Publ. Math. Debrecen
72/1-2 (2008), 141–154

Sequences of algebraic numbers and density modulo 1

By Roman Urban

Abstract. We prove density modulo 1 of the sets of the form

$$\{\mu^m\lambda^n\xi + r_m : m,\ n \in \mathbb{N}\}$$

and

$$\{\mu^m\lambda^n\xi + r^{m+n}\beta : m,\ n \in \mathbb{N}\},$$

where λ, μ is a pair of rationally independent real algebraic numbers, satisfying some additional assumptions, $\xi \neq 0$, $r, \beta \in \mathbb{R}$ and r_m is any sequence of real numbers.

1. Introduction

It is a very well known result in the theory of distribution modulo 1 that for every irrational ξ the sequence $\{n\xi : n \in \mathbb{N}\}$ is dense modulo 1 (and even uniformly distributed modulo 1) [13].

In 1967, in his seminal paper [5], Furstenberg proved the following

Theorem 1.1 (Furstenberg, [5]). *If $p, q > 1$ are rationally independent integers (i.e., they are not both integer powers of the same integer) then for every irrational ξ the set*

$$\{p^m q^n\xi : m,\ n \in \mathbb{N}\} \tag{1.1}$$

is dense modulo 1.

Mathematics Subject Classification: 11J71, 54H20.
Key words and phrases: Distribution modulo 1, algebraic numbers, topological dynamics, ergodicity, ID-semigroups.
Research supported in part by the European Commission Marie Curie Host Fellowship for the Transfer of Knowledge "Harmonic Analysis, Nonlinear Analysis and Probability" MTKD-CT-2004-013389 and by MNiSW research grant N 201 012 31/1020.

One possible direction of generalization is to consider p and q in Theorem 1.1 that are not necessarily integers. This was done by Berend in [4].

According to [16], Furstenberg conjectured, that under conditions of Theorem 1.1, the set $\{(p^m + q^n)\xi : m,\ n \in \mathbb{N}\}$ is dense modulo 1. As far as we know, this conjecture is still open, however there are some results concerning related questions. For example, B. Kra in [12], proved the following

Theorem 1.2 (Kra, [12, Theorem 1.2 and Corollary 2.2])**.** *For $i = 1, 2$, let p_i, q_i be two multiplicatively independent integers whose absolute values are bigger than 1. Assume that $p_1 \neq p_2$ or $q_1 \neq q_2$. Then, for every $\xi_1, \xi_2 \in \mathbb{R}$ with at least one $\xi_i \notin \mathbb{Q}$, the set*

$$\{p_1^m q_1^n \xi_1 + p_2^m q_2^n \xi_2 : m,\ n \in \mathbb{N}\}$$

is dense modulo 1.

Furthermore, let r_m be any sequence of real numbers and $\xi \notin \mathbb{Q}$. Then, the set

$$\{p_1^m q_1^n \xi + r_m : m,\ n \in \mathbb{N}\} \tag{1.2}$$

is dense modulo 1.

Inspired by Berend's result [4], we prove some kind of generalization of the second part of Theorem 1.2[1] Namely, we allow algebraic numbers, satisfying some additional assumption, to appear in (1.2) instead of integers, and prove the following two theorems.

Theorem 1.3. *Let λ, μ be a pair of rationally independent real algebraic numbers (with conjugates $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_d$ and $\mu = \mu_1, \mu_2, \ldots, \mu_r$) such that $\mu \in \mathbb{Q}(\lambda)$, i.e., $\mu = g(\lambda)$ for some $g \in \mathbb{Q}[x]$.*

Assume that λ has the property that for every $n \in \mathbb{N}$, λ^n has the same degree over $\mathbb{Q}$ as λ.

Let $S = \{\infty, p_1, p_2, \ldots, p_s\}$, where $p_k \geq 2$, $1 \leq k \leq s$, are the primes appearing in the denominators of the coefficients of $g \in \mathbb{Q}[x]$ and the minimal polynomial $P_\lambda \in \mathbb{Q}[x]$ of λ.

Assume further that

$$\min_{p\in S} \min_{1\leq i\leq d} |\lambda_i|_p > 1 \quad \text{and} \quad \min_{p\in S} \min_{1\leq j\leq r} |\mu_j|_p > 1, \tag{1.3}$$

where $|\cdot|_p$ is the p-adic norm, whereas $|\cdot|_\infty$ stands for the usual absolute value.[2]

[1] [22] and [24] contain some extensions of the first part of Theorem 1.2 to the setting of algebraic numbers of degree 2.

[2] See subsection 2.2 for the definition of the p-adic norm.

Then, for any non-zero ξ and any sequence of real numbers r_m, the set

$$\{\mu^m\lambda^n\xi + r_m : m,\ n \in \mathbb{N}\}$$

is dense modulo 1.

Theorem 1.4. *Let λ, μ be a pair of rationally independent real algebraic numbers satisfying conditions of Theorem 1.3. Then, for any non-zero ξ and any two real numbers r, β, the set*

$$\{\mu^m\lambda^n\xi + r^{m+n}\beta : m,\ n \in \mathbb{N}\} \tag{1.4}$$

are dense modulo 1.

The sets of the form (1.4) with λ and $\mu \in \mathbb{N}$ have been considered by Berend in [3]. Here we generalize his proof to our setting. Theorem 1.3 for algebraic integers was proved in [23].

