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Minimal coverings of completely reducible groups

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Abstract. Let G be a group that is a set-theortic union of finitely many proper subgroups. Cohn defined $\sigma(G)$ to be the least integer m such that G is the union of m proper subgroups. Determining σ is an open problem for most non-solvable groups. In this paper we give a formula for $\sigma(G)$, where G is a completely reducible group.

1. Introduction and results

Let G be a group that is a set-theoretic union of finitely many proper subgroups and by a cover (or covering) of G we mean any finite set of proper subgroups whose set-theoretic union is the whole group G. Cohn [4] defined $\sigma(G)$ to be the least integer m (if it exists) such that G has a covering with m subgroups (we call any such covering minimal) and otherwise $\sigma(G) = \infty$. A result of Neumann [12] states that if G is a union of m proper subgroups, then the intersection of these subgroups is of finite index in G. It follows that in study of $\sigma(G)$, we may assume that G is finite. It is an easy exercise that $\sigma(G)$ can never be 2, so $\sigma(G) \geq 3$. Groups that are the union of three proper subgroups, as $C_2 \times C_2$ is for example, are investigated in papers [6], [7], [14]. Also groups G with $\sigma(G) \in \{3, 4, 5\}$ and $\sigma(G) = 6$ are characterized in [4] and [1], respectively. However Tomkinson [15] proved that there is no group with $\sigma(G) = 7$. Cohn [4] showed that for any prime power p^a there exists

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a solvable group G with $\sigma(G) = p^a + 1$. In fact, Tomkinson [15] established that $\sigma(G) - 1$ is always a prime power for solvable groups G. It is natural to ask what can be said about $\sigma(G)$ for non-solvable groups. Bryce, Fedri and Serena begun this project in [3], where they calculated $\sigma(G)$ for the linear groups $G \in \{PSL_2(q), PGL_2(q), SL_2(q), PGL_2(q)\}$. They obtained the formula $\frac{1}{2}q(q+1)$ for even prime powers $q \geq 4$ and the formula $\frac{1}{2}q(q+1)+1$ for odd prime powers $q \geq 5$. Moreover Lucido [10] studied this problem for the simple Suzuki groups and found that $\sigma(Sz(q)) = \frac{1}{2}q^2(q^2+1)$, where $q = 2^{2m+1}$. Maróti [11] gave exact or asymptotic formulas for $\sigma(\mathrm{Sym}_n)$ and $\sigma(\mathrm{Alt}_n)$. In particular, it is shown in [11] that if $n > 1$ is odd, then $\sigma(\mathrm{Sym}_n) = 2^{n-1}$ unless $n = 9$ and $\sigma(\mathrm{Sym}_n) \leq 2^{n-2}$ if n is even. Also Maróti proved that if $n \neq 7, 9$, then $\sigma(\mathrm{Alt}_n) \geq 2^{n-2}$ with equality if and only if n is even but not divisible by 4. Holmes in [8] obtained $\sigma(S)$ for some sporadic simple groups S. See also [9] for some related results. Thus the situation for non-solvable groups seems to be totally different from solvable ones.

A group G is called *completely reducible* if it is a direct product of simple groups. In the sequel a completely reducible group will be called a CR-group. Note that in a CR-group, every normal subgroup is a direct factor (see [13, Theorem 3.3.12]). A CR-group is centerless if and only if it is a direct product of non-abelian simple groups. A finite group G contains a normal centerless CR-subgroup which contains all normal centerless CR-subgroups; this subgroup is called the centerless CR-radical of G. For more details concerning CR-groups, see [13, pp. 88–89]. In this paper we prove the following results.

Theorem 1.1. *Let G be a finite group. If $G = A_1 \times A_2 \times \cdots \times A_n$, where A_i is a non-abelian simple group for each i, then $\sigma(G) = \min\{\sigma(A_1), \ldots, \sigma(A_n)\}$.*

Theorem 1.2. *Let G be a finite CR-group. Then $\sigma(G) = \min\left\{\sigma(R), \sigma\left(\frac{G}{R}\right)\right\}$, where R is the centerless CR-radical of G.*

2. Proofs

We begin with the following easy lemma.

Lemma 2.1. *Let G be a finite non-cyclic group. If M is a maximal subgroup of G such that $\sigma(G) < \sigma(M)$, then either M is a normal subgroup of G or $|G : M| \leq \sigma(G) - 1$.*

Proof. Suppose that $M \ntrianglelefteq G$. Then M has $|G : M|$ conjugates in G. There are maximal subgroups A_i of G for which $G = \cup_{i=1}^{\sigma(G)} A_i$ and $M = \cup_{i=1}^{\sigma(G)} (M \cap A_i)$.

Since $\sigma(G) < \sigma(M)$, then there exists $j \in \{1, \ldots, \sigma(G)\}$ such that $M = M \cap A_j$. Hence for every $x \in G$, there exist $i_x \in \{1, \ldots, \sigma(G)\}$ such that $M^x = A_{i_x}$. Therefore $|G : M| \leq \sigma(G)$. Now since $G \neq \cup_{g \in G} M^g$, $|G : M| \leq \sigma(G) - 1$. □

The following result which will be useful in the sequel, is a generalization of Lemma 4 of [4]. Its proof is similar to that of Lemma 4 of [4] and we give it for the reader's convenience.

