Lineability and coneability of discontinuous functions on $\mathbb{R}$

By A. AIZPURU (Cádiz) C. PÉREZ-ESLAVA (Cádiz), F. J. GARCÍA-PACHECO (Kent)
and J. B. SEOANE-SEPÚLVEDA (Madrid)

Abstract. We construct infinite dimensional vector spaces and positive cones of discontinuous functions on $\mathbb{R}$ enjoying some *special* properties, such as functions with an arbitrary $\mathcal{F}_\sigma$ set of points of discontinuity, discontinuous Riemann-integrable functions, or functions having either jump or removable discontinuities at a given point. We show that these special phenomena occur *more often* than one could expect, i.e. in a *linear* or *algebraic* way.

1. Preliminaries and background

This paper is a contribution to the study of certain *pathological* properties on subsets of functions on $\mathbb{R}$. We show that these properties occur more often than one could expect, existing large algebraic structures enjoying them (infinite dimensional vector spaces, algebras, or positive cones every non-zero element of which enjoys this given property). Examples of such kind are continuous nowhere differentiable functions, everywhere surjective functions, or differentiable nowhere monotone functions.

Given such a special property, we say that the subset $M \subset \mathcal{F}(\mathbb{R}, \mathbb{R})$ of functions which satisfies it is *lineable* if $M \cup \{0\}$ contains an infinite dimensional vector space. At times, we will say that the set M is μ-lineable if it contains a vector space of dimension μ. Also, we let $\lambda(M)$ be the maximum cardinality (if it exists) of such a vector space. These terminologies of lineable and lineability were first introduced in [7] and, later, in [2], [3].

In [3], it was shown that the set of everywhere surjective functions is 2^c-lineable. Moreover, there exists an infinitely generated algebra every non-zero element of which is an everywhere surjective function from $\mathbb{C}$ to $\mathbb{C}$ ([5]). In [3] it was also shown that the set of differentiable functions on $\mathbb{R}$ which are nowhere monotone is lineable in $\mathcal{C}(\mathbb{R})$. FONF, GURARIY, and KADEČ showed ([8]) that there exists a closed and infinite dimensional vector space of continuous nowhere differentiable functions on $[0, 1]$, as a subspace of $\mathcal{C}[0, 1]$. Some of these pathological behaviors occur in really interesting ways. ARON, PÉREZ-GARCÍA, and the fourth author ([4]) constructed, given any set $E \subset \mathbb{T}$ of measure zero, an infinite dimensional, infinitely generated dense subalgebra of $\mathcal{C}(\mathbb{T})$ every non-zero element of which has a Fourier series expansion divergent in E.

Besides vector spaces or algebras, we could also study the existence of positive or negative cones, introducing the following concept:

Definition 1.1 (Coneability). A set of functions in $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is said to be coneable if it possesses a positive (or negative) cone containing an infinite linearly independent set.

This paper is divided in three sections. In each of them we focus on a particular property and type of function. These properties are:

(1) Functions whose points of discontinuity form an arbitrary given $\mathcal{F}_\sigma$ set.

(2) Riemann-integrable functions with an arbitrary $\mathcal{F}_\sigma$ set of measure zero as their set of points of discontinuity.

(3) Functions having either jump or removable discontinuities at a given point.

Set theoretical considerations, cardinal theory, topology, Borel structures, matrix theory, abstract algebra, and usual real analysis techniques are involved. From now on, $\aleph_0$ and c will denote the cardinalities of $\mathbb{N}$ and the power set of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$, respectively

2. $\mathcal{F}_\sigma$ sets of points of discontinuity

As known from the works of BAIRE, BOREL et al. (1905) [6], for every function $f : \mathbb{R} \longrightarrow \mathbb{R}$, the set of points of discontinuity of f is an $\mathcal{F}_\sigma$ set. Also, given any $\mathcal{F}_\sigma$ set there exists a function from $\mathbb{R}$ to $\mathbb{R}$ whose set of points of discontinuity is exactly this set. Modern references for these results are, for example, [9], [10, p. 30, ex. 22 and 23], and [13]; We will be using these latest references along this paper.

In this section we will study the lineability and coneability of the set of all functions whose set of points of discontinuity is a given $\mathcal{F}_\sigma$ set F. We will distinguish two cases depending on whether F is closed or not.

