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Algebraic approach to equivariance of solutions for an iterative equation

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Abstract. Describing the symmetry of a mapping by equivariance with respect to a linear transformation group, the reference [Proc. Roy. Soc. Edinburgh **A130** (2000), 1153–1163] gave the existence of equivariant solutions of the polynomial-like iterative equation under the action of topologically finitely generated subgroups of $GL(\mathbb{R})$ on $\mathbb{R}$ and the orthogonal group $\mathbf{O}(N)$ on $\mathbb{R}^N$ ($N \geq 2$). In this paper, based on the algebraic structure of closed subgroups of $GL(\mathbb{R})$, we prove the equivariance of solutions on $\mathbb{R}$ with respect to closed subgroups of $GL(\mathbb{R})$ and extend the result of $\mathbf{O}(N)$-equivariance of solutions to the group $\mathbf{O}(N) \times \langle c\mathcal{I}_N \rangle$ on $\mathbb{R}^N$.

1. Introduction

Related to problems of iterative roots (see [9], [22]), invariant curves (see [9], [14], [17]) and normal forms of dynamical systems (see (2.16) in [1]), equations involving iteration become interesting. For a self-mapping f on a Banach space X over $\mathbb{R}$ and a positive integer n, the n-th iterate f^n is defined by $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) \equiv x$. An interesting form of such equations is the so-called polynomial-like iterative equation, a linear combination of iterates of the unknown mapping f, i.e.,

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \cdots + \lambda_n f^n(x) = F(x), \quad x \in X, \tag{1.1}$$

where $F : X \to X$ is a given mapping and all coefficients λ_i ($i = 1, 2, \ldots, n$) are real constants. For linear F, equation (1.1) on $\mathbb{R}$ was investigated in [2], [8],

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[13], [15], [16], [20], [21]. For nonlinear F, equation (1.1) on $\mathbb{R}$ was discussed in [12], [28] for $n = 2$ and in [23, 24] for general n. In [22] and [27] the open problems on the C^m smoothness and the leading coefficient were put forwarded and later discussed in [11] and [26]. Solutions in $\mathbb{R}^n$ and analytic solutions in $\mathbb{C}$ were discussed in [10], [19]. In many of those works fixed points of mappings f and F are involved, that the normalization condition

$$\sum_{i=1}^{n} \lambda_i = 1 \tag{1.2}$$

is imposed naturally.

As in many references [4], [5], [18], symmetry of a mapping is described by equivariance of the mapping with respect to a Lie group Γ of linear transformations. The reason why one prefers the terminology of Lie group to the general one is, as told on p. 13 of [18], that "this combination of algebra and calculus leads to powerful techniques for the study of symmetry which are not available for, say, finite groups". For a Lie group Γ of linear transformations of X, say that $f : X \to X$ is Γ-*equivariant* if

$$f(\gamma x) = \gamma f(x), \quad \forall x \in X,\ \gamma \in \Gamma.$$

Sometimes we also say that $f : A \subset X \to X$ is of Γ-equivariance if f is a restriction of a Γ-equivariant mapping on the subset A. In [25] equivariance of continuous solutions for equation (1.1) was discussed under the action of topologically finitely generated subgroups of $GL(\mathbb{R})$ on $\mathbb{R}$ and the orthogonal group $\mathbf{O}(N)$ on $\mathbb{R}^N$ ($N \geq 2$).

In this paper, based on the algebraic structure of closed subgroups of $GL(\mathbb{R})$, we prove the equivariance of solutions of equation (1.1) on $\mathbb{R}$ with respect to closed subgroups of $GL(\mathbb{R})$, so a more general version of equivariance of solutions of equation (1.1) is obtained by a different proof. The idea of this proof is to reduce the equivariant problem by 'factoring out' the group action algebraically to a non-equivariant one. Then, we discuss equation (1.1) on $\mathbb{R}^N$ and extend the result of $\mathbf{O}(N)$-equivariance of solutions to the group $\mathbf{O}(N) \times \langle c\mathcal{I}_N \rangle$, where $\langle c\mathcal{I}_N \rangle$ is the group of positive dilations.

