Publ. Math. Debrecen
72/1-2 (2008), 155–166

A new characterization of the reduced minimum modulus of an operator on Banach spaces

By YIFENG XUE (Shanghai)

Abstract. Let X, Y be Banach spaces and let $B(X,Y)$ (resp. $C(X,Y)$) denote the set of all bounded (resp. nonzero densely defined and closed) linear operators T from X (resp. $\mathfrak{D}(T)$) to Y. We prove that the reduced minimum modulus $\gamma(T)$ of $T \in C(X,Y)$ is $\inf\{\|A\| \mid \operatorname{Ker} T \subsetneqq \operatorname{Ker}(T+A),\ A \in B(X,Y)\}$. Using this result, we give various estimates of the upper bound of $|\gamma(T+A) - \gamma(T)|$ for any $T \in C(X,Y)$ and $A \in B(X,Y)$.

1. Introduction

Throughout this paper, $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ denote Banach spaces over $\mathbb{C}$ and $B(X,Y)$ is the Banach space of all bounded linear operators from X to Y. Put $X^* = B(X,\mathbb{C})$. Let $C(X,Y)$ be the set of all nonzero closed linear operators T from $\mathfrak{D}(T)$ to Y with $\mathfrak{D}(T)$ dense in X. According to [11], for $T \in C(X,Y)$ the null space $\operatorname{Ker} T$ of T is a closed subspace of X and the reduced minimum modulus $\gamma(T)$ of T is given by

$$\gamma(T) = \inf\{\|Tx\| \mid \operatorname{dist}(x, \operatorname{Ker} T) = 1,\ x \in \mathfrak{D}(T)\}. \tag{1.1}$$

Let $T \in B(X,Y)$; the adjoint operator T^* defined by

$$(T^*y^*)(x) = y^*(Tx), \quad \forall\, x \in X,\ y^* \in Y^*,$$

Mathematics Subject Classification: 47A05.
Key words and phrases: reduced minimum modulus, gap of two subspaces, range of an operator, kernel of an operator.
Research supported by Natural Science Foundation of China.

is in $B(Y^*, X^*)$ with $\|T\| = \|T^*\|$. If $T \in C(X, Y)$, then there is a unique closed operator T^* from $\mathfrak{D}(T^*) \subset Y^*$ to X^* such that $(T^*y^*)(x) = y^*(Tx)$, $\forall\, x \in \mathfrak{D}(T)$, $y^* \in \mathfrak{D}(T^*)$. We have that $\mathrm{Ran}(T) = \{Tx \mid x \in \mathfrak{D}(T)\}$ is closed iff $\gamma(T) > 0$ and $\gamma(T^*) = \gamma(T)$. From (1.1), we have

$$\|Tx\| \geq \gamma(T)\,\mathrm{dist}(x, \mathrm{Ker}\, T), \quad \forall\, T \in C(X, Y) \text{ and } x \in \mathfrak{D}(T). \tag{1.2}$$

The reduced minimum modulus of an operator on Banach spaces plays a very important role in the study not only of the spectral properties of operators but also of the generalized inverses of bounded linear operators and in the perturbation analysis of the solutions of operator equations in Banach spaces. For example, if $T \in B(X) = B(X, X)$ and 0 is in the boundary of $\sigma(T)$, then $\overline{\lim}_{n\to\infty}\gamma(T^n)^{\frac{1}{n}} > 0$ implies that 0 is isolated in $\sigma(T)$ (cf. [14]). Furthermore, if T is a Fredholm operator on X with 0 in its generalized resolvent set, then

$$\lim_{n\to\infty} \gamma(T^n)^{\frac{1}{n}} = \sup\left\{ \frac{1}{r(L)} \;\Big|\; TLT = T \right\}, \tag{1.3}$$

where $r(L)$ is the spectral radius of L. When X is a Hilbert space, (1.3) is true even without the condition of Fredholmness for T ([3], [4]). Some other applications of the reduced minimum modulus can be seen in [5], [6], [7], [8], [12], [15], [16].

Let M, N be two subspaces of X. Put

$$\delta(M, N) = \begin{cases} \sup\{\mathrm{dist}(x, N) \mid x \in M,\ \|x\| = 1\} & M \neq \{0\} \\ 0 & M = \{0\} \end{cases}.$$

Let $V(X)$ denote the set of all closed linear subspaces of X. For $M, N \in V(X)$, set $\mathrm{gap}(M, N) = \max\{\delta(M, N),\, \delta(N, M)\}$. $\mathrm{gap}(M, N)$ is called the gap between the subspaces M and N (cf. [11]).

In [13], A. Markus showed that if $S, T \in B(X)$ with $\mathrm{Ran}(S)$ and $\mathrm{Ran}(T)$ closed, then

$$|\gamma(S) - \gamma(T)| \leq \frac{3\|S - T\|}{1 - 2\,\mathrm{gap}(\mathrm{Ker}\, S, \mathrm{Ker}\, T)}, \qquad \mathrm{gap}(\mathrm{Ker}\, S, \mathrm{Ker}\, T) < \frac{1}{2}$$

$$|\gamma(S) - \gamma(T)| \leq \frac{3\|S - T\|}{1 - 2\,\mathrm{gap}(\mathrm{Ran}(S), \mathrm{Ran}(T))} \qquad \mathrm{gap}(\mathrm{Ran}(S), \mathrm{Ran}(T)) < \frac{1}{2}$$