Proposition 2.2. *Let G be a finite group such that $G = H \times K$ for two subgroups H and K of G. If every maximal subgroup of G contains either H or K, then $\sigma(G) = \min\{\sigma(H), \sigma(K)\}$.*

Proof. Since every maximal subgroup M of G contains either H or K, M is equal to either $H_0 \times K$ or $H \times K_0$, where H_0 is maximal in H and K_0 maximal in K. Thus we may assume that $G = (\cup_{i=1}^{p} H \times M_i) \bigcup (\cup_{j=1}^{q} M_j \times K)$, where $p + q = \sigma(G)$, $p, q \geq 0$ and M_i is maximal in K and N_j is maximal in H. Now we claim that one of p and q must be zero.

Let $G_1 = \cup_{i=1}^{p} H \times M_i$ and $G_2 = \cup_{j=1}^{q} N_j \times K$. If $q \neq 0$, then $G_1 \neq G$ and so there exists an element $a_2 \in G \backslash G_1$. Therefore $a_2 \notin M_i$ for all$i \in \{1, \ldots, p\}$ and so $aa_2 \notin G_1$ for all $a \in H$. Hence $aa_2 \in G_2$ for all $a \in H$. Thus $aa' \in G_2$ for all $a \in H$ and $a' \in K$. Hence $G_2 = G$ and $p = 0$.

Now if $p = 0$, then $G = G_2 = (\cup_{j=1}^{q} N_j)K$, whence $H = \cup_{j=1}^{q} N_j$. This implies that $\sigma(H) \leq \sigma(G) = q$. Similarly if $q = 0$, then $\sigma(K) \leq p = \sigma(G)$. But $\sigma(G) \leq \min\{\sigma(H), \sigma(K)\}$ − see for example Lemma 2 in [4] − which gives the result. □

Recall that a finite group G is said to be *primitive* if it has a maximal subgroup M such that the core of M in G, $M_G = \cap_{g \in G} M^g$ is trivial. In this situation we call M a stabilizer of G. We need the following trichotomy of R. Baer on primitive groups.

Theorem 2.3 (Baer [2])**.** *Let G be a finite primitive group with a stabilizer M. Then exactly one of the following three statements holds:*

(1) *G has a unique minimal normal subgroup N, this subgroup N is self-centralizing (in particular, abelian), and N is complemented by M in G.*

(2) *G has a unique minimal normal subgroup N, this N is non-abelian, and N is supplemented by M in G.*

(3) *G has exactly two minimal normal subgroups N and N^*, and each of them is complemented by M in G. Also $C_G(N) = N^*$, $C_G(N^*) = N$ and $N \cong N^* \cong NN^* \cap M$.*

Remark 2.4 (see Example 15.3(3) in p. 54 of [5]). Let G be a finite group.

(1) If M is a maximal subgroup of G, then $\frac{G}{M_G}$ is a primitive group.

(2) If G is a non-abelian simple group, then $G \times G$ is a primitive group in which the diagonal subgroup $D = \{(g, g) : g \in G\}$ is a stabilizer.

Lemma 2.5. *Let H and K be non-abelian simple groups. If $G = H \times K$, then $\sigma(G) = \min\{\sigma(H), \sigma(K)\}$.*

Proof. If $H \cong K$, then $G \cong H \times H$ is a primitive group with stabilizer diagonal subgroup $D = \{(h, h) : h \in H\}$. We have $D \cong H$ and D is a maximal subgroup of G which is not normal in G. If $\sigma(G) < \sigma(H) = \sigma(D)$, then by Lemma 2.1, $|G : D| \leq \sigma(G) - 1$. Since $|G : D| = |H|$, we have $|H| < \sigma(H)$ which is a contradiction. Thus $\sigma(G) \geq \sigma(H)$. Now the corollary to Lemma 2 of [4] completes the proof.

Thus we may assume that $H \ncong K$. Then by Theorem 2.3 G is not a primitive group and so M_G is non-trivial for every maximal subgroup M of G. Therefore $M_G = H$ or $M_G = K$ and so $H \leq M$ or $K \leq M$. The proof is now complete by Proposition 2.2. □

Proof of Theorem 1.1. We argue by induction on n. If $n = 1$, then the result is clear and if $n = 2$, then the result follows from Lemma 2.5. So we may assume that $n \geq 3$. If there exist distinct $i, j \in \{1, \dots, n\}$ such that $A_i \cong A_j$ and $i < j$, then $G \cong G_1 = N \times A_i \times A_i$, where

$$N = \prod_{k \in \{1,\dots,n\} \setminus \{i,j\}} A_k.$$

Now consider $M = N \times D$, where $D = \{(a, a) : a \in A_i\}$ is the diagonal subgroup of $A_i \times A_i$. Then M is a maximal subgroup of G_1 which is not normal in G_1, since $D \ntrianglelefteq A_i \times A_i$. On the other hand, since $D \cong A_i$, by the induction hypothesis we have $\sigma(M) = \min\{\sigma(A_1), \dots, \sigma(A_n)\}$. It follows from the corollary to Lemma 2 of [4] that $\sigma(G_1) \leq \sigma(M)$. Now suppose, aiminig for a contradiction, that $\sigma(G_1) < \sigma(M)$. Then Lemma 2.1 implies that $|G_1 : M| < \sigma(G)$. Therefore $\sigma(G) > |A_i| > \sigma(A_i)$, which is the contradiction we sought. Hence $\sigma(G) = \sigma(M) = \min\{\sigma(A_1), \dots, \sigma(A_n)\}$.