2. Equivariance to closed subgroups of $GL(\mathbb{R})$

Consider Lie group Γ of linear transformations on $\mathbb{R}^N$. and refer to standard group theory texts such as Fuchs [3] and Hall [6] for group-theoretic background. As in [4], [5], for any $x \in \mathbb{R}^N$ the subgroup $\Sigma_x := \{\gamma \in \Gamma : \gamma x = x\}$, called the *isotropy group*, is a *closed* subgroup of Γ by continuity. Our discussion is focused at closed subgroups of Γ.

In the case $N = 1$, invertible linear transformations of $\mathbb{R}$ take the form $x \mapsto \gamma x$ where $0 \neq \gamma \in \mathbb{R}$. Without loss of generality, any Lie group acting linearly on $\mathbb{R}$ can therefore be identified with a subgroup of $GL(\mathbb{R})$, the multiplicative topological group of nonzero reals, which we can identify with $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$. All such groups are Abelian.

Let $\mathcal{C}(I)$ consist of all continuous real-valued functions on $I := [-1, 1]$ and

$$\begin{aligned}
\mathcal{F}_\Gamma(I) &= \{f \in \mathcal{C}(I) \mid f(\gamma x) = \gamma f(x), \forall \gamma \in \Gamma \text{ and } \forall x, \gamma x \in I\},\\
\mathcal{F}(I; m, M) &= \{f \in \mathcal{C}(I) \mid f(1) = 1,\ f(-1) = -1,\ \text{and}\\
&\qquad m(y - x) \leq f(y) - f(x) \leq M(y - x),\ \forall y > x \in I\},\\
\mathcal{F}_\Gamma(I; m, M) &= \mathcal{F}(I; m, M) \cap \mathcal{F}_\Gamma(I),
\end{aligned}$$

where $M \geq 1 \geq m \geq 0$. The main result in this section is the following.

Theorem 1. *Suppose that Γ is a closed subgroup of $GL(\mathbb{R})$ and that $M > 1$. If $F \in \mathcal{F}_\Gamma(I; 0, M)$ and* (1.2) *holds with $\lambda_1 > 0, \lambda_i \geq 0$ ($i = 2, \ldots, n$), then equation* (1.1) *has a continuous solution $f \in \mathcal{F}_\Gamma(I; 0, M/\lambda_1)$, which possesses Γ-equivariance.*

Before proving the theorem, observe that $GL(\mathbb{R}) = \{c\mathcal{I} \mid 0 \neq c \in \mathbb{R}\}$, where $\mathcal{I} = \mathrm{id}_{\mathbb{R}}$, the identity on $\mathbb{R}$. The following lemma shows the algebraic structure of closed subgroups of $GL(\mathbb{R})$.

Lemma 1. *The closed subgroups of $G = GL(\mathbb{R})$ are:*

(a) $\mathbf{1}$.

(b) $\langle c \rangle$ *where $0 \neq c \in \mathbb{R}$ and (without loss of generality) $|c| > 1$.*

(c) $G^o = \{c\mathcal{I} \mid c > 0\}$.

(d) $\{-1, 1\}$.

(e) $\{-1, 1\} \times \langle c \rangle$ *where $0 \neq c \in \mathbb{R}$ and (without loss of generality) $c > 1$.*

(f) G.

In order to prove this lemma, we need the following well-known result, which is Theorem 438 in [7] p. 375.

Lemma 2 (Kronecker's Theorem)**.** *Suppose that $a_1, a_2 \in \mathbb{R}$.* (i) *If the ratio a_1/a_2 is rational then $\{ka_1 + la_2 : k,\ l \in \mathbb{Z}\} = \{ka : k \in \mathbb{Z}\}$ for a constant $a \in \mathbb{R}$.* (ii) *If the ratio a_1/a_2 is irrational then the closure of $\{ka_1 + la_2 : k,\ l \in \mathbb{Z}\}$ is $\mathbb{R}$.*

Proof of Lemma 1. Observe that $GL(\mathbb{R})$ is isomorphic to $GL^+(\mathbb{R}) \times \mathbb{Z}_2$ where $GL^+(\mathbb{R})$ is the group of *dilations* $x \mapsto ax$ for real $a > 0$ and $\mathbb{Z}_2 = \{-1, 1\}$. The subgroups of $GL(\mathbb{R})$ therefore fall into three classes:

Case (1) those contained in $GL^+(\mathbb{R})$,

Case (2) those that contain $\mathbb{Z}_2$, and

Case (3) those that satisfy neither of these conditions.