(cf. [12, Lemma 3.4]). These inequalities may be the earliest estimate pertaining to the reduced minimum moduli of operators. Of much later date is an alternate form of the estimate $|\gamma(S) - \gamma(T)|$ given by

$$|\gamma(S) - \gamma(T)| \leq \max\{\gamma(S), \gamma(T)\}\,\mathrm{gap}(\mathrm{Ker}\, S, \mathrm{Ker}\, T) + \|S - T\| \tag{1.4}$$

in case X, Y are Hilbert spaces and $S, T \in B(X,Y)$ (cf. [2] or [5, Lemma 2.3]). If X, Y are Banach spaces and $S, T \in B(X,Y)$, the above is rewritten as

$$|\gamma(S) - \gamma(T)| \leq 2\max\{\gamma(S), \gamma(T)\}\,\mathrm{gap}(\mathrm{Ker}\, S, \mathrm{Ker}\, T) + \|S - T\|$$

(cf. [18]).

In this paper, we first give two new characterizations of the reduced minimum modulus of a closed operator. These two results improve [6, Theorem 2.3] and lead us to define a reduced minimum modulus of a nonzero element in a unital C^*-algebra (see [17]). Then we give some estimates of the bounded operator perturbation of the reduced minimum modulus of a closed operator. Finally, we discuss the continuity of the reduced minimum modulus.

2. Some equivalent descriptions of the reduced minimum modulus

We begin with four lemmas.

Lemma 2.1. *Let $M, N \in V(X)$. Then we have*

(1) $\delta(M,N) = 0$ *iff* $M \subset N$;

(2) $\delta(M,N) < 1$ *implies that* $\dim M \leq \dim N$;

(3) *If* $N \subsetneqq M$, *then* $\delta(M,N) = 1$.

Proof. From the definition of $\delta(\cdot\,,\cdot)$, we can get (1); (2) comes from [11, Corollary IV.2.6] and (3) is [11, Lemma III.1.12]. □

Lemma 2.2 ([9, Lemma 3.2])**.** *Let M, N be two subspaces of X. Then $\delta(M,N) = \delta(\overline{N}, \overline{M})$, where $\overline{M}$ (resp. $\overline{N}$) represents the closure of M (resp. N).*

Lemma 2.3. *Let $T \in C(X,Y)$ and $A \in B(X,Y)$. Then*

$$\gamma(T)\delta(\mathrm{Ker}(T+A), \mathrm{Ker}\, T) \leq \|A\|, \quad \gamma(T)\delta(\mathrm{Ran}(T), \mathrm{Ran}(T+A)) \leq \|A\|.$$

Proof. The proof of the statement is the same as in [7, Lemma 2.3]. □

Lemma 2.4. *Let $T \in C(X,Y)$. Then there is a sequence of operators $\{A_n\} \subset B(X,Y)$ such that*

(1) $\mathrm{Ker}\, T \subsetneqq \mathrm{Ker}(T + A_n)$, $\mathrm{Ran}(T + A_n) \subset \mathrm{Ran}(T)$, $\forall\, n \geq 1$;

(2) $\lim_{n\to\infty} \|A_n\| = \gamma(T)$.

Moreover, if $\mathrm{Ran}(T)$ *is closed, then* $\overline{\mathrm{Ran}(T + A_n)} \neq \mathrm{Ran}(T)$, $\forall\, n \geq 1$.

Proof. By (1.1), we can find $\{x_n\} \subset \mathfrak{D}(T)$ such that

$$\lim_{n\to\infty} \|Tx_n\| = \gamma(T) \quad \text{and} \quad \operatorname{dist}(x_n, \operatorname{Ker} T) = 1,\ \forall\, n \geq 1.$$

Thus there is a sequence $\{f_n\} \subset X^*$ with $\|f_n\| = 1$ and

$$f_n(x_n) = \operatorname{dist}(x_n, \operatorname{Ker} T) = 1,\ f_n(x) = 0,\ \forall\, x \in \operatorname{Ker} T, \quad \forall\, n \geq 1.$$

Put $A_n x = -(Tx_n) f_n(x)$, $\forall\, x \in X$. Then $A_n \in B(X, Y)$, $\lim_{n\to\infty} \|A_n\| = \gamma(T)$ and

$$\operatorname{Ker} T \subset \operatorname{Ker}(T + A_n),\ \operatorname{Ran}(T + A_n) \subset \operatorname{Ran}(T), \quad \forall\, n \geq 1.$$

Noting that $x_n \in \operatorname{Ker}(T + A_n)$ and $x_n \notin \operatorname{Ker} T$, we have $\operatorname{Ker} T \subsetneqq \operatorname{Ker}(T + A_n)$, $\forall\, n \geq 1$. This proves (1) and (2).