As an example illustrating Theorem 1.3 and Theorem 1.4, consider the following expressions containing algebraic numbers of degree 2,

$$\left(\frac{17}{\sqrt{2}} + \frac{1}{3\cdot 5\cdot 7}\right)^m \left(\frac{11\cdot 17}{\sqrt{2}} + \frac{11}{3\cdot 5\cdot 7} + \frac{1}{7^2}\right)^n + 7^m$$

and

$$2\left(\frac{5^3}{\sqrt{3}} + \frac{1}{2\cdot 11}\right)^m \left(\frac{7\cdot 5^3}{\sqrt{3}} + \frac{7}{2\cdot 11} + \frac{1}{2^3}\right)^n + 3^{m+n}\pi.$$

Another kind of generalization of Furstenberg's Theorem 1.1, which we are going to use in the proof of our results, is to consider higher-dimensional analogues. A generalization to a commutative semigroup of non-singular $d \times d$-matrices with integer coefficients acting by endomorphisms on the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, and to the commutative semigroups of continuous endomorphisms acting on a-adic solenoids and on other compact abelian groups was given by Berend in [1] and [2], respectively (see Section 2.3.4). Recently some generalizations for non-commutative semigroups of endomorphisms acting on $\mathbb{T}^d$ have been obtained in [6], [7], [17].

The structure of the paper is as follows. In Section 2 we recall some notions and facts from ergodic theory, topological dynamics and some elementary definitions and notions concerning p-adic numbers and a-adic solenoids. Following Berend [1], [2], we recall the definition of an ID-semigroup of endomorphisms of a compact group and state Berend's theorem, [2], which gives conditions that guarantee that a given semigroup of endomorphisms of an a-adic solenoid is an ID-semigroup. Finally in Section 3, using some ideas from [12], [4] we prove Theorem 1.3 and Theorem 1.4.

2. Preliminaries

2.1. Algebraic numbers. We say that $P \in \mathbb{Z}[x]$ is *monic* if the leading coefficient of P is one, and *reduced* if its coefficients are relatively prime. A *real algebraic integer* is any real root of a monic polynomial $P \in \mathbb{Z}[x]$, whereas an *algebraic number* is any root (real or complex) of a (not necessarily monic) non-constant polynomial $P \in \mathbb{Z}[x]$. The *minimal polynomial* of an algebraic number θ is the reduced element P of $\mathbb{Z}[x]$ of the least degree such that $P(\theta) = 0$. If θ is an algebraic number, the roots of its minimal polynomial are simple. The *degree* of an algebraic number is the degree of its minimal polynomial.

Let θ be an algebraic integer of degree n and let $P \in \mathbb{Z}[x]$ be the minimal polynomial of θ. The $n-1$ other distinct (real or complex) roots $\theta_2, \ldots, \theta_n$ of P are called *conjugates* of θ.

2.2. p-adic numbers. The basic references for this subsection are [10], [14], [18]. By $\mathbb{P} \subset \mathbb{N}$ we denote the set of primes. Let $p \in \mathbb{P}$ be a prime number. The *p-adic norm* $|\cdot|_p$ on the field $\mathbb{Q}$ is defined by $|0|_p = 0$ and $|p^k \frac{n}{m}|_p = p^{-k}$ for $k, n, m \in \mathbb{Z}$ and $p \nmid nm$. The *p-adic field of rational numbers* $\mathbb{Q}_p$ is defined as the completion of $\mathbb{Q}$ with respect to the norm $|\cdot|_p$. It is easy to see that the p-adic norm $|\cdot|_p$ on $\mathbb{Q}$ and its extension to $\mathbb{Q}_p$ satisfy:

(i) $|x|_p \in \{p^k : k \in \mathbb{Z}\} \cup \{0\}$,

(ii) $|xy|_p = |x|_p |y|_p$,

(iii) $|x+y|_p \leq \max\{|x|_p, |y|_p\}$, (*ultrametric triangle inequality*)

for all $x, y \in \mathbb{Q}_p$.

For simplicity of notation we write $\mathbb{Q}_\infty = \mathbb{R}$, $|\cdot|_\infty = |\cdot|$ for the usual absolute value, and $\{x\}_\infty = \{x\}$ for the fractional part of $x \in \mathbb{R}$.

The p-adic field $\mathbb{Q}_p$ is a locally compact field and every $x \in \mathbb{Q}_p$ can be uniquely expressed as a convergent, in $|\cdot|_p$-norm, sum (*Hensel representation*),

$$x = \sum_{k=t}^{\infty} x_k p^k, \tag{2.1}$$

for some $t \in \mathbb{Z}$ and $x_k \in \{0, 1, \ldots, p-1\}$. The *fractional part* of $x \in \mathbb{Q}_p$, denoted by $\{x\}_p$, is 0 if the number t in the Hensel representation (2.1) is greater than or equal to 0, and equal to $\sum_{k<0} x_k p^k$, if $t < 0$.

The *integral part* $[x]_p$ of an element $x \in \mathbb{Q}_p$ is $\sum_{k \geq 0} x_k p^k$.

The closure of $\mathbb{Z}$ in $\mathbb{Q}_p$ is the compact ring $\mathbb{Z}_p$ of *p-adic integers*. An element $x \in \mathbb{Q}_p$ is a p-adic integer if it has a Hensel representation (2.1) with $t \geq 0$, that is, its fractional part $\{x\}_p = 0$.

For a positive integer a, denote by $\mathbb{Z}[1/a]$ the ring obtained from $\mathbb{Z}$ by adjoining $1/a$. Thus, any $x \in \mathbb{Q}_p$ can be uniquely written as $x = [x]_p + \{x\}_p$, where $[x] \in \mathbb{Z}_p$ and the fractional part $\{x\}_p \in \mathbb{Z}[1/p] \cap [0, 1)$.