The logarithm function provides an isomorphism between $GL^+(\mathbb{R})$ and the additive group of $\mathbb{R}$.

Let H be a closed subgroup of $GL^+(\mathbb{R})$, that is, $H^* = \{\log h : h \in H\}$, the image of H under logarithm is a closed subgroup of the additive group $\mathbb{R}$. Then, by Lemma 2, either $H^* = \{0\}$, or H^* is generated by one element a and hence is cyclic, or H^* contains a non-cyclic subgroup with a generating set containing two elements a, a' where a' is not a rational multiple of a and the closure of the group $\langle a, a' \rangle$ generated by a, a' is the whole of $\mathbb{R}$. This proves (a), (c) and part of (b) where $c > 1$.

If $H \supset \mathbb{Z}_2$ then it is clear that $H = H_0 \times \mathbb{Z}_2$ where H_0 is a closed subgroup of $GL^+(\mathbb{R})$. This proves (d), (e) and (f).

In the third case, H must be of the form $H = \{h, \sigma(h)\}$ where $h \in H' \subset GL^+(\mathbb{R})$ and $\sigma : H' \to \mathbb{Z}_2$ is a surjective homeomorphism. This is possible only when $H' = \langle c_0 \rangle$ is cyclic and $c_0 > 1$, in which case $\sigma(c_0^n) = (-1)^n$ and we can express H as $\langle -c_0 \rangle$. This proves the other part of (b) where $c < -1$.

Therefore, we have completed the proof of Lemma 1. □

The following known result of continuous solutions is also useful.

Lemma 3 ([23])**.** *Suppose that $F : J = [a, b] \to J$ (where $a < b$) is an increasing function with fixed points at a and b and Lipschitz constant $M > 1$ and that (1.2) holds with $\lambda_1 > 0, \lambda_i \geq 0$ ($i = 2, \dots, n$). Then (1.1) has an increasing continuous solution f on J which has the Lipschitz constant M/λ_1 and fixes a and b.*

Proof of Theorem 1. It suffices to prove Theorem 1 for each of these six cases provided in Lemma 1.

Case (a) is just the non-equivariant case. Note that $F \in \mathcal{F}_\Gamma(I; 0, M)$ implies in particular that F is monotonic increasing, a condition that occurs already in the non-equivariant case as in [23] and [24], so we can obtain our result directly from Lemma 3.

In Case (b), there is no loss of generality in assuming that $\Gamma = \langle c \rangle$ where

$c > 1$. It follows that

$$F\left(\pm\frac{1}{c^k}x\right) = \pm\frac{1}{c^k}F(x), \quad k = 0, 1, 2\ldots, \tag{2.3}$$

since $F(cx) = cF(x)$, $\forall x \in I$. Moreover, $F(1) = 1$ and $F(-1) = -1$ for any $F \in \mathcal{F}_\Gamma(I; 0, M)$. By continuity, (2.3) implies that $F(0) = 0$. Notice that the actions of Γ on $I_+ = [0, 1]$ and $I_- = [-1, 0]$ are independent of each other because $-1 \notin \Gamma$. So, it suffices to observe $\mathcal{F}_\Gamma(I_+; 0, M)$. From (2.3), we see that

$$F\left(\frac{1}{c^k}\right) = \frac{1}{c^k}, \quad k = 0, 1, 2\ldots,$$

Let $J_k := [1/c^{k+1}, 1/c^k]$. Then the mapping F restricted on each J_k is in a non-equivariant case and satisfies the conditions in Lemma 3.

In case (c) G^o-equivariance implies that F is a scalar multiple of the identity, and the condition on fixed points ± 1 implies that F is the identity. Now f can (and must) be chosen to be the identity.

Cases (d,e,f) are similar but with the additional constraint that the function f must be odd; this can be achieved by working on the interval $[0, 1]$ and extending to $[-1, 0]$ using equivariance under $\mathbb{Z}_2 = \{-1, 1\}$.

The proofs for cases (b,d,e) can be seen as 'factoring out' the group action by working on the orbit space

$$\mathbb{R}/\Gamma = \{\Gamma(x) : x \in \mathbb{R}\},$$

where $\Gamma(x) = \{\gamma x : \gamma \in \Gamma\}$. Since this is topologically equivalent to a bounded closed interval, the non-equivariant theorem Lemma 3 can be applied; then the resulting function is lifted back to the original space (uniquely). □

3. Equivariance to $\mathbf{O}(N) \times \langle c\mathcal{I}_N \rangle$

In this section consider the action of the group $\mathbf{O}(N) \times \langle c\mathcal{I}_N \rangle$ on $\mathbb{R}^N$ ($N \geq 2$), where $\mathcal{I}_N$ is the identity on $\mathbb{R}^N$ and $0 < c \in \mathbb{R}$, and generalize the result in [25] for $\mathbf{O}(N)$-equivariance.