Now suppose that $\operatorname{Ran}(T)$ is closed. Let $\{x_n\}$, $\{f_n\}$ and $\{A_n\}$ be as above. Define linear functionals g_n on $\operatorname{Ran}(T)$ by $g_n(Tx) = f_n(x)$, $x \in \mathfrak{D}(T)$, $n \geq 1$. g_n is well-defined since $f_n(x) = 0$, $\forall\, x \in \operatorname{Ker} T$. Moreover, for any $x \in \mathfrak{D}(T)$, any $z \in \operatorname{Ker} T$ and $\forall\, n \geq 1$, we have $|g_n(Tx)| = |f_n(x - z)| \leq \|f_n\| \|x - z\|$. Thus

$$|g_n(Tx)| \leq \operatorname{dist}(x, \operatorname{Ker} T) \leq \frac{1}{\gamma(T)} \|Tx\|$$

by (1.2), i.e., g_n is bounded on $\operatorname{Ran}(T)$ and hence by the Hahn–Banach theorem, there is $\{\hat{g_n}\}_1^\infty \subset Y^*$ such that

$$\hat{g_n}(y) = g_n(y), \quad \forall\, y \in \operatorname{Ran}(T) \text{ and } \|\hat{g_n}\| \leq \frac{1}{\gamma(T)}$$

for any $n \geq 1$. Since $\hat{g_n}((T + A_n)x) = 0$, $\forall\, x \in \mathfrak{D}(T) = \mathfrak{D}(T + A_n)$ and $\hat{g_n}(Tx_n) = 1$, we conclude that $\overline{\operatorname{Ran}(T + A_n)} \subsetneqq \operatorname{Ran}(T)$, $\forall\, n \geq 1$. □

Let $T \in C(X, Y)$ and set

$$M_1(T) = \{A \in B(X, Y) \mid \overline{\operatorname{Ran}(T + A)} \subset \overline{\operatorname{Ran}(T)},\ \operatorname{Ker} T \subsetneqq \operatorname{Ker}(T + A)\},$$

$$M_2(T) = \{A \in B(X, Y) \mid \overline{\operatorname{Ran}(T + A)} \subsetneqq \overline{\operatorname{Ran}(T)},\ \operatorname{Ker} T \subset \operatorname{Ker}(T + A)\}.$$

We now present our main result as follows:

Theorem 2.5. *Let $T \in B(X, Y)$ and $M_1(T)$, $M_2(T)$ be as above. Then*

$$\begin{aligned}\gamma(T) &= \inf\{\, \|A\| \mid A \in M_1(T)\,\} \\ &= \inf\left\{ \frac{\|A\|}{\delta(\operatorname{Ker}(T + A), \operatorname{Ker} T)} \;\Big|\; \operatorname{Ker}(T + A) \not\subseteq \operatorname{Ker} T,\ A \in B(X, Y) \right\} \\ &= \inf\{\, \|A\| \mid \operatorname{Ker} T \subsetneqq \operatorname{Ker}(T + A),\ A \in B(X, Y)\,\}.\end{aligned}$$

In addition, if $\operatorname{Ran}(T)$ *is closed, then*

$$\begin{aligned}\gamma(T) &= \inf\{\, \|A\| \mid A \in M_2(T)\,\} \\ &= \inf\left\{ \frac{\|A\|}{\delta(\operatorname{Ran}(T), \operatorname{Ran}(T+A))} \;\middle|\; \operatorname{Ran}(T) \not\subseteq \overline{\operatorname{Ran}(T+A)},\ A \in B(X,Y) \right\} \\ &= \inf\{\, \|A\| \mid \overline{\operatorname{Ran}(T+A)} \subsetneqq \operatorname{Ran}(T),\ A \in B(X,Y)\}.\end{aligned}$$

Proof. Set

$$\begin{aligned} S_1(T) &= \{A \in B(X,Y) \mid \operatorname{Ker}(T+A) \not\subseteq \operatorname{Ker} T\}, \\ S_2(T) &= \{A \in B(X,Y) \mid \overline{\operatorname{Ran}(T)} \not\subseteq \overline{\operatorname{Ran}(T+A)}\}, \\ S_3(T) &= \{A \in B(X,Y) \mid \operatorname{Ker} T \subsetneqq \operatorname{Ker}(T+A)\}, \\ S_4(T) &= \{A \in B(X,Y) \mid \overline{\operatorname{Ran}(T+A)} \subsetneqq \overline{\operatorname{Ran}(T)}\}. \end{aligned}$$

Clearly, $M_1(T) \subset S_3(T) \subset S_1(T)$, $M_2(T) \subset S_4(T) \subset S_2(T)$. By Lemma 2.1 (3), $\delta(\operatorname{Ker}(T+A), \operatorname{Ker} T) = 1$ when $A \in M_1(T)$ or $A \in S_3(T)$; by Lemma 2.1 (3) and Lemma 2.2, $\delta(\operatorname{Ran}(T), \operatorname{Ran}(T+A)) = 1$ when $A \in M_2(T)$ or $A \in S_4(T)$. Thus we have by Lemma 2.3,

$$\begin{aligned}\gamma(T) &\le \inf\left\{ \frac{\|A\|}{\delta(\operatorname{Ran}(T), \operatorname{Ran}(T+A))} \;\middle|\; A \in S_2(T) \right\} \\ &\le \inf\{\|A\| \mid A \in S_4(T)\} \le \inf\{\|A\| \mid A \in M_2(T)\} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\gamma(T) &\le \inf\left\{ \frac{\|A\|}{\delta(\operatorname{Ker}(T+A), \operatorname{Ker} T)} \;\middle|\; A \in S_1(T) \right\} \\ &\le \inf\{\|A\| \mid A \in S_3(T)\} \le \inf\{\|A\| \mid A \in M_1(T)\} \end{aligned} \tag{2.2}$$