2.1. The $\mathcal{F}_\sigma$ set is closed. As we mentioned above, given any closed set F, there exists a function from $\mathbb{R}$ to $\mathbb{R}$ whose set of points of discontinuity is exactly F. Here, our purpose is to construct a vector space of dimension at least c every non-zero element of which is a function whose set of points of discontinuity is exactly F. We start writing

$$\mathbb{R}\backslash F = \bigcup_{n\in N}^{\cdot} (a_n, b_n) \quad \text{and} \quad \text{int}(F) = \bigcup_{m\in M}^{\cdot} (c_m, d_m),$$

where N and M are countable sets. For every $n \in N$ and every $m \in M$ let $\phi_n : (a_n, b_n) \longrightarrow \mathbb{R}$ and $\psi_m : (c_m, d_m) \longrightarrow \mathbb{R}$ be homeomorphisms. Now, it is known ([3]) the existence of a vector space U of dimension 2^c contained in the set

$$\{f : \mathbb{R} \longrightarrow \mathbb{R}/f \text{ is everywhere surjective}\} \cup \{0\}.$$

Let us recall that the everywhere surjective functions are exactly the functions $f : \mathbb{R} \to \mathbb{R}$ so that, for every non-void interval (a, b), $f((a, b)) = \mathbb{R}$ and, as a consequence, these type of functions are nowhere continuous and their graph is dense in $\mathbb{R}^2$. The existence of this class of functions was first noticed by LEBESGUE (1904) in [12] (for further study of this class of functions, see [3], [5]). Let us denote by V any vector space of dimension c contained in the set

$$\left\{f : \mathbb{R} \longrightarrow \mathbb{R}/f \text{ is continuous and } \lim_{|x|\to\infty} |f(x)| = \infty\right\} \cup \{0\}.$$

The existence of such a vector space is shown in, for instance, [1]. For every $(u, v, r) \in U \times V \times \mathbb{R}$, consider the function

$$f_{uvr} : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto f_{uvr}(x) = \begin{cases} r & x \in \text{bd}(F), \\ u(\psi_m(x)) & x \in (c_m, d_m), \ m \in M, \\ v(\phi_n(x)) & x \in (a_n, b_n), \ n \in N. \end{cases}$$

Let us see that the set of points of discontinuity of f_{uvr} is exactly F. If $x \in \mathbb{R}\backslash F$ then f_{uvr} is continuous at x, since $\mathbb{R}\backslash F$ is open and v and ϕ_n are continuous functions. If $x \in \text{int}(F)$ then f_{uvr} is discontinuous at x, since $u \circ \psi_m$ is an every-

where surjective function in (c_m, d_m). If $x \in \mathrm{bd}(F)$ then f_{uvr} is discontinuous at x, since $x \in \overline{\{a_n, b_n : n \in N\}}$ and we know that, for every $n \in N$,

$$\lim_{y \to a_n} |v(\phi_n(y))| = \lim_{y \to b_n} |v(\phi_n(y))| = \infty.$$

Now, $X = \{f_{uvr} : u \in U, \ v \in V, \ r \in \mathbb{R}\}$ is a vector space, since if $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, $u_1, \ldots, u_k \in U$, $v_1, \ldots, v_k \in V$, and $r_1, \ldots, r_k \in \mathbb{R}$, then

$$\lambda_1 f_{u_1 v_1 r_1} + \cdots + \lambda_k f_{u_k v_k r_k} = f_{\lambda_1 u_1 + \cdots + \lambda_k u_k, \, \lambda_1 v_1 + \cdots + \lambda_k v_k, \, \lambda_1 r_1 + \cdots + \lambda_k r_k}.$$

Finally, suppose that $\mathrm{int}(F) = \varnothing$, then $\mathbb{R} \backslash F \neq \varnothing$ and, therefore, X has dimension c. Indeed, if $\{v_l\}_{l \in L}$ is a basis for V then $\{f_{v_l 1}\}_{k \in K}$ is a basis for X. On the other hand, if $\mathrm{int}(F) \neq \varnothing$ then X has dimension 2^c. Indeed, if $\{u_k\}_{k \in K}$ and $\{v_l\}_{l \in L}$ are basis for U and V respectively, then $\{f_{u_k v_l 1} : k \in K, \ l \in L\}$ is a basis for X. As a consequence, we have the following result.