Let $\mathbf{O}_c$ denote $\mathbf{O}(N) \times \langle c\mathcal{I}_N \rangle$ for short. In standard representation,

$$\mathbf{O}(N) = \{A \in GL(N) : AA^T = \mathcal{I}_N\},$$

where A^T denotes the transpose of A. For example, $\mathbf{O}(2)$ is generated by rotations on $\mathbb{R}^2$ and the flip

$$\kappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $B = B^N = \{x \in \mathbb{R}^N \mid \|x\| \le 1\}$ and $\langle \cdot\, , \cdot \rangle$ denote the usual inner product on $\mathbb{R}^N$. Define

$$\mathcal{F}_{\mathbf{O}_c}(B) = \{f \in \mathcal{C}(B) \mid f(\gamma x) = \gamma f(x),\ \forall \gamma \in \mathbf{O}_c \text{ and } \forall x, \gamma x \in B\},$$

$$\begin{aligned} \mathcal{F}(B; m, M) = \{f \in \mathcal{C}(B) \mid\ & f \text{ fixes } \partial B \text{ pointwise, and for any } v \in B, \\ & m(t_2 - t_1)\|v\|^2 \le \langle f(t_2 v) - f(t_1 v), v\rangle \le M(t_2 - t_1)\|v\|^2 \\ & \text{when } t_2 \ge t_1 \text{ and } t_1 v, t_2 v \in B\} \end{aligned}$$

and

$$\mathcal{F}_{\mathbf{O}_c}(B; m, M) = \mathcal{F}(B; m, M) \cap \mathcal{F}_{\mathbf{O}_c}(B)$$

when $M \ge 1 \ge m \ge 0$.

Theorem 2. *Let $F \in \mathcal{F}_{\mathbf{O}_c}(B; 0, M)$ where $M > 1$. Then the equation (1.1) where $\lambda_1 > 0, \lambda_i \ge 0$ $(i = 2, \dots, n)$ and $\sum_{i=1}^n \lambda_i = 1$ has a solution $f \in \mathcal{F}_{\mathbf{O}_c}(B; 0, M/\lambda_1)$, which is continuous and possesses $\mathbf{O}(N) \times \langle c\mathcal{I}_N \rangle$-equivariance.*

Although $\mathbf{O}_c$ is not compact, the theory of fixed-point spaces in [5] can still be applied directly here.

Lemma 4. *Suppose f be a $\mathbf{O}_c$-equivariant mapping on $\mathbb{R}^N$. If Σ is a subgroup of $\mathbf{O}_c$ then the fixed-point space* $\mathrm{Fix}(\Sigma)$, *defined by* $\mathrm{Fix}(\Sigma) = \{x \in \mathbb{R}^N \mid \gamma x = x,\ \forall \gamma \in \Sigma\}$, *is invariant under f.*

In fact, the proof is not related to the compactness of the group. For any $x \in \mathrm{Fix}(\Sigma)$, by the equivariance we see that $\gamma f(x) = f(\gamma x) = f(x),\ \forall \gamma \in \Sigma$, that is, $f(x) \in \mathrm{Fix}(\Sigma)$. The next is to characterize $\mathbf{O}_c$-equivariant mappings.

Lemma 5. (a) *Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be an $\mathbf{O}_c$-equivariant mapping. Then there exists a function $f^* : \mathbb{R}^+ := \{x \in \mathbb{R} : x \ge 0\} \to \mathbb{R}$ such that $f^*(\|x\|)$ is $\mathbf{O}_c$-invariant and*

$$f(x) = f^*(\|x\|)x, \qquad \forall x \in \mathbb{R}^N. \tag{3.4}$$

(b) *Conversely if f is of the form* (3.4) *then f is $\mathbf{O}_c$-equivariant.*