Define

$$\tau_p : \mathbb{Q}_p \to \mathbb{C} : \ x \mapsto \exp(2\pi i \{x\}_p). \tag{2.2}$$

It is easy to see that the map τ_p is a homomorphism and the additive group $\mathbb{Q}_p/\mathbb{Z}_p$ is isomorphic with the group μ_{p^∞} of p-th power roots of unity in the complex field $\mathbb{C}$ (see [18]).

2.3. a-adic solenoids and Berend's Theorem. In this subsection we recall the definition and basic facts about *a-adic solenoids*. We follow the presentation of [2] (see also [8]).

Consider $\mathbb{Z}[1/a]$ as a topological group with the discrete topology. We assume that a is square-free, that is $a = p_1 p_2 \dots p_s$, where p_j's are distinct primes. The dual group $\widehat{\mathbb{Z}[1/a]}$ of $\mathbb{Z}[1/a]$ is called the *a-adic solenoid* and we denote it by Ω_a (see [8]). The compact abelian group Ω_a may be considered as a quotient group of the additive group $\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_s}$ by a discrete subgroup

$$B = \{(b, -b, \dots, -b) : b \in \mathbb{Z}[1/a]\}. \tag{2.3}$$

That is,

$$\Omega_a = \mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_s}/B. \tag{2.4}$$

In fact, let

$$i : \mathbb{Z}[1/a] \to \mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_s}$$

be a discrete embedding of $\mathbb{Z}[1/a]$ into $\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_s}$, given by $i(x) = (x, x, \dots, x)$. For $p \in \mathbb{P} \cup \{\infty\}$, the dual group $\hat{\mathbb{Q}}_p$ is topologically isomorphic with $\mathbb{Q}_p$ and the action of the character $\chi_x \in \hat{\mathbb{Q}}_p$ corresponding to $x \in \mathbb{Q}_p$ is $\chi_x(y) = \exp(2\pi i \{xy\}_p)$, where $\{\cdot\}_p$ stands for the fractional part defined in subsection 2.2. The dual endomorphism $\hat{i} : \mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_s} \to \Omega_a$ gives Ω_a as a quotient group of $\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_s}/\ker \hat{i}$. But it is not difficult to see that $\ker \hat{i} = B$ (see [2] for details). Since the image of $\mathbb{R}$ by $\hat{i}$ is dense in Ω_a (see [8]), it follows that Ω_a is connected.

By (2.4) it follows that for any non-negative integer d,

$$\Omega_a^d = \mathbb{R}^d \times \mathbb{Q}_{p_1}^d \times \cdots \times \mathbb{Q}_{p_s}^d/B^d,$$

where

$$B^d = \{(b, -b, \dots, -b) : b \in \mathbb{Z}[1/a]^d\}. \tag{2.5}$$

2.3.1. *Continuous endomorphism of an a-adic solenoid Ω_a.* Now, we recall the description of the ring of continuous endomorphisms of an a-adic solenoid Ω_a, $a = p_1 \dots p_s$, and its d-fold Cartesian product Ω_a^d. To simplify the notations we write $p_0 = \infty$. Thus, according to the notations introduced in subsection 2.2, $\mathbb{Q}_{p_0} = \mathbb{Q}_\infty = \mathbb{R}$.

Any $c \in \mathbb{Z}[1/a]$ gives rise to an endomorphism φ_c of $\prod_{j=0}^s \mathbb{Q}_{p_j}$ defined by

$$\varphi_c(x_0, x_1, \ldots, x_s) = (cx_0, cx_1, \ldots, cx_s),$$

$(x_0, x_1, \ldots, x_s) \in \prod_{j=0}^s \mathbb{Q}_{p_j}$. Clearly, φ_c leaves the subgroup B, defined in (2.3), invariant. Thus, φ_c induces an endomorphism of Ω_a. Moreover, all the endomorphisms of Ω_a are of this form. Thus the ring $\mathrm{End}(\Omega_a^d)$ of endomorphisms of Ω_a^d is isomorphic to $\mathrm{M}(d, \mathbb{Z}[1/a])$, where $\mathrm{M}(d, R)$ denotes the ring of $d \times d$ matrices over a ring R. The action of the matrix $C \in \mathrm{M}(d, \mathbb{Z}[1/a])$ on $\prod_{j=0}^s \mathbb{Q}_{p_j}^d$ is given by

$$C(x_0, x_1, \ldots, x_s) = (Cx_0, Cx_1, \ldots, Cx_s), \tag{2.6}$$

$(x_0, x_1, \ldots, x_s) \in \prod_{j=0}^s \mathbb{Q}_{p_j}^d$.

If C is an endomorphism of Ω_a^d, then the dual endomorphism $\hat{C}$ is given by the same matrix acting from the right on $\mathbb{Z}[1/a]^d$.

2.3.2. *Norms.* The norm of the vector $x = (x_1, \ldots, x_d)$ belonging to the p-adic vector space $\mathbb{Q}_p^d$, is defined by

$$\|x\|_p = \max_{1 \le j \le d} |x_j|_p. \tag{2.7}$$

The p-adic absolute value $|\cdot|_p$ has a unique extension to any finite algebraic extension K of $\mathbb{Q}_p$. The norm in K^d is defined as in (2.7). For a $\mathbb{Q}_p$-linear map $A : \mathbb{Q}_p^d \to \mathbb{Q}_p^d$ its norm is defined as $\|A\|_p = \sup_{\|x\|_p \le 1} \|Ax\|_p$. For a K-linear map from K^d to K^d, where K is a finite algebraic extension of K, we define its norm similarly.