Theorem 2.1. *Given a closed set F, the set H of all functions whose set of points of discontinuity is F is lineable with $\lambda(H) \geq c$. Moreover, if $\mathrm{int}(F) \neq \varnothing$ then $\lambda(H) = 2^c$.*

2.2. The $\mathcal{F}_\sigma$ set is not closed. As we said at the beginning of this section, given any non-closed $\mathcal{F}_\sigma$ set F, there exists a function from $\mathbb{R}$ to $\mathbb{R}$ whose set of points of discontinuity is exactly F (see, e.g., [10, p. 30, ex. 23], for a more modern reference). Here, our purpose is to construct a positive cone C every non-zero element of which is a function from $\mathbb{R}$ to $\mathbb{R}$ whose set of points of discontinuity is exactly F. Let us call f to such a function, that we know exists ([10, p. 30, ex. 23]). Next, consider the set

$$C = \left\{ \lambda_1 f^{k_1} + \cdots + \lambda_p f^{k_p} : p \in \mathbb{N}, \ \lambda_1, \ldots, \lambda_p \geq 0, \ k_1, \ldots, k_p \in \mathbb{N}^* \right\}$$

where $\mathbb{N}^*$ denotes the set of odd natural numbers. Now, let us take any finite positive linear combination

$$\lambda_1 f^{k_1} + \cdots + \lambda_p f^{k_p}, \tag{1}$$

in which $\lambda_1, \ldots, \lambda_p > 0$. In order to see that the set of points of discontinuity of the function given by (1) is F, it suffices to prove that this function is not continuous at every point in F. Take any $x \in F$. There exist $\varepsilon > 0$ and a sequence $(x_n)_{n \in \mathbb{N}}$ convergent to x such that $|f(x_n) - f(x)| \geq \varepsilon$. We can assume, without loss of generality, that either $f(x_n) < f(x) - \varepsilon$ for every $n \in \mathbb{N}$ or $f(x_n) > f(x) + \varepsilon$ for

every $n \in \mathbb{N}$. We will assume the first possibility, the other case is similar. With our assumption we have that, for every $n \in \mathbb{N}$,

$$\lambda_1 f^{k_1}(x_n) + \cdots + \lambda_p f^{k_p}(x_n) < \lambda_1 (f(x) - \varepsilon)^{k_1} + \cdots + \lambda_p (f(x) - \varepsilon)^{k_p}.$$

If the function given in (1) is continuous at x, then we obtain the following contradiction

$$\lambda_1 f^{k_1}(x) + \cdots + \lambda_p f^{k_p}(x) \leq \lambda_1 (f(x) - \varepsilon)^{k_1} + \cdots + \lambda_p (f(x) - \varepsilon)^{k_p}$$
$$< \lambda_1 f^{k_1}(x) + \cdots + \lambda_p f^{k_p}(x).$$

Next, let us see that $\{f^k : k \in \mathbb{N}^*\}$ is linearly independent. Take any identically zero linear combination

$$\lambda_1 f^{k_1} + \cdots + \lambda_p f^{k_p} = 0. \tag{2}$$

By the construction of the function f given in [10, p. 30, ex. 23], there exists a countable family of non-empty disjoint sets $\{B_n\}_{n \in \mathbb{N}}$ so that $f(x) = 2^{-n}$ for every $x \in B_n$ and every $n \in \mathbb{N}$. For every $i \in \{1, \ldots, p\}$ let us take an element $x_i \in B_i$. By evaluating the equation (2) at the points $x_1, \ldots, x_p$ we obtain the following linear system of equations:

$$\begin{pmatrix} \dfrac{1}{2^{k_1}} & \dfrac{1}{2^{k_2}} & \dfrac{1}{2^{k_3}} & \cdots & \dfrac{1}{2^{k_p}} \\ \left(\dfrac{1}{2^{k_1}}\right)^2 & \left(\dfrac{1}{2^{k_2}}\right)^2 & \left(\dfrac{1}{2^{k_3}}\right)^2 & \cdots & \left(\dfrac{1}{2^{k_p}}\right)^2 \\ \left(\dfrac{1}{2^{k_1}}\right)^3 & \left(\dfrac{1}{2^{k_2}}\right)^3 & \left(\dfrac{1}{2^{k_3}}\right)^3 & \cdots & \left(\dfrac{1}{2^{k_p}}\right)^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\dfrac{1}{2^{k_1}}\right)^p & \left(\dfrac{1}{2^{k_2}}\right)^p & \left(\dfrac{1}{2^{k_3}}\right)^p & \cdots & \left(\dfrac{1}{2^{k_p}}\right)^p \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The matrix of the previous system is non-singular (it is a Van der Monde-type matrix). Therefore we have that, necessarily, $\lambda_1 = \lambda_2 = \cdots = \lambda_p = 0$ and the family $\{f^k : k \in \mathbb{N}^*\}$ is linearly independent. We have, thus, proved the following result.