On the other hand, by Lemma 2.4, there is a sequence of operators $\{A_n\} \subset B(X,Y)$ such that $\lim_{n\to\infty} \|A_n\| = \gamma(T)$ and $\operatorname{Ran}(T+A_n) \subset \operatorname{Ran}(T)$, $\operatorname{Ker} T \subsetneqq \operatorname{Ker}(T+A_n)$, $n \ge 1$ and moreover, $\overline{\operatorname{Ran}(T+A_n)} \subsetneqq \operatorname{Ran}(T)$ if $\operatorname{Ran}(T)$ is closed. Since $\{A_n\}_1^\infty \subset M_1(T)$ and $\{A_n\}_1^\infty \subset M_2(T)$ if $\operatorname{Ran}(T)$ is closed, it follows that

$$\inf\{\|A\| \mid A \in M_1(T)\} \le \gamma(T), \tag{2.3}$$

$$\inf\{\|A\| \mid A \in M_2(T)\} \le \gamma(T) \quad \text{(when } \operatorname{Ran}(T) \text{ is closed)}. \tag{2.4}$$

Therefore, combining (2.1) with (2.4) and (2.2) with (2.3), we get the results. $\square$

Corollary 2.6. *Let $T \in C(X,Y)$*

(1) *If* $\dim \operatorname{Ker} T < \infty$, *then*

$$\gamma(T) = \inf\{\|A\| \mid \dim \operatorname{Ker}(T+A) > \dim \operatorname{Ker} T,\ A \in B(X,Y)\};$$

(2) *If* $\dim \operatorname{Ran}(T) < \infty$, *then*

$$\gamma(T) = \inf\{\|A\| \mid \dim \operatorname{Ran}(T) > \dim \operatorname{Ran}(T+A),\ A \in B(X,Y)\}.$$

(3) *If* $\operatorname{codim} \operatorname{Ran}(T) < \infty$, *then*

$$\gamma(T) = \inf\{\|A\| \mid \operatorname{codim} \operatorname{Ran}(T+A) > \operatorname{codim} \operatorname{Ran}(T),\ A \in B(X,Y)\}.$$

Proof. (1) Let $A \in B(X,Y)$ with $\dim \operatorname{Ker}(T+A) > \dim \operatorname{Ker} T$. Then by Lemma 2.1 (2), $\delta(\operatorname{Ker}(T+A), \operatorname{Ker} T) = 1$. Noting that

$$M_1(T) \subset \{A \in B(X,Y) \mid \dim \operatorname{Ker}(T+A) > \dim \operatorname{Ker} T\} \subset S_1(T),$$

we obtain that by Theorem 2.5,

$$\gamma(T) \le \inf\{\|A\| \mid \dim \operatorname{Ker}(T+A) > \dim \operatorname{Ker} T,\ A \in B(X,Y)\} \le \gamma(T),$$

i.e., $\gamma(T) = \inf\{\|A\| \mid \dim \operatorname{Ker}(T+A) > \dim \operatorname{Ker} T,\ A \in B(X,Y)\}$.

(2) The proof is similar to the proof of (1).

(3) $\operatorname{codim} T < \infty$ implies that $\operatorname{Ran}(T)$ is closed. Let $A \in B(X,Y)$ with $\operatorname{codim} T < \operatorname{codim}(T+A)$. Since

$$\operatorname{codim}(T+A) = \dim \operatorname{Ker}(T+A)^* \quad \text{and} \quad \operatorname{codim} T = \dim \operatorname{Ker} T^*$$

by [11, Theorem IV. 5.13], it follows from Corollary 2.6 (1) that

$$\gamma(T) = \gamma(T^*) \le \|A^*\| = \|A\|. \tag{2.5}$$

Now, for any $\epsilon > 0$ we can choose $B \in B(X,Y)$ with $\overline{\operatorname{Ran}(T+B)} \subsetneqq \operatorname{Ran}(T)$ such that $\gamma(T) > \|B\| - \epsilon$ by Theorem 2.5. From

$$\operatorname{Ker} T^* = \operatorname{Ran}(T)^\perp = \{f \in Y^* \mid f(y) = 0,\ \forall\, y \in \operatorname{Ran}(T)\},$$

$$\operatorname{Ker}(T+B)^* = \operatorname{Ran}(T+B)^\perp, \quad \overline{\operatorname{Ran}(T+B)} \subsetneqq \operatorname{Ran}(T)$$

we deduce that $\operatorname{Ker} T^* \subsetneqq \operatorname{Ker}(T+B)^*$. Thus, $\operatorname{codim} T < \operatorname{codim}(T+B)$. This means that

$$\{\|A\| \mid \operatorname{codim}(T+A) > \operatorname{codim} T\} \le \|B\| < \gamma(T) + \epsilon. \tag{2.6}$$

Combining (2.5) with (2.6), we can obtain the assertion. □

Corollary 2.7. *Let $G(X)$ denote the set of all invertible operators in $B(X)$. Then* $\operatorname{dist}(T, B(X)\backslash G(X)) = \|T^{-1}\|^{-1}$, $\forall\, T \in G(X)$.

Proof. If there is $A \in B(X)$ such that, $\|T - A\| < \|T^{-1}\|^{-1}$, then $\|I - T^{-1}A\| < 1$ so that $A \in G(X)$. This indicates that $\operatorname{dist}(T, B(X)\backslash G(X)) \geq \|T^{-1}\|^{-1}$.