Proof. (a) Choose a fixed unit vector $u \in \mathbb{R}^N$ and let Σ be the isotropy subgroup of u, that is, $\Sigma = \{\gamma \in \mathbf{O}_c \mid \gamma u = u\}$. By definition $\mathrm{Fix}(\Sigma) = \mathbb{R}u$. Let $r \in \mathbb{R}^+$. Since f is $\mathbf{O}_c$-equivariant, by Lemma 4 it maps $\mathrm{Fix}(\Sigma)$ to itself, therefore $f(ru) = \phi(r)u$ for some $\phi : \mathbb{R}^+ \to \mathbb{R}$. Let $x \in \mathbb{R}^N$ and $r = \|x\|$, and define a real function $f^* : \mathbb{R}^+ \to \mathbb{R}$ by

$$f^*(s) = \begin{cases} \phi(s)/s & \text{as } s > 0, \\ 0 & s = 0. \end{cases}$$

If $x \neq 0$ then there exists $\gamma \in \mathbf{O}(N) \subset \mathbf{O}_c$ such that $\gamma(ru) = x$. Therefore

$$f(x) = f(\gamma(ru)) = \gamma f(ru) = \gamma\phi(r)u = \phi(r)\gamma u = \frac{\phi(r)}{r}x = f^*(\|x\|)x$$

as required. If $x = 0$ then $f(0) = f(x) = f(\gamma x) = \gamma f(0)$ for all $\gamma \in \mathbf{O}_c$. The fact that $\mathrm{Fix}(\mathbf{O}_c) = \{0\}$ implies $f(0) = 0$. Clearly (3.4) holds for $x = 0$.

Furthermore, for all $\gamma \in \mathbf{O}_c$, from (3.4) we have

$$\gamma f^*(\|x\|)x = \gamma f(x) = f(\gamma x) = f^*(\|\gamma x\|)\gamma x$$

for all $x \in \mathbb{R}^N$, whence $f^*(\|x\|) = f^*(\|\gamma x\|)$ and $f^*(\|x\|)$ is $\mathbf{O}_c$-invariant.

(b) If $\gamma \in \mathbf{O}_c$ then

$$f(\gamma x) = f^*(\|\gamma x\|)\gamma x = f^*(\|x\|)\gamma x = \gamma f^*(\|x\|)x = \gamma f(x), \quad \forall x \in \mathbb{R}^N,$$

that is, f is $\mathbf{O}_c$-equivariant. □

Proof of Theorem 2. Let U be any 1-dimensional linear subspace of $\mathbb{R}^N$. By continuity and the fact that F fixes ∂B pointwise, F maps $U \cap B$ into itself, where $B = B^N$ is the unit ball. Let $u \in U$ be a unit vector. Then $U \cap B = \{tu \mid t \in [-1, 1]\}$. By Lemma 5, $F(tu) = F^*(|t|)tu$ for a function $F^* : \mathbb{R}^+ \to \mathbb{R}$. Let

$$\tilde{F}(t) = tF^*(|t|), \quad \forall t \in [-1, 1]. \tag{3.5}$$

The continuity of F guarantees $\tilde{F}$ is continuous on $[-1.1]$. In fact, from the proof of Lemma 5 it is easy to guarantee the continuity of $F^*(t)$ and $\tilde{F}(t)$ at $t \neq 0$. Since $F(x)$ is continuous at $x = 0$, it follows from (3.4) that

$$\lim_{t\to 0_+} F^*(t)t = 0.$$

This ensures the continuity of $\tilde{F}$ on the whole interval $[-1, 1]$.

Now we claim that $\tilde{F} \in \mathcal{F}_{\mathbb{Z}_2\times\langle c\rangle}(I; 0, M)$ where $I = [-1, 1]$, $\mathbb{Z}_2 = \{-1, 1\}$ and $c > 0$. Clearly $\tilde{F} \in \mathcal{C}(I)$ and is odd, that is, $\tilde{F}$ is $\mathbb{Z}_2$-equivariant. By Lemma 5, $F^*(\|x\|)$ is $\mathbf{O}(N) \times \langle c\mathcal{I}_N\rangle$-invariant. Then

$$\tilde{F}(ct) = ctF^*(|ct|) = ctF^*(\|ctu\|) = ctF^*(\|tu\|) = ctF^*(|t|) = c\tilde{F}(t),$$