Theorem 2.2. *Given any non-closed $\mathcal{F}_\sigma$ set F, the set of functions whose set of points of discontinuity is exactly F is coneable.*

3. Discontinuous Riemann-integrable functions

Taking into account the results of BAIRE, BOREL, et al. ([6]) mentioned at the beginning of the previous section, along with the well-known Lebesgue theorem characterizing Riemann-integrable functions in terms of their discontinuities (see [12], or [14] for a modern reference), it is clear that if $f : [a, b] \longrightarrow \mathbb{R}$ is a Riemann-integrable function then the set of points of discontinuity of f is an $\mathcal{F}_\sigma$ set of measure zero. Also, given any $\mathcal{F}_\sigma$ set of measure zero contained in an interval $[a, b]$, there exists a Riemann-integrable function from $[a, b]$ to $\mathbb{R}$ whose set of points of discontinuity is exactly this $\mathcal{F}_\sigma$ set.

In this section we study the lineability and coneability properties of the set H of all Riemann-integrable functions whose set of points of discontinuity is an $\mathcal{F}_\sigma$ set F of measure zero. As we did in the previous section, we will distinguish two cases depending on whether F is closed or not.

3.1. The $\mathcal{F}_\sigma$ set is closed. If F is a closed set contained in an interval $[a, b]$, then it is easy to see that the function

$$[a, b] \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} 1 & x \in F, \\ 0 & x \in [a, b] \backslash F \end{cases}$$

is Riemann-integrable and verifies that its set of points of discontinuity is exactly F. Here, our purpose is to construct a vector space of dimension, at least, $\aleph_0$ every non-zero element of which is a Riemann-integrable function from $[a, b]$ to $\mathbb{R}$ whose set of points of discontinuity is exactly F. First, let us see the existence of a vector space U with $\dim(U) = \aleph_0$, and so that every non-zero element of which is a continuous bounded function from $\mathbb{R}$ to $\mathbb{R}$ oscillating at both ∞ and $-\infty$. As usual, P will denote the set of all odd prime numbers. For every $p \in P$ let us consider the continuous bounded function

$$u_p : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto u_p(x) = \sin(x/p),$$

which is oscillating at both ∞ and $-\infty$. Now, let us consider the vector space $U = \text{span}\{u_p : p \in P\}$. From the fact that the family $\{\sin(x/p) : p \in P\}$ is linearly independent (see, e.g. [1]), it can be proved that the set $\{u_p : p \in P\}$ is also linearly independent. Finally, it is clear that every non-zero element of U is

continuous and bounded, and if

$$\phi\left(\cdot\right) = \sum_{j=1}^{k} \alpha_j \sin\left(\frac{\cdot}{p_j}\right)$$

is a non-zero function of U, then, by considering the sequence

$$\left(x_n = p_2 \cdot \cdots \cdot p_k \cdot \pi \cdot n\right)_{n \in \mathbb{Z}},$$

it can be seen that ϕ is oscillating at both ∞ and $-\infty$, since

$$\phi\left(x_n\right) = \alpha_1 \sin\left(\frac{p_2 \cdot \cdots \cdot p_k}{p_1} \cdot \pi \cdot n\right)$$

does not converge as n goes to ∞ or $-\infty$ (here we are assuming, without loss, that $\alpha_1 \neq 0$).

Now, once we know the existence of such an U, we start by writing

$$[a,b] \setminus F = \bigcup_{n \in N}^{\cdot} I_n,$$

where for every $n \in N$, I_n is an open interval in $[a,b]$ and N is countable. At the moment, we will take care of the case in which $a, b \in F$. For every $n \in N$ let $\phi_n : I_n \longrightarrow \mathbb{R}$ be an homeomorphism. Now, for every $(u,r) \in U \times \mathbb{R}$ consider the function

$$f_{ur} : [a,b] \longrightarrow \mathbb{R}$$

$$x \longmapsto f_{ur}(x) = \begin{cases} r & x \in F, \\ u(\phi_n(x)) & x \in I_n, \ n \in N. \end{cases}$$