Now, for every $\epsilon > 0$ we can find $S \in B(X)$ such that $\operatorname{Ker} S \neq \{0\}$ and

$$\|T^{-1}\|^{-1} = \gamma(T) > \|T - S\| - \epsilon$$

by Theorem 2.5. Since $S \in B(X)\backslash G(X)$, we have

$$\|T^{-1}\|^{-1} \leq \operatorname{dist}(T, B(X)\backslash G(X)) < \|T^{-1}\|^{-1} + \epsilon.$$

The assertion follows. □

3. The perturbation analysis of the reduced minimum modulus

Let $T \in C(X,Y)$ and $A \in B(X,Y)$. In this section, we will consider the relationship between $\gamma(T + A)$ and $\gamma(T)$ and then discuss the continuity of the functional $T \mapsto \gamma(T)$ on $C(X,Y)$.

Lemma 3.1. ([11, Lemma IV.2.2]) *Let X be a Banach space and $V_1, V_2, V_3 \in V(X)$. Then*

$$\delta(V_1, V_2) \geq \frac{\delta(V_1, V_3) - \delta(V_2, V_3)}{1 + \delta(V_2, V_3)}, \quad \delta(V_2, V_3) \geq \frac{\delta(V_1, V_3) - \delta(V_1, V_2)}{1 + \delta(V_1, V_2)}.$$

Proposition 3.2. *$T \in C(X,Y)$ and $A \in B(X,Y)$. Then*

$$\gamma(T + A) \geq \gamma(T)\frac{1 - \delta(\operatorname{Ker} T, \operatorname{Ker}(T + A))}{1 + \delta(\operatorname{Ker} T, \operatorname{Ker}(T + A))} - \|A\|; \tag{3.1}$$

in addition, if $\operatorname{Ran}(T + A)$ *is closed, then*

$$\gamma(T + A) \geq \gamma(T)\frac{1 - \delta(\operatorname{Ran}(T + A), \operatorname{Ran}(T))}{1 + \delta(\operatorname{Ran}(T + A), \operatorname{Ran}(T))} - \|A\|. \tag{3.2}$$

Proof. By Theorem 2.5, there is a sequence of operators $\{B_n\} \subset S_3(T+A)$ (or $\{B_n\} \subset S_4(T + A)$ when $\operatorname{Ran}(T + A)$ is closed) such that $\lim_{n\to\infty} \|B_n\| = \gamma(T + A)$. Consequently, $\delta(\operatorname{Ker}(T + A + B_n), \operatorname{Ker}(T + A)) = 1$ (or $\delta(\operatorname{Ran}(T + A),$

$\mathrm{Ran}(T+A+B_n)) = 1$ when $\mathrm{Ran}(T+A)$ is closed), $n = 1, 2, \ldots$. It follows from Lemma 2.3 and Lemma 3.1 that

$$\begin{aligned}\|A\| + \|B_n\| &\geq \|B_n + A\| \geq \gamma(T)\delta(\mathrm{Ker}(T+A+B_n), \mathrm{Ker}\, T)\\ &\geq \frac{\delta(\mathrm{Ker}(T+A+B_n), \mathrm{Ker}(T+A) - \delta(\mathrm{Ker}\, T, \mathrm{Ker}(T+A))}{1+\delta(\mathrm{Ker}\, T, \mathrm{Ker}(T+A))},\end{aligned}$$

$n \geq 1$. Letting $n \to \infty$, we obtain the (3.1).

When $\mathrm{Ran}(T+A)$ is closed we have, also by Lemma 2.3 and Lemma 3.1,

$$\begin{aligned}\|A\| + \|B_n\| &\geq \|B_n + A\| \geq \gamma(T)\delta(\mathrm{Ran}(T), \mathrm{Ran}(T+A+B_n))\\ &\geq \frac{\delta(\mathrm{Ran}(T+A), \mathrm{Ran}(T+A+B_n)) - \delta(\mathrm{Ran}(T+A), \mathrm{Ran}(T))}{1+\delta(\mathrm{Ran}(T+A), \mathrm{Ran}(T))},\end{aligned}$$

$n \geq 1$. Now let $n \to \infty$, and we get the inequality (3.2). □

Proposition 3.3. *Let $T \in C(X,Y)$ and $A \in B(X,Y)$. If one of the following conditions is satisfied, then $|\gamma(T+A) - \gamma(T)| \leq \|A\|$.*

(1) $\dim \mathrm{Ker}(T+A) = \dim \mathrm{Ker}\, T < \infty$;

(2) $\dim \mathrm{Ran}(T+A) = \dim \mathrm{Ran}(T) < \infty$;

(3) $\mathrm{codim}(T+A) = \mathrm{codim}\, T < \infty$.

Proof. For any $\epsilon > 0$, there is $C \in B(X,Y)$ such that

$$\dim \mathrm{Ker}(T+A) = \dim \mathrm{Ker}\, T < \dim \mathrm{Ker}(T+A+C),\ \ \gamma(T+A) > \|C\| - \epsilon$$

by Corollary 2.6. Thus, by using Corollary 2.6 again, we have

$$\gamma(T) \leq \|A+C\| \leq \|A\| + \|C\| < \gamma(T+A) + \|A\| + \epsilon.$$

Then $\gamma(T) - \gamma(T+A) \leq \|A\|$ as $\epsilon \to 0$. Similarly, we have $\gamma(T+A) - \gamma(T) \leq \|A\|$. So $|\gamma(T+A) - \gamma(T)| \leq \|A\|$.

Similarly, we can obtain the result when T and $T+A$ satisfy (2) or (3). □

The following corollary presents two estimates of the perturbation of $\gamma(\cdot)$ in the general case.