Now assume that $A_i \ncong A_j$ for any two distinct $i, j \in \{1, \dots, n\}$ and let $H = A_1 \times A_2 \times \cdots \times A_{n-1}$. We claim that every maximal subgroup S of G contains either H or A_n. If $A_n \nleq S$, then $A_n \nleq S_G$ and so $S_G = A_{i_1} \times \cdots \times A_{i_k}$, where $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n-1\}$. Since $\frac{G}{S_G}$ is a primitive group, Theorem 2.3 implies that $k = n - 1$ and so $S_G = H \leq S$. The proof is now complete by Proposition 2.2 and induction hypothesis. □

Proof of Theorem 1.2. Suppose that $G = A \times R$ such that A is an abelian CR-subgroup of G and R is the centerless CR-radical of G. We may assume that both A and R are non-trivial. We claim that every maximal subgroup M of G contains either A or R. If $A \nleq M$, then $A \nleq M_G$. Thus there exists a normal subgroup N of prime order such that $N \nleq M_G$. Since $\frac{G}{M_G}$ is a primitive group and $\frac{NM_G}{M_G}$ is a minimal normal subgroup of $\frac{G}{M_G}$, it follows from Theorem 2.3 that $\frac{G}{M_G}$ contains a unique minimal normal abelian subgroup. If $R \nleq M_G$, then there exists a non-abelian simple normal subgroup $S \leq R$ of G such that $S \nleq M_G$. Thus $\frac{SM_G}{M_G}$ is a minimal normal subgroup of $\frac{G}{M_G}$, and so it is abelian, a contradiction. This implies that $R \leq M_G \leq M$. Now the proof follows from Proposition 2.2. □

Proposition 2.6. *Let H be a finite CR-group whose center is of odd order and let Sym_n be the symmetric group of degree $n \geq 5$. Then $\sigma(H \times \mathrm{Sym}_n) = \min\{\sigma(H), \sigma(\mathrm{Sym}_n)\}$.*

Proof. By hypothesis and Proposition 2.2, it is enough to show that every maximal subgroup M of $G = H \times \mathrm{Sym}_n$ contains either H or Sym_n. If $H \not< M$, then $H \not< M_G$ and so, as H is a CR-group, there exists a (non-abelian or abelian) simple normal subgroup S contained in H such that $S \not< M_G$. Therefore $S \cap M_G = 1$ and $\frac{SM_G}{M_G} \cong S$ is a (simple) minimal normal subgroup of $\frac{G}{M_G}$. Also $M_G \cap \mathrm{Sym}_n = 1$, Alt_n or Sym_n.
We dismiss the first two of these possibilities.

(1) If $M_G \cap \mathrm{Sym}_n = 1$, then $\mathrm{Sym}_n \cong \frac{M_G \mathrm{Sym}_n}{M_G} \trianglelefteq \frac{G}{M_G}$. Since $\mathrm{Alt}_n \trianglelefteq \mathrm{Sym}_n$, $\overline{K} = \frac{M_G \mathrm{Alt}_n}{M_G}$ is a minimal normal subgroup of $\overline{G} = \frac{G}{M_G}$. Now we claim that $\overline{K} \neq \frac{SM_G}{M_G}$; if $X = \mathrm{Alt}_n M_G = SM_G$ and each product is direct. Now $C_X(M_G) = Z(M_G)\mathrm{Alt}_n = Z(M_G)S$ so $C_X(M_G)' = \mathrm{Alt}_n = S' \leq H$, a contradiction. Since $\frac{G}{M_G}$ is primitive, Theorem 2.3 implies that $C_{\overline{G}}(\frac{SM_G}{M_G}) = \overline{K}$. Thus $\mathrm{Sym}_n \cong \frac{M_G \mathrm{Sym}_n}{M_G} \leq \overline{K} \cong \mathrm{Alt}_n$, which is a contradiction.

(2) In this case $M_G \cap \mathrm{Sym}_n = \mathrm{Alt}_n$ and so $\frac{M_G \mathrm{Sym}_n}{M_G}$ is a normal subgroup of order 2, therefore central in the primitive group $\frac{G}{M_G}$. Thus by Theorem 2.3, $\frac{G}{M_G} \cong C_2$. Since $S \cong \frac{M_G S}{M_G} \leq \frac{G}{M_G}$, we have that $S \cong C_2$ and so the center of H is of even order, contradicting the hypothesis.

Hence $M_G \cap \mathrm{Sym}_n = \mathrm{Sym}_n \leq M_G \leq M$. This completes the proof. □

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