Let us see that the set of points of discontinuity of f_{ur} is exactly F. If $x \in I_n$ for some $n \in N$ then f_{ur} is continuous at x, since u and ϕ_n are continuous functions and I_n is an open interval. If $x \in F$ then f_{ur} is discontinuous at x, since x can be approximated by a subsequence of extreme points of the intervals I_n, and we know that for every $n \in N$, $u \circ \phi_n$ is oscillating at both extremes of I_n. Now, $X = \{f_{ur} : u \in U, \ r \in \mathbb{R}\}$ is a vector space, since if $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, $u_1, \ldots, u_k \in U$ and $r_1, \ldots, r_k \in \mathbb{R}$, then

$$\lambda_1 f_{u_1 r_1} + \cdots + \lambda_k f_{u_k r_k} = f_{\lambda_1 u_1 + \cdots + \lambda_k u_k, \lambda_1 r_1 + \cdots + \lambda_k r_k}.$$

Finally, X has dimension $\aleph_0$. Indeed, if $\{u_k t\}_{k \in \mathbb{N}}$ is a basis for U, then $\{f_{u_k 1}\}_{k \in \mathbb{N}}$ is a basis for X.

For the case in which either a or b are in $[a,b]\backslash F$, it suffices to consider the vector spaces $V = \text{span}\{v_p : p \in P\}$ and $W = \text{span}\{w_p : p \in P\}$, where for every $p \in P$

$$v_p : [-\infty, \infty) \longrightarrow \mathbb{R}$$

$$x \longmapsto v_p(x) = \begin{cases} 0 & x \leq 0, \\ \sin(x/p) & x \geq 0, \end{cases}$$

and

$$w_p : (-\infty, \infty] \longrightarrow \mathbb{R}$$

$$x \longmapsto w_p(x) = \begin{cases} \sin(x/p) & x \leq 0, \\ 0 & x \geq 0. \end{cases}$$

Let us also consider homeomorphisms $\phi_a : I_a \longrightarrow [-\infty, \infty)$ and $\phi_b : I_b \longrightarrow (-\infty, \infty]$, where I_a and I_b are, respectively, the connected components of a and b in $[a,b]\backslash F$.

As a consequence of all of this, we have the following result.

Theorem 3.1. *Given a closed set F of measure zero contained in an interval $[a,b]$, the set H of all Riemann-integrable functions whose set of points of discontinuity is F is lineable.*

3.2. The $\mathcal{F}_\sigma$ set is not closed. In [10, p. 44, ex. 26] there can be found a proof of the fact that given any non-closed $\mathcal{F}_\sigma$ set F of measure zero contained in an interval $[a,b]$, there exists a Riemann-integrable function from $[a,b]$ to $\mathbb{R}$ whose set of points of discontinuity is exactly F. Here, our purpose is to construct a positive cone C every non-zero element of which is a Riemann-integrable function from $[a,b]$ to $\mathbb{R}$ whose set of points of discontinuity is exactly F. Let us call f to such a function, that we know exists ([10, p. 44, ex. 26]). Next, consider the set

$$C = \left\{ \lambda_1 f^{k_1} + \cdots + \lambda_p f^{k_p} : p \in \mathbb{N}, \ \lambda_1, \ldots, \lambda_p \geq 0, \ k_1, \ldots, k_p \in \mathbb{N}^* \right\}$$

where $\mathbb{N}^*$ denotes the set of odd natural numbers. Clearly, every non-zero function of C is bounded and, from the previous section, we know that every non-zero element of C is a function from $[a,b]$ to $\mathbb{R}$ whose set of points of discontinuity is exactly F. Now, to see that the family $\{f^k : k \in \mathbb{N}^*\}$ is linearly independent we proceed as in the previous section, since in the construction of the function f given in [10, p. 44, ex. 26], it can be found a countable family of non-empty disjoint sets $\{B_n\}_{n \in \mathbb{N}}$ so that $f(x) = 2^{-n}$ for every $x \in B_n$ and every $n \in \mathbb{N}$. Finally, we have the corresponding theorem.

Theorem 3.2. *Given any non-closed $\mathcal{F}_\sigma$ set F of measure zero contained in an interval $[a,b]$, the set C of all Riemann-integrable functions whose set of points of discontinuity is exactly F is coneable.*

4. Removable discontinuities and jumps

Let us consider an interval I and a point $a \in I$. Let $f : I \longrightarrow \mathbb{R}$ be a function for which both limits from the left and from the right at a exist. Then, we will use the following notation:

$$f(a+) := \lim_{x \to a^+} f(x),$$

$$f(a-) := \lim_{x \to a^-} f(x),$$

$$f(a*) := \frac{f(a+) + f(a-)}{2}.$$

In this section, we study the lineability of the set H of all functions from I to $\mathbb{R}$ having either jump or removable discontinuities at a.