Corollary 3.4. *Let $T \in C(X,Y)$ and $A \in B(X,Y)$. Then*

$$\begin{aligned}&|\gamma(T+A) - \gamma(T)|\\ &\qquad \leq \max\{\gamma(T+A), \gamma(T)\}\frac{2\,\mathrm{gap}(\mathrm{Ker}\, T, \mathrm{Ker}(T+A))}{1+\mathrm{gap}(\mathrm{Ker}\, T, \mathrm{Ker}(T+A))} + \|A\|. \qquad (3.3)\end{aligned}$$

If $\mathrm{gap}(\mathrm{Ker}\, T, \mathrm{Ker}(T+A)) < 1$, *then*

$$|\gamma(T+A) - \gamma(T)| \leq \frac{4\|A\|}{1 - \mathrm{gap}(\mathrm{Ker}\, T, \mathrm{Ker}(T+A))}. \tag{3.4}$$

If $\mathrm{Ran}(T)$ *and* $\mathrm{Ran}(T+A)$ *are both closed, then*

$$\begin{aligned} &|\gamma(T+A) - \gamma(T)| \\ &\qquad \leq \max\{\gamma(T+A),\, \gamma(T)\} \frac{2\ \mathrm{gap}(\mathrm{Ran}(T), \mathrm{Ran}(T+A))}{1 + \mathrm{gap}(\mathrm{Ran}(T), \mathrm{Ran}(T+A))} + \|A\|. \end{aligned} \tag{3.5}$$

If $\mathrm{Ran}(T)$ *and* $\mathrm{Ran}(T+A)$ *are both closed and* $\mathrm{gap}(\mathrm{Ran}(T), \mathrm{Ran}(T+A)) < 1$, *then*

$$|\gamma(T+A) - \gamma(T)| \leq \frac{4\|A\|}{1 - \mathrm{gap}(\mathrm{Ran}(T), \mathrm{Ran}(T+A))}. \tag{3.6}$$

PROOF. By (3.1),

$$\begin{aligned} \gamma(T) - \gamma(T+A) &\leq \gamma(T) \frac{2\delta(\mathrm{Ker}\, T, \mathrm{Ker}(T+A))}{1 + \delta(\mathrm{Ker}\, T, \mathrm{Ker}(T+A))} + \|A\| \\ &\leq \gamma(T) \frac{2\ \mathrm{gap}(\mathrm{Ker}\, T, \mathrm{Ker}(T+A))}{1 + \mathrm{gap}(\mathrm{Ker}\, T, \mathrm{Ker}(T+A))} + \|A\|. \end{aligned}$$

Interchanging T and $T+A$ in the above inequality, we get

$$\gamma(T+A) - \gamma(T) \leq \gamma(T+A) \frac{2\ \mathrm{gap}(\mathrm{Ker}\, T, \mathrm{Ker}(T+A))}{1 + \mathrm{gap}(\mathrm{Ker}\, T, \mathrm{Ker}(T+A))} + \|A\|.$$

Thus we have (3.3).

By (3.1) we have

$$\gamma(T) \geq \gamma(T+A) \frac{1 - \delta(\mathrm{Ker}(T+A), \mathrm{Ker}\, T)}{1 + \delta(\mathrm{Ker}(T+A), \mathrm{Ker}\, T)} - \|A\|.$$

Thus, by Lemma 2.3,

$$\begin{aligned} \gamma(T+A) - \gamma(T) \leq& \frac{(\gamma(T) + \|A\|)(1 + \delta(\mathrm{Ker}(T+A), \mathrm{Ker}\, T)}{1 - \delta(\mathrm{Ker}(T+A), \mathrm{Ker}\, T)} - \gamma(T) \\ \leq& \frac{4\|A\|}{1 - \delta(\mathrm{Ker}(T+A), \mathrm{Ker}\, T)} \leq \frac{4\|A\|}{1 - \mathrm{gap}(\mathrm{Ker}\, T, \mathrm{Ker}(T+A))}. \end{aligned}$$

Similarly, we also have

$$\gamma(T) - \gamma(T+A) \leq \frac{4\|A\|}{1 - \mathrm{gap}(\mathrm{Ker}\, T, \mathrm{Ker}(T+A))}.$$

So we get (3.4).

The remaining proofs are similar. □

Remark 3.5. The author proved in [5] that

$$\gamma(T) \geq \gamma(S)[1 - \delta(\operatorname{Ker} S, \operatorname{Ker} T)] - \|S - T\|, \quad S, T \in B(X, Y)$$

when X, Y are Hilbert spaces. From this inequality we can deduce (1.4) and

$$|\gamma(S) - \gamma(T)| \leq \frac{2\|S - T\|}{1 - \operatorname{gap}(\operatorname{Ker} S, \operatorname{Ker} T)}, \ \operatorname{gap}(\operatorname{Ker} S, \operatorname{Ker} T) < 1.$$

Finally, we discuss the behavior of $\lim_{n\to\infty} \gamma(T + A_n)$ for $T \in C(X, Y)$ and $\{A_n\} \subset B(X, Y)$ with $\lim_{n\to\infty} \|A_n\| = 0$.

Lemma 3.6. *Let X, Y be Banach spaces.*

(1) *For given $\alpha > 0$, the set $\{T \in B(X, Y) \mid \gamma(T) \geq \alpha\}$ is norm–closed in $B(X, Y)$;*

(2) *Assume that X, Y are reflexive. Let $T \in C(X, Y)$ and $\{A_n\} \subset B(X, Y)$ with $\lim_{n\to\infty} \|A_n\| = 0$. If $\gamma = \inf_{n\geq 1} \gamma(T + A_n) > 0$, then $\gamma(T) \geq \gamma$.*

Proof. (1) is Lemma 1.9 of [1]. We now prove (2).