By $\mathbb{Q}_a^d$ we denote the *"covering space"* of Ω_a^d, that is

$$\mathbb{Q}_a^d = \prod_{j=0}^s \mathbb{Q}_{p_j}^d,$$

where a is a product of primes $a = p_1 \ldots p_s$ and $p_0 = \infty$, $\mathbb{Q}_\infty = \mathbb{R}$.

Next, we define the "norm" $\|\cdot\|$ on $\mathbb{Q}_a^d$. For $\mathbf{x} = (x_0, x_1, \ldots, x_s) \in \mathbb{Q}_a^d$, let us put

$$\|\mathbf{x}\| = \max_{0 \le j \le s} \|x_j\|_{p_j}. \tag{2.8}$$

The space $\mathbb{Q}_a^d$ becomes a metric space with the metric induced by (2.8).

2.3.3. *Homomorphism* $\chi_d : \Omega_a^d \to \mathbb{T}^d$. Define

$$\chi_1 : \Omega_a = \mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_s}/B \to \mathbb{T},$$

by the following formula

$$\begin{aligned}\chi_1\big((x_0, x_1, \ldots, x_s) + B\big) &= e^{2\pi i x_0} e^{2\pi i \{x_1\}_{p_1}} \cdots e^{2\pi i \{x_s\}_{p_s}} \\ &= e^{2\pi i x_0} \tau_{p_1}(x_1) \cdots \tau_{p_s}(x_s),\end{aligned} \tag{2.9}$$

where τ_p is defined in (2.2). Since τ_p is a homomorphism from $\mathbb{Q}_p$ to the 1-torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, it is easy to check that the map χ_1 is well defined, i.e., for every $r \in \mathbb{Z}[1/a]$, we have

$$\chi_1\left((x_0 + r, x_1 - r, \ldots, x_s - r) + B\right) = \chi_1\left((x_0, x_1, \ldots, x_s) + B\right).$$

Now, we extend the map χ_1 defined in (2.9) to Ω_a^d, $d > 1$. For $j = 0, \ldots, s$, we denote

$$x^j = (x_1^j, \ldots, x_d^j) \in \mathbb{Q}_{p_j}^d.$$

Now we define a homomorphism

$$\chi_d : \Omega_a^d \to \mathbb{T}^d$$

by formula

$$\chi_d\big((x^0, x^1, \ldots, x^s) + B^d\big) = \big(\chi_1(x_1^0, x_1^1, \ldots, x_1^s), \ldots, \chi_1(x_d^0, x_d^1, \ldots, x_d^s)\big). \tag{2.10}$$

2.3.4. *ID-semigroups and Berend's Theorem.* Following [1], [2], we say that the semigroup Σ of endomorphisms of a compact group G has the *ID-property* if the only infinite closed Σ-invariant subset of G is G itself.[3] Recall that that a subset $A \subset G$ is said to be Σ-*invariant* if $\Sigma A \subset A$.

We say, as we do in the case of real numbers, that two endomorphisms σ and τ are *rationally dependent* if there exist integers m and n, not simultaneously equal to 0, such that $\sigma^m = \tau^n$. Otherwise, we say that σ and τ are *rationally independent*.

Berend in [2] gave necessary and sufficient conditions in arithmetical terms for a commutative semigroup Σ of endomorphisms of Ω_a^d to have the ID-property. Namely, he proved the following.

Theorem 2.1 (Berend, [2, Theorem II.1]). *A commutative semigroup Σ of continuous endomorphisms of Ω_a^d has the ID-property if and only if the following hold:*

[3] ID stands for *infinite invariant is dense.*

(i) *There exists an endomorphism* $\sigma \in \Sigma$ *such that the characteristic polynomial* f_{σ^n} *of* σ^n *is irreducible over* $\mathbb{Q}$ *for every positive integer* n.

(ii) *For every common eigenvector* v *of* Σ *there exists an endomorphism* $\sigma_v \in \Sigma$ *whose eigenvalue in the direction of* v *is of norm greater than* 1.

(iii) Σ *contains a pair of rationally independent endomorphisms.*

Let us explain in more details how to understand the statement of the condition (ii). It is proved in [2] that the condition (i) implies that the roots $\lambda_{1,\sigma}, \dots \lambda_{d,\sigma}$ of σ are distinct and that there exists a basis $v^{(i)} \in \mathbb{Q}(\lambda_{i,\sigma})^d$, $i = 1, \dots, d$, in which Σ has a diagonal form. Let K_j be the splitting field of the characteristic polynomial f_σ of σ over $\mathbb{Q}_{p_j}$, $j = 0, \dots, s$, and let $v^{1,j}, \dots, v^{d,j}$ be a basis of K_j^d corresponding to $v^{(i)}$, $i = 1, \dots, d$. The vectors $v^{i,j}$, $i = 1, \dots, d$, $j = 0, \dots, s$, are the common eigenvectors of Σ. Denote by $\lambda_{i,j,\tau}$ $i = 1, \dots, d$, the eigenvalues of any $\tau \in \Sigma$, considered as a linear map of K_j^d with respect to the basis $v^{1,j}, \dots, v^{d,j}$. Then the condition (ii) says that for every $1 \le i \le d$ and $0 \le j \le s$ there exists a $\sigma_{i,j} \in \Sigma$ such that $|\lambda_{i,j,\sigma_{i,j}}|_{p_j} > 1$.