4.1. Functions having only removable discontinuities. Let I be any non-trivial interval and consider a point $a \in I$. Let $f, g : I \longrightarrow \mathbb{R}$ be linearly independent functions having removable discontinuities at a. Let us take

$$\alpha = \frac{f(a*) - f(a)}{g(a) - g(a*)}$$

and consider the function $h = f + \alpha g$. We have that h is continuous at a. As a consequence we have the following result.

Theorem 4.1. *Let I be any non-trivial interval and consider a point $a \in I$. Let K denote the set of all functions from I to $\mathbb{R}$ having a removable discontinuity at a. Then, $\lambda(K) = 1$.*

4.2. Functions having only jump discontinuities. Let I be any non-trivial interval and consider a point $a \in I$. Let $f, g : I \longrightarrow \mathbb{R}$ be linearly independent functions having jump discontinuities at a. Let us take

$$\alpha = \frac{f(a+) - f(a-)}{g(a-) - g(a+)}$$

and consider the function $h = f + \alpha g$. We have that h is either continuous or has a removable singularity at a. As a consequence we have the following result.

Theorem 4.2. *Let I be any non-trivial interval and consider a point $a \in I$. Let L denote the set of all functions from I to $\mathbb{R}$ having a jump discontinuity at a. Then, $\lambda(L) = 1$.*

4.3. Functions having either removable or jump discontinuities. Let I be an arbitrary given non-trivial interval and consider a point $a \in I$. Let H denote the set of all functions from I to $\mathbb{R}$ having either a removable or jump discontinuity at a. Assume that X is a vector space of dimension greater or equal

to 3 contained in $H \cup \{0\}$. Let us consider $\{f, g, h\}$ a linearly independent set contained in X. We will reach a contradiction. Taking into account theorem 4.1, it is clear that, at least, two of three functions f, g, and h must have a jump discontinuity at a. Then, let us assume that this two functions are f and g. Let us take

$$\alpha = \frac{f(a-) - f(a+)}{g(a+) - g(a-)}$$

and consider the function $u = f + \alpha g$. We have that u has a removable discontinuity at a. Now, applying again theorem 4.1, k cannot have a removable discontinuity at a. Let us take

$$\beta = \frac{g(a-) - g(a+)}{h(a+) - h(a-)}$$

and consider the function $v = g + \beta h$. We have that v has a removable discontinuity at a. Since u and v are linearly independent, we have a contradiction according to theorem 4.1. Now, consider the vector space $Y = \text{span}\{f, g\}$ where

$$f : I \longrightarrow \mathbb{R}$$
$$x \longmapsto f(x) = \begin{cases} -5 & x \neq a, \\ 0 & x = a, \end{cases}$$

and

$$g : I \longrightarrow \mathbb{R}$$
$$x \longmapsto g(x) = \begin{cases} (x-a)^3 + 2 & x > a, \\ (x-a)^3 - 7 & x \leq a. \end{cases}$$

It can be easily seen that Y has dimension 2 and $Y \subseteq H \cup \{0\}$. As a consequence of all of this, we have the following theorem.

Theorem 4.3. *Let I be any non-trivial interval and consider a point $a \in I$. Let H denote the set of all functions from I to $\mathbb{R}$ having either a removable or jump discontinuity at a. Then, $\lambda(H) = 2$.*

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ANTONIO AIZPURU
DEPARTAMENTO DE MATEMÁTICAS
FACULTAD DE CIENCIAS
UNIVERSIDAD DE CÁDIZ
11510 - PUERTO REAL, CÁDIZ
SPAIN

E-mail: antonio.aizpuru@uca.es

C. PÉREZ-ESLAVA
DEPARTAMENTO DE MATEMÁTICAS
FACULTAD DE CIENCIAS
UNIVERSIDAD DE CÁDIZ
11510 - PUERTO REAL, CÁDIZ
SPAIN

E-mail: sonriencanto@hotmail.com

FRANCISCO J. GARCÍA-PACHECO
DEPARTMENT OF MATHEMATICS
KENT STATE UNIVERSITY
KENT, OHIO, 44242
USA

E-mail: fgarcia@math.kent.edu

JUAN B. SEOANE-SEPÚLVEDA
DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
UNIVERSIDAD COMPLUTENSE DE MADRID
PLAZA DE LAS CIENCIAS 3
28040 - MADRID
SPAIN

E-mail: jseoane@mat.ucm.es