partial t}\right|$ and $\left|\dfrac{\partial h}{\partial z_i}\right|$ are bounded by B_0' and B_i', respectively, $(i = 1,2,\dots,n)$. Let us fix ε, V, W and y_0. We shall prove that f is locally Hölder continuous on V with exponent $2\alpha/(1+\alpha)$. Abusing notation let t_0 denote an arbitrary element of V and let $0 < r$, $R < \varepsilon$. Fixing t_0 let $\bar{f}$ denote the mean value of f on the interval with endpoints $t_0 - r$, $t_0 + r$. Let us integrate the two sides of the functional equation over the interval with endpoints $y_0 - R$, $y_0 + R$. We have

$$2Rf(t) = \int_{y_0-R}^{y_0+R} h(t,y,f_1(g_1(t,y)),\dots,f_n(g_n(t,y)))\,dy,$$

and

$$2R\bar{f} = \frac{1}{2r}\int_{t_0-r}^{t_0+r}\int_{y_0-R}^{y_0+R} h(t',y,f_1(g_1(t',y)),\dots,f_n(g_n(t',y)))\,dy\,dt'\,.$$

Hence

$$\begin{aligned}|f(t)-\bar{f}| = \frac{1}{2R}\Bigg|\int_{y_0-R}^{y_0+R} & h(t,y,f_1(g_1(t,y)),\dots,f_n(g_n(t,y))) \\ & -\frac{1}{2r}\int_{t_0-r}^{t_0+r} h(t',y,f_1(g_1(t',y)),\dots,f_n(g_n(t',y)))\,dt'\,dy\Bigg|.\end{aligned}$$

To get a good upper estimate for the left hand side we need an upper estimate for the difference

$$h(t,y,f_1(g_1(t,y)),\dots,f_n(g_n(t,y))) - h(t',y,f_1(g_1(t',y)),\dots \\ \dots, f_n(g_n(t',y))).$$

We may apply the Taylor theorem for the function h with points

$$z=(t,y,z_1,\dots,z_n) \quad \text{and} \quad z'=(t',y,z_1',\dots,z_n')$$

where $t',t\in V$, $y\in W$, $z_i=f_i(g_i(t,y))$ and $z_i'=f_i(g_i(t',y))$ for $i=1,\dots,n$. We have

$$h(z)-h(z')=\int_0^1 \frac{\partial h}{\partial t}(\tau z+(1-\tau)z')(t-t')\,d\tau \\ +\sum_{i=1}^n \int_0^1 \frac{\partial h}{\partial z_i}(\tau z+(1-\tau)z')(z_i-z_i')\,d\tau.$$

Using this and omitting variables we have

$$4rR|f(t)-\bar{f}| = \Bigg| \int_{y_0-R}^{y_0+R}\int_{t_0-r}^{t_0+r}\Bigg(\int_0^1 \frac{\partial h}{\partial t}(\tau z+(1-\tau)z')(t-t')\,d\tau \\ +\sum_{i=1}^n \int_0^1 \frac{\partial h}{\partial z_i}(\tau z+(1-\tau)z')(z_i-z_i')d\tau\Bigg)dt'\,dy\Bigg|.$$

Using the triangle inequality, we get $n+1$ terms on the right hand side. For the first term we get the trivial upper bound $4RrB_0'2r$, where B_0' is an upper bound of $\left|\frac{\partial h}{\partial t}\right|$. If $\bar{h}_i'$ denotes the mean value of the partial derivative $\frac{\partial h}{\partial z_i}$, that is

$$\bar{h}_i'=\frac{1}{4rR}\int_{y_0-R}^{y_0+R}\int_{t_0-r}^{t_0+r}\int_0^1 \frac{\partial h}{\partial z_i}(z)\,d\tau\,dt\,dy,$$

then the other terms can be rewritten in the form

$$\int_{y_0-R}^{y_0+R}\int_{t_0-r}^{t_0+r}\int_0^1\left(\frac{\partial h}{\partial z_i}(\tau z+(1-\tau)z')-\bar{h}_i'\right)(z_i-z_i')\,d\tau\,dt'\,dy \\ +\bar{h}_i'\int_{y_0-R}^{y_0+R}\int_{t_0-r}^{t_0+r}(z_i-z_i')\,dt'dy.$$

First we give an upper estimate for the absolute value of the first term of this sum. An upper estimate of $|z_i-z_i'|$ is $H(L2r)^\alpha$, where H is a

Hölder-constant for f_i and L is a Lipschitz-constant for g_i. Hence

$$\left|\int_{y_0-R}^{y_0+R}\int_{t_0-r}^{t_0+r}\int_0^1 \frac{\partial h}{\partial z_i}(\tau z+(1-\tau)z')-\bar h_i')(z_i-z_i')\,d\tau\,dt'\,dy\right|$$
$$\le H(2rL)^\alpha\int_{y_0-R}^{y_0+R}\int_{t_0-r}^{t_0+r}\int_0^1\left|\frac{\partial h}{\partial z_i}(\tau z+(1-\tau)z')-\bar h_i'\right|d\tau\,dt'\,dy.$$

Because the difference between the value and the mean value of a function is not greater then the difference between any two values, we need to estimate the difference $\left|\frac{\partial h}{\partial z_i}(\tau z+(1-\tau)z')-\frac{\partial h}{\partial z_i}(z'')\right|$. This is not greater than L_i' multiplied by the norm of $\tau z+(1-\tau)z'-z''$, that is, L_i' times the maximal distance between the vectors z and $z''=(t'',y'',z_1'',\dots,z_n'')$, where $z_i''=f_i(g_i(t'',y''))$ and L_i' is a Lipschitz-constant for $\frac{\partial h}{\partial z_i}$. The maximal distance between z and z'' can be estimated by $r+R+nH(L(2r+2R))^\alpha$. Hence we get the upper bound

$$4rRH(2rL)^\alpha L_i'(r+R+nH(L(2r+2R))^\alpha)$$

for the first term.

To get an upper bound for the second term, we need an upper bound for the absolute value of

$$\int_{y_0-R}^{y_0+R}\int_{t_0-r}^{t_0+r}(z_i-z_i')\,dt'\,dy=$$
$$=\int_{y_0-R}^{y_0+R}\int_{t_0-r}^{t_0+r}f_i(g_i(t,y))-f_i(g_i(t',y))\,dt'\,dy,$$

because $|\bar h_i'|$ is trivially bounded by the upper bound B_i' of $\left|\frac{\partial h}{\partial z_i}\right|$. From Lemma 2 we get the upper bound $8LBB'r^2+8LBL'r^2(r+R)$ for this integral.

Summing up all these estimates, we get

$$|f(t)-\bar f|\le 2B_0'r+H(2rL)^\alpha\sum_{i=1}^n L_i'(r+R+nH(L(2r+2R))^\alpha)$$
$$+\sum_{i=1}^n B_i'(2LBB'r+2LBL'r(r+R))\,/\,R.$$

If $r\le R$ this can be rewritten as

$$|f(t)-\bar f|\le C_0r+C_1r^\alpha R^\alpha+C_2r/R,$$

where C_0, C_1 and C_2 do not depend on t_0, r and R. If we choose r and R such that they satisfy the condition $R = r^{(1-\alpha)/(1+\alpha)}$, then we have

$$|f(t) - \bar{f}| \le (C_0 + C_1 + C_2) r^{2\alpha/(1+\alpha)}$$

whenever $0 < r < r_0 = \varepsilon^{(1+\alpha)/(1-\alpha)}$ and $|t - t_0| \le r$. Integrating and using Lemma 1, we get that f is locally Hölder continuous on V which implies the theorem.

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