2.4. Topological transitivity and ergodicity. Let us start with some basic definitions given in [15], [9]. We consider a *discrete topological dynamical system* (X, f) given by a metric space X and a continuous map $f : X \to X$. We say that a topological dynamical system (X, f) (or simply that a map f) is *topologically transitive* if for any two nonempty open sets $U, V \subset X$ there exists $n = n(U, V) \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. One can show that f is topologically transitive if for every nonempty open set U in X, $\bigcup_{n\ge 0} f^{-n}(U)$ is dense in X (see [11] for other equivalent definitions). If there exists a point $x \in X$ such that its orbit $\{f^n(x) : n \in \mathbb{N}\}$ is dense in X, then we say that x is a *transitive point*. Under some additional assumptions on X, the map f is topologically transitive if and only if there is a transitive point $x \in X$. Namely, we have the following:

Proposition 2.2 ([20])**.** *If* X *has no isolated point and* f *has a transitive point then* f *is topologically transitive. If* X *is separable, second category and* f *is topologically transitive then* f *has a transitive point.*

Consider a probability space $(X, \mathcal{B}, \mu)$ and a continuous transformation $f : X \to X$. We say that the map f is *measure preserving*, and that μ is f*-invariant*, if for every $A \in \mathcal{B}$ we have $\mu(f^{-1}(A)) = \mu(A)$. Recall that f is said to be *ergodic* if every set A such that $f^{-1}(A) = A$ has measure 0 or 1.

Let G be a compact abelian group and let m denote the normalized Haar measure on G. It is known (see e.g. [15]) that m is invariant under surjective continuous homomorphisms. Recall that the *dual group* (or *character group*) $\hat{G}$

of G consists of all continuous homomorphisms χ of G into the group of comlex numbers of modulus one. Given a continuous endomorphism θ of G, the induced homomorphisms θ on $\hat{G}$ is defined by $\hat{\theta}(\chi)(x) = \chi(\theta(x))$ for all $x \in G$.

Theorem 2.3. *Let G ba a compact abelian group with normalized Haar measure m, and let θ be a continuous surjective endomorphism of G. Then the following are equivalent:*

(i) *The endomorphism θ is ergodic.*

(ii) *The induced homomorphism $\hat{\theta}$ has no non-trivial finite orbits on the character group $\hat{G}$.*

(iii) *For every $n \geq 1$ the endomorphism* $\mathrm{Id} - \theta^n$ *of G is serjective.*

(iv) *The dual endomorphism $\hat{\theta}$ is* aperiodic, *i.e., $\hat{\theta}^n - \mathrm{Id}$ is injective for all $n \geq 1$.*

Proof. See e.g. [19], where also other equivalent statements are given. □

We will need the following lemma which is a particular case of classical result giving relation between ergodicity and topological transitivity (see e.g. [15] for the proof).

Lemma 2.4. *If $A \in \mathrm{End}(\Omega_a^d)$ is ergodic then it is topologically transitive. In particular, A has a transitive point $t \in \Omega_a^d$, i.e., $\{A^n t : n \in \mathbb{N}\}$ is dense in Ω_a^d.*

The next lemma characterizes finite invariant sets of ergodic endomorphisms of Ω_a^d.

Lemma 2.5 ([2, Lemma II.15])**.** *Let σ be an ergodic endomorphism of Ω_a^d. A finite σ-invariant set consists only of torsion elements.*

Recall that a closed Σ-invariant set $A \subset \Omega_a^d$ is Σ-*minimal* if it has no proper closed invariant subsets.

Proposition 2.6 ([2, Proposition II.7])**.** *Let Σ be a semigroup of endomorphisms of Ω_a^d satysfying the conditions of Theorms 2.1. Let M be a Σ-minimal set. Then M is a finite set of torsion elements.*

3. Proof of Theorem 1.3 and 1.4

Let $\lambda > 1$ be a fixed real algebraic number of degree $d > 1$ with minimal (monic) polynomial $P_\lambda \in \mathbb{Q}[x]$,

$$P_\lambda(x) = x^d + c_{d-1}x^{d-1} + \cdots + c_1 x + c_0.$$

We associate with λ the following *companion matrix* σ_λ of P_λ,

$$\sigma_\lambda = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -c_0 & -c_1 & -c_2 & \dots & -c_{d-1} \end{pmatrix}.$$

Remark. (i) We can think of σ_λ as a matrix of multiplication by λ in the basis of the algebraic number field $\mathbb{Q}(\lambda)$ consisting of $1, \lambda, \dots, \lambda^{d-1}$, that is, if $x \in \mathbb{Q}(\lambda)$ has coordinates $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{d-1})$ in the basis $\{1, \lambda, \dots, \lambda^{d-1}\}$, then $\lambda x \in \mathbb{Q}(\lambda)$ has coordinates $\alpha\sigma_\lambda$.

(ii) Notice that the characteristic polynomial f_{σ_λ} of σ_λ is equal to P_λ.

For an arbitrary element $\mu \in \mathbb{Q}(\lambda)$, let $g \in \mathbb{Q}[x]$ be such that $\mu = g(\lambda)$. We define the matrix

$$\sigma_\mu = g(\sigma_\lambda). \tag{3.1}$$

Let a be the product of all primes dividing the denominator of some entry of either σ_λ or σ_μ. Then the matrices $\sigma_\lambda, \sigma_\mu \in \mathrm{M}(d, \mathbb{Z}[1/a])$, act on $\mathbb{Z}[1/a]^d$ by multiplication from the right and on $\Omega_a^d = \widehat{\mathbb{Z}[1/a]^d}$ by multiplication from the left. Denote by Σ the semigroup of endomorphisms of Ω_a^d generated by σ_λ and σ_μ. The vector $(1, \lambda, \dots, \lambda^{d-1})^t$ is an eigenvector of the matrix σ_λ with an eigenvalue λ, that is $\sigma_\lambda(1, \lambda, \dots, \lambda^{d-1})^t = \lambda(1, \lambda, \dots, \lambda^{d-1})^t \in \mathbb{R}^d$. Since Σ is a commutative semigroup it follows that