We have by (1.2)

$$\|(T + A_n)x\| \geq \gamma \operatorname{dist}(x, \operatorname{Ker}(T + A_n)), \quad \forall\, x \in \mathfrak{D}(T), \ n \geq 1. \tag{3.7}$$

Since X is reflexive, we can pick $z_n \in \operatorname{Ker}(T + A_n)$ such that $\|x - z_n\| = \operatorname{dist}(x, \operatorname{Ker}(T + A_n))$. Then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ and $z \in X$ such that $z_{n_k} \xrightarrow{w} z$. Consequently, $\|x - z\| \leq \underline{\lim}_{k\to\infty} \|x - z_{n_k}\|$. Noting that

$$\lim_{k\to\infty} \|A_{n_k} z_{n_k}\| \to 0, \quad T z_{n_k} = -A_{n_k} z_{n_k}, \ k \geq 1 \text{ and}$$
$$\lim_{k\to\infty} f(z_{n_k}) \to f(z_0), \quad \forall\, f \in \operatorname{Ran}(T^*),$$

we have $f(z) = 0$, $\forall\, f \in \operatorname{Ran}(T^*)$. Since X, Y are reflexive and T is densely defined and closed, it follows from [11, Problem III.5.27, Theorem III.5.29] that $z \in \operatorname{Ker} T$. Therefore we deduce from (3.7) that

$$\|Tx\| \geq \gamma \underline{\lim}_{k\to\infty} \|x - z_{n_k}\| \geq \gamma \|x - z\| \geq \gamma \operatorname{dist}(x, \operatorname{Ker} T), \quad x \in \mathfrak{D}(T),$$

which implies $\gamma(T) \geq \gamma$. □

Corollary 3.7. *Let X, Y be Banach spaces, $T \in C(X, Y)$ and $\{A_n\} \subset B(X, Y)$.*

(1) *If* $\gamma(T) > 0$ *and* $\operatorname{Ker} T = \{0\}$ *or* $\operatorname{Ran}(T) = Y$*, then* $\lim_{n\to\infty} \gamma(T + A_n) = \gamma(T)$*;*

(2) *Let* X*,* Y *be reflexive. If* $\gamma(T) = 0$*, then* $\lim_{n\to\infty} \gamma(T + A_n) = 0$*;*

(3) *If* $\operatorname{Ran}(T)$ *is closed and* $\operatorname{Ker} T \neq \{0\}$*,* $\operatorname{Ran}(T) \neq Y$*, then there is* $\{B_n\} \subset B(X, Y)$ *with* $\lim_{n\to\infty} \|B_n\| = 0$ *such that* $\lim_{n\to\infty} \gamma(T + B_n) = 0$.

Proof. (1) Let n be large enough so that $\|A_n\| < \gamma(T)$. Then $\operatorname{Ker} T = \{0\}$ indicates that $\operatorname{Ker}(T + A_n) = \{0\}$. Thus, by Proposition 3.3, $|\gamma(T + A_n) - \gamma(T)| \leq \|A_n\|$.

If $\operatorname{Ran}(T) = Y$, then $\operatorname{Ker} T^* = \{0\}$ and $\gamma(T^*) = \gamma(T) > 0$. By applying the above argument to $(T + A_n)^*$ and T^*, we also have

$$|\gamma(T + A_n) - \gamma(T)| = |\gamma((T + A_n)^*) - \gamma(T^*)| \leq \|A_n^*\| = \|A_n\|.$$

(2) If $\lim_{n\to\infty} \gamma(T + A_n) \neq 0$, then there exist an $\epsilon_0 > 0$ and a subsequence $\{\gamma(T + A_{n_k})\}$ of $\{\gamma(T + A_n)\}$ such that $\gamma(T + A_{n_k}) \geq \epsilon_0$, $\forall k \geq 1$. Thus $\gamma(T) \geq \epsilon_0$ by Lemma 3.6 (2), which contradicts the assumption $\gamma(T) = 0$.

(3) Pick $x_0 \in \operatorname{Ker} T$ with $\|x_0\| = 1$ and $y_0 \in Y \setminus \operatorname{Ran}(T)$ with $\|y_0\| = 1$. Let $x_0^* \in X^*$ such that $\|x_0^*\| = x_0^*(x_0) = 1$ and put $B_n(x) = n^{-1} x_0^*(x)\, y_0$, $\forall x \in X$, $n \geq 1$. Then $\operatorname{Ker}(T + B_n) = \operatorname{Ker} T \cap \operatorname{Ker} B_n$, and $\operatorname{Ker}(T + B_n) \subsetneqq \operatorname{Ker} T$, $\forall n \geq 1$. So, by Theorem 2.5, $\gamma(T + B_n) \leq \|-B_n\| = n^{-1}$ hence $\lim_{n\to\infty} \gamma(T + B_n) = 0 \neq \gamma(T)$. □

Combining Lemma 2.3, and Theorem 2.5 with Lemma 3.6 (1), we have

Corollary 3.8. *Let* $\{T_n\} \subset B(X, Y)$ *and* $T \in B(X, Y)$ *with* $\lim\limits_{n\to\infty} \|T_n - T\| = 0$.