$$v = (1, \lambda, \dots, \lambda^{d-1}, \overbrace{0, \dots, 0}^{ds})^t \in \mathbb{R}^d \times \mathbb{Q}_{p_1}^d \times \dots \times \mathbb{Q}_{p_s}^d$$

is a common eigenvector of Σ acting on $\mathbb{Q}_a^d$ (the action is given by (2.6)). In particular,

$$\sigma_\mu v = g(\sigma_\lambda)v = g(\lambda)v = \mu v.$$

Lemma 3.1. *Let $\mu \in \mathbb{Q}(\lambda)$, i.e., $\mu = g(\lambda)$ for some $g \in \mathbb{Q}[x]$. Let $\lambda_1, \dots, \lambda_d$ and $\mu_1, \dots, \mu_r$ denote the conjugates of $\lambda = \lambda_1$ and $\mu = \mu_1$. Then, for every $j \le d$, there is a $k \le r$, such that $g(\lambda_j) = \mu_k$.*

Proof. For $j = 1, \dots, d$ we define an isomorphism $\varphi_j : \mathbb{Q}(\lambda) \to \mathbb{Q}(\lambda_j)$ by setting $\varphi_j(h(\lambda)) = h(\lambda_j)$ when $h \in \mathbb{Q}[x]$. It is known that for each $j \le d$, μ and $\varphi_j(\mu)$ have the same minimal polynomial (see e.g. [21]). Since $\mu = g(\lambda)$ and $\varphi_j(\mu) = \varphi_j(g(\lambda)) = g(\lambda_j)$, it follows that for each $j \le d$, $g(\lambda)$ and $g(\lambda_j)$ have the same minimal polynomial. But the characteristic polynomial f_{σ_μ} of the matrix $\sigma_\mu = g(\sigma_\lambda)$ has a root $g(\lambda) = \mu$, hence for all $j \le r$, $g(\lambda_j)$ are the zeros of f_{σ_μ}, and the lemma follows. □

Clearly, under the assumptions of Theorem 1.3, the operators σ_λ and σ_μ are rationally independent endomorphisms of Ω_a^d. Since λ^n has degree d over $\mathbb{Q}$ and is a root of the characteristic polynomial $f_{\sigma_\lambda^n}$ of σ_λ^n, it follows that $f_{\sigma_\lambda^n}$ is irreducible over $\mathbb{Q}$ for every $n \in \mathbb{N}$. Furthermore, by (1.3) and Lemma 3.1, all the $|\cdot|_p$-norms ($p \in S$) of λ_i, μ_j, $1 \le i \le d$, $1 \le j \le r$ are greater than 1. Hence, the condition (ii) of Theorem 2.1 is also satisfied. Thus we have proved the following

Lemma 3.2. *Under the assumptions of Theorem 1.3, the semigroup Σ of continuous endomorphisms of Ω_a^d generated by σ_λ and σ_μ is the ID-semigroup.*

Let X be a compact metric space with a distance d. Consider the space $\mathcal{C}_X$ of all closed subsets of X. The *Hausdorff metric* d_H on the space $\mathcal{C}_X$ is defined as

$$d_H(A,B) = \max\{\max_{x\in A} d(x,B), \max_{x\in B} d(x,A)\},$$

where $d(x,B) = \min_{y\in B} d(x,y)$ is the distance of x from the set B. It is known that if X is a compact metric space then $\mathcal{C}_X$ is also compact.

The next lemma generalizes to our setting the corresponding results from [12, Lemma 2.1] and [23], where the semigroup generated by the two maps of the 1-torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} : x \mapsto px \mod 1$ and $x \mapsto qx \mod 1$, and the semigroup of endomorphisms of $\mathbb{T}^d$ were considered. For clarity of exposition, we give detailed proof.

Lemma 3.3. *Let σ, τ be a pair of rationally independent and commuting endomorphisms of Ω_a^d. Assume that the semigroups $\Sigma = \langle \sigma, \tau \rangle$ generated by σ and τ satisfies the conditions of Theorem 2.1, and σ is an ergodic endomorphism of Ω_a^d. Let A be an infinite σ-invariant subset of Ω_a^d. Then for every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that the set $\tau^m A$ is ε-dense.*

Proof. It is clear that, taking the closure of A if necessary, we can assume that A is closed. We consider the space $\mathcal{C}_{\Omega_a^d}$ of all closed subsets of Ω_a^d, with the Hausdorff metric d_H. Let

$$\mathcal{F} := \overline{\{\tau^n A : n \in \mathbb{N}\}} \subset \mathcal{C}_{\Omega_a^d}.$$

Since the set A is σ-invariant, it follows that every element (set) $F \in \mathcal{F}$ is also σ-invariant. Define,

$$T = \bigcup_{F\in\mathcal{F}} F \subset \Omega_a^d.$$

Since A is an infinite set and $A \subset T$, it follows that T is infinite. Notice that T is closed in Ω_a^d, since $\mathcal{F}$ is closed in $\mathcal{C}_{\Omega_a^d}$. Moreover, T is σ- and τ-invariant. Hence, by Theorem 2.1, we get

$$T = \Omega_a^d.$$

Since σ is an ergodic endomorphism, it follows by Lemma 2.4, that there exists $t \in T$ such that the orbit $\{\sigma^n t : n \in \mathbb{N}\}$ is dense in Ω_a^d, i.e.,

$$\overline{\{\sigma^n t : n \in \mathbb{N}\}} = \Omega_a^d. \tag{3.2}$$

Clearly, $t \in F$ for some $F \in \mathcal{F}$. By definition of $\mathcal{F}$, there is a sequence $\{n_k\} \subset \mathbb{N}$ such that $F = \lim_k \tau^{n_k} A$, and the limit is taken in the Hausdorff metric d_H. Since $t \in F$ and F is σ-invariant, we get, $F \supset \overline{\{\sigma^n t : n \in \mathbb{N}\}} = \Omega_a^d$, by (3.2). Hence, $F = \Omega_a^d$. Therefore, for sufficiently large k, $\tau^{n_k} A$ is ε-dense. □

Now we are ready to give

Proof of Theorem 1.3. Let $\alpha = \xi(1, \lambda, \lambda^2, \ldots, \lambda^{d-1}, \overbrace{0, \ldots, 0}^{ds})^t \in \mathbb{Q}_a^d$ be a common eigenvector of the semigroup $\Sigma = \langle \sigma_\lambda, \sigma_\mu \rangle$ (acting on $\mathbb{Q}_a^d$). Consider the following subset of Ω_a^d,

$$A = \{\sigma_\lambda^n \pi(\alpha) : n \in \mathbb{N}\} = \{\pi(\lambda^n \xi, \lambda^{n+1}\xi, \ldots, \lambda^{n+d-1}\xi, \overbrace{0, \ldots, 0}^{ds}) : n \in \mathbb{N}\},$$

where $\pi : \mathbb{Q}_a^d \to \Omega_a^d$ is the canonical projection.

Notice that A is infinite. In fact, suppose that A is finite. Using Theorem 2.3 we check that σ_λ is ergodic. Clearly, A is σ_λ-invariant. Hence, by Lemma 2.5, A consists only of torsion elements. However, $\lambda \notin \mathbb{Q}$, so $\pi(\alpha)$ is not a torsion element, and we get a contradiction. By Lemma 3.2, $\Sigma = \langle \sigma_\lambda, \sigma_\mu \rangle$ is the ID-semigroup of Ω_a^d. Thus, by Lemma 3.3 applied to σ_λ and σ_μ, there exists $m \in \mathbb{N}$ such that $\sigma_\mu^m A$ is ε-dense. Let $v_m = \pi(r_m, 0, \ldots, 0)$. Since

$$\sigma_\mu^m A + v_m = \{\pi(\mu^m \lambda^n \xi + r_m, \mu^m \lambda^{n+1}\xi, \ldots, \mu^m \lambda^{n+d-1}\xi, 0, \ldots, 0) : n \in \mathbb{N}\}$$

is a translate of an ε-dense set, it is also ε-dense. Now, taking the image of the set $\sigma_\mu^m A + v_m$ by the homomorphism $\chi_d : \Omega_a^d \to \mathbb{T}^d$, defined in (2.10), and then projecting on the first coordinate, the result follows. □

Proof of Theorem 1.4. Assume that $\lambda > \mu$. The result will follow if we were able to show that for every $\varepsilon > 0$ there is $M \in \mathbb{N}$ such that the set $\{\lambda^m \mu^{M-m} \xi : 0 \le m \le M\}$ is ε-dense modulo 1. In order to do this we consider the companion matrices $\sigma_\lambda, \sigma_\mu$ and $\sigma_{\lambda/\mu}$ acting on Ω_a^d (since $\lambda/\mu \in \mathbb{Q}(\lambda)$, we define $\sigma_{\lambda/\mu}$ in the same way as σ_μ, i.e., by (3.1)). Observe that none of the eigenvalues of $\sigma_{\lambda/\mu}$ is a root of unity. In fact, the eigenvalues of $\sigma_{\lambda/\mu}$ are of the form $\lambda_i/g(\lambda_i)$. Suppose that $\lambda_i/g(\lambda_i) \in \mathbb{Q}(\lambda_i)$ is a root of unity. Applying the isomorphism $\varphi_i^{-1} : \mathbb{Q}(\lambda_i) \to \mathbb{Q}(\lambda)$, defined in the proof of Lemma 3.1, to the ratio $\lambda_i/g(\lambda_i)$ we get

$$\varphi_i^{-1}\left(\frac{\lambda_i}{g(\lambda_i)}\right) = \frac{\lambda}{\varphi_i^{-1}(\varphi_i(g(\lambda)))} = \frac{\lambda}{\mu}.$$

Hence λ/μ is a root of unity, and suitable powers of λ and μ are equal. But λ and μ are rationally independent. Hence, we get a contradiction, and by Theorem 2.3 (iii) we conclude that the operator $\sigma_{\lambda/\mu}$ is ergodic. Now, by Lemma 2.4, it follows that there is an element $t \in \Omega_a^d$ such that its orbit $\{\sigma_{\lambda/\mu}^n t : n \in \mathbb{N}\}$ is dense in Ω_a^d. Thus, by compactness, for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$, and a neighborhood U of t such that for every $s \in U$, $\{\sigma_{\lambda/\mu}^n s : 0 \le n \le N\}$ is ε-dense in Ω_a^d. Let $U_0 = \sigma_{\lambda/\mu}^{-N}(U)$. Let $\alpha = \xi(1, \lambda, \lambda^2, \dots, \lambda^{d-1}, \overbrace{0, \dots, 0}^{ds})^t$. Since $\pi(\alpha)$ is not a torsion element, by Proposition 2.6 we conclude that the Σ-orbit of α is infinite. Thus, by Lemma 3.2, we can take, m_0 and $n_0 \in \mathbb{N}$ such that

$$\sigma_\lambda^{m_0}\sigma_\mu^{n_0}\pi(\alpha) = \theta + b, \tag{3.3}$$

where $b \in B^d$ (B^d is defined in (2.5)) and $\theta \in U_0$. Now, consider the set

$$A_N = \{\sigma_\lambda^{m_0+j}\sigma_\mu^{n_0+N-j}\pi(\alpha) : 0 \le j \le N\} = \{\sigma_{\lambda/\mu}^j \sigma_\mu^N \sigma_\lambda^{m_0}\sigma_\mu^{n_0}\pi(\alpha) : 0 \le j \le N\}.$$