(1) *If* $\inf_{n\geq 1} \gamma(T_n) > 0$*, then* $\gamma(T) > 0$ *and* $\lim_{n\to\infty} \gamma(T_n) = \gamma(T)$*;*

(2) *If* $\gamma(T) > 0$ *and* $\gamma(T_n) > 0$*,* $\forall n \geq 1$*, then* $\lim_{n\to\infty} \gamma(T_n) = \gamma(T)$ *iff* $\inf_{n\geq 1} \gamma(T_n) > 0$.

Remark 3.9. (1) Harte and Mbekhta proved that if $T \in B(X, Y)$ with $\operatorname{Ran}(T)$ closed satisfies the condition: $\operatorname{Ker} T = \{0\}$ or $\operatorname{Ran}(T) = Y$, then $\gamma(\cdot)$ is continuous at T; if T satisfies condition $\operatorname{Ker} T \neq \{0\}$ and $\operatorname{Ran}(T) \neq Y$, then $\gamma(\cdot)$ is discontinuous at T ([10, Theorem 9]). By Lemma 3.6 (1), if $T \in B(X, Y)$ is such that $\operatorname{Ran}(T)$ is not closed, then $\gamma(\cdot)$ is continuous at T. All this proves the continuity of $\gamma(\cdot)$ on $B(X, Y)$.

(2) Let X, Y be Hilbert spaces and let $\{T_n\} \subset B(X, Y)$ and $T \in B(X, Y)$ with $\operatorname{Ran}(T)$ and $\operatorname{Ran}(T_n)$ closed, $n \geq 1$. Assume that $\lim_{n\to\infty} \|T_n - T\| = 0$. Then Corollary 3.8 (2) can be rewritten as

$$\lim_{n\to\infty} \gamma(T_n) = \gamma(T) \quad \text{iff} \quad \inf_{n\geq 1} \gamma(T_n) > 0 \quad \text{and}$$

$$\operatorname{Ran}(T_n) \cap \operatorname{Ran}(T)^\perp = \{0\} \quad \text{iff} \quad \operatorname{Ker} T \cap \operatorname{Ker} T_n^\perp = \{0\}$$

for n large enough (cf. [5]).

Acknowledgement. The author is grateful to the referee for very helpful comments and suggestions.

References

[1] C. Apostol, Inner Derivations with closed range, *Rev. Roum. Math. Pures et Appl.* **21**(3) (1976), 249–265.

[2] C. Apostol, The reduced minimum modulus, *Mich. Math. J.* **32**(3) (1985), 279–294.

[3] C. Badea and M. Mbekhta, Generalized inverses ant the maximal radius of regularity of a Fredholm operator, *Integral Equations operator Theory* **28** (1997), 133–146.

[4] C. Badea and M. Mbekhta, Compressions of resolvents and maximal radius of regularity, *Trans. Amer. Math. Soc.* **351** (1999), 2949–2960.

[5] G. Chen, M. Wei and Y. Xue, Perturbation analysis of the least squares solution in Hilbert spaces, *Linear Algebra Appl.* **244** (1996), 69–80.

[6] G. Chen, Y. Wei and Y. Xue, The generalized condition numbers of bounded linear operators in Banach spaces, *J. Australian Math. Soc.* **76** (2004), 281–290.

[7] G. Chen and Y. Xue, Perturbation analysis for the operator equation $Tx = b$ in Banach spaces, *J. Math. Anal. Appl.* **212** (1997), 107–125.

[8] G. Chen and Y. Xue, The expression of the generalized inverse of the perturbed operator under Type I perturbation in Hilbert spaces, *Linear Algebra Appl.* **285** (1998), 1–6.

[9] J. Ding and L.J. Huang, Perturbation of generalized inverses of linear operators in Hilbert spaces, *J. Math. Anal. Appl.* **198** (1996), 506–515.

[10] R. Harte and M. Mbekhta, Generalized inverses inverses in C^*-algebras II, *Studia Math.* **106** (1993), 129–138.

[11] T. Kato, Perturbation Theory for Linear Operators, (2nd Edition), *Berlin, Springer*, 1984.

[12] J. J. Koliha and V. Rakočević, Continuity of the Drazin inverse II, *Studia Math.* **131** (1998), 167–177.

[13] A. Markus, On some properties of linear operators connected with the notion of the gap, *UChen. Zap. Kishinev. Gos. Univ.* **39** (1959), 265–272 (in *Russian*).

[14] M. Mbekhta, Generalized spectrum and a problem of Apostol, *Proc. Amer. Math. Soc.* **118** (1993), 857–859.

[15] M. Mbekhta, Partial isometries and generalized inverses, *Acta Sci. Math.* (Szeged) **70** (2004), 767–781.

[16] C. Schmoeger, Perturbation properties of some classes of operators, *Rendiconti di Matematica (Serie VII)* **14** (1994), 533–541.

[17] Y. Xue, The reduced minimum modulus in C^*-algebras, *Integral Equations operator Theory* (*to appear*).

[18] C. Zhu, J. Cai and G. Chen, Perturbation analysis for the reduced minimum modulus of bounded linear operator in Banach spaces, *Appl. Math. Comp.* **145** (2003), 13–23.

YIFENG XUE
DEPARTMENT OF MATHEMATICS
EAST CHINA NORMAL UNIVERSITY
SHANGHAI 200062
P.R. CHINA

E-mail: yfxue@math.ecnu.edu.cn or xyf63071@public9.sta.net.cn

(Received September 29, 2006; revised February 12, 2007)