By (3.3) $A_N = \{\sigma_{\lambda/\mu}^j\sigma_\mu^N\theta : 0 \le j \le N\}$, $\theta \in U_0$. Since $\sigma_\mu^N\theta \in U$, we conclude that A_N is ε-dense. Taking $M = m_0 + n_0 + N$ we get ε-dense set

$$\{\sigma_\lambda^m\sigma_\mu^{M-m}\pi(\alpha) : 0 \le m \le M\}$$
$$= \{\pi(\lambda^m\mu^{M-m}\xi, \lambda^{m+1}\mu^{M-m}\xi, \dots, \lambda^{m+d-1}\mu^{M-m}\xi, \overbrace{0, \dots, 0}^{ds}) : 0 \le m \le M\}.$$

Now, taking the image of the above set by $\chi_d : \Omega_a^d \to \mathbb{T}^d$, defined in (2.10), we get the result. $\square$

Acknowledgements. The author wishes to thank the anonymous referees for their careful reading of the manuscript and their remarks which significantly improved the paper.

References

[1] D. Berend, Multi-invariant sets on tori, *Trans. Amer. Math. Soc.* **280** (1983), 509–532.

[2] D. Berend, Multi-invariant sets on compact abelian groups, *Trans. Amer. Math. Soc.* **286** (1984), 505–535.

[3] D. Berend, Actions of sets of integers on irrationals, *Acta Arith.* **48** (1987), 275–306.

[4] D. Berend, Dense (mod 1) dilated semigroups of algebraic numbers, *J. Number Theory* **26** (1987), 246–256.

[5] H. Furstenberg, Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation, *Math. Systems Theory* **1** (1967), 1–49.

[6] Y. Guivarc'h and A. N. Starkov, Orbits of linear group actions, random walk on homogeneous spaces, and toral automorphisms, *Ergodic Theory Dynam. Systems* **24** (2004), 767–802.

[7] Y. Guivarc'h and R. Urban, Semigroup actions on tori and stationary measures on projective spaces, *Studia Math.* **171** (2005), 33–66.

[8] E. Hewitt and K. A. Ross, Abstract Harmonic Analysis, Vol. 1, *Springer*, *Berlin*, 1994.

[9] A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Encyclopedia of Mathematics and its Applications, 54, *Cambridge University Press*, *Cambridge*, 1995.

[10] N. Koblitz, p-adic Numbers, p-adic Analysis, and Zeta-Functions. Second edition, Graduate Texts in Mathematics, 58, *Springer-Verlag*, *New York*, 1984.

[11] S. Kolyada and L. Snoha, Some aspects of topological transitivity – a survey, *Grazer Math. Ber.* **334** (1997), 3–35.

[12] B. Kra, A generalization of Furstenberg's Diophantine theorem, *Proc. Amer. Math. Soc.* **127** (1999), 1951–1956.

[13] L. Kuipers and H. Niederreiter, Uniform Distribution of Sequences, Pure and Applied Mathematics, *Wiley-Interscience [John Wiley & Sons]*, *New York – London – Sydney*, 1974.

[14] K. Mahler, p-adic Numbers and their Functions. Second edition, Cambridge Tracts in Mathematics, 76, *Cambridge University Press*, *Cambridge – New York*, 1981.

[15] R. Mañé, Ergodic Theory and Differentiable Dynamics, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), *Springer-Verlag*, *Berlin*, 1987.

[16] D. Meiri, Entropy and uniform distribution of orbits in $\mathbb{T}^d$, *Israel J. Math.* **105** (1998), 155–183.

[17] R. Muchnik, Semigroup actions on $\mathbb{T}^n$, *Geometriae Dedicata* **110** (2005), 1–47.

[18] A. M. Robert, A Course in p-adic Analysis, Graduate Texts in Mathematics, 198, *Springer-Verlag*, *New York*, 2000.

[19] M. Shirvani and T. D. Rogers, Ergodic endomorphisms of compact Abelian groups, *Commun. Math. Phys.* **118** (1988), 401–410.

[20] S. Silverman, On maps with dense orbits and the definition of chaos, *Rocky Mt. J. Math.* **22** (1992), 353–375.

[21] I. Stewart and D. Tall, Algebraic Number Theory and Fermat's Last Theorem. Third edition., *A. K. Peters, Ltd.*, *Natick, MA*, 2002.

[22] R. Urban, On density modulo 1 of some expressions containing algebraic integers, *Acta Arith.* **127** (2007), 217–229.

[23] R. Urban, Sequences of algebraic integers and density modulo 1, *J. Théor. Nombres Bordeaux* (*to appear*).

[24] R. Urban, Algebraic numbers and density modulo 1, *J. Number Theory* (*to appear*).

ROMAN URBAN
INSTITUTE OF MATHEMATICS
WROCLAW UNIVERSITY
PLAC GRUNWALDZKI 2/4
50-384 WROCLAW, POLAND

E-mail: urban@math.uni.wroc.pl

(Received September 26, 2006; revised May 10, 2007)