for all $t \in I$, where $u \in U$ is the unit vector. Hence $\tilde{F} \in \mathcal{F}_{\mathbb{Z}_2 \times \langle c \rangle}(I)$. Moreover, since u and $-u$ belong to ∂B we have $F^*(1) = 1$ and $\tilde{F}(\pm 1) = \pm 1$. Note that for any $t_1, t_2 \in I$ with $t_2 > t_1$,

$$F(t_2 u) - F(t_1 u) = \tilde{F}(t_2)u - \tilde{F}(t_1)u = (\tilde{F}(t_2) - \tilde{F}(t_1))u,$$

and $\langle F(t_2 u) - F(t_1 u), u \rangle = \tilde{F}(t_2) - \tilde{F}(t_1)$. Thus $F \in \mathcal{F}(B; 0, M)$ implies $\tilde{F} \in \mathcal{F}(I; 0, M)$. Thus what we claimed is true.

From (3.5) we see

$$F(tu) = \tilde{F}(t)u. \tag{3.6}$$

By Theorem 1, there exists a function $\tilde{f} \in \mathcal{F}_{\mathbb{Z}_2 \times \langle c \rangle}(I; 0, M/\lambda_1)$ such that

$$\lambda_1 \tilde{f}(x) + \lambda_2 \tilde{f}^2(x) + \cdots + \lambda_n \tilde{f}^n(x) = \tilde{F}(x) \tag{3.7}$$

for $t \in I$. Extend $\tilde{f}$ to $f : B^N \to \mathbb{R}^N$ by setting

$$f(x) = f^*(\|x\|)x \tag{3.8}$$

where

$$f^*(t) = \begin{cases} \tilde{f}(t)/t & \text{if } t > 0, \\ 0 & t = 0. \end{cases}$$

Clearly f is continuous for $x \neq 0$ because of the continuity of $\tilde{f}$. At $x = 0$ it is obvious that $\lim_{x \to 0} \|f(x)\| = \lim_{x \to 0} |f^*(\|x\|)| \, \|x\| = \lim_{x \to 0} |\tilde{f}(\|x\|)| = 0$. Therefore f is continuous on B^N. For any $0 \neq x \in B^N$ let $t = \|x\|$ and $v = x/\|x\|$. Then $x = tv$ and $f(x) = f(tv) = \tilde{f}(t)v$ as in (3.6). Clearly $f^n(x) = \tilde{f}^n(t)v$ for any integer $n > 0$. Therefore (3.7) implies that

$$\lambda_1 \tilde{f}(t)v + \lambda_2 \tilde{f}^2(t)v + \cdots + \lambda_n \tilde{f}^n(t)v = \tilde{F}(t)v,$$

so that

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \cdots + \lambda_n f^n(x) = F(x).$$

It is easy to verify that f defined in (3.8) is of $\mathbf{O}(N) \times \langle c\mathcal{I}_N \rangle$-equivariance, since $\tilde{f}$ is of $\mathbb{Z}_2 \times \langle c \rangle$-equivariance. Hence we have obtained a solution f to (1.1) in $\mathcal{F}_{\mathbf{O}_c}(B; 0, M/\lambda_1)$. $\square$

The corresponding results on uniqueness and stability can be given similarly.

4. Applications

Theorem 2, being the main result of this paper and proved on the basis of Theorem 1, generalizes the N-dimensional result of equivariance given in [25] from the group $O(N)$ to $O(N) \times \langle c\mathcal{I}_N \rangle$. In order to demonstrate how Theorem 2 works on a practical example, let us simply consider a mapping F on the unit disk of $\mathbb{R}^2$ defined in polar coordinates by $F:(r,\theta)\mapsto(\Phi(r),\theta)$, where Φ is a C^1-smooth function on $I_+=[0,1]$ of $\langle\frac{1}{2}\rangle$-equivariance. Function Φ can be constructed by linking C^1-smooth functions Φ_k, each of which is defined on $[\frac{1}{2^{k+1}},\frac{1}{2^k}]$ for $k=0,1,2,\dots$ and satisfies

(i) $\Phi_k\big(\frac{1}{2^k}\big)=\frac{1}{2^k},\Phi_k\big(\frac{1}{2^{k+1}}\big)=\frac{1}{2^{k+1}}$,

(ii) $m\le\Phi_k'(r)\le M,\forall r\in\big(\frac{1}{2^{k+1}},\frac{1}{2^k}\big)$, where $0\le m<1<M$, and

(iii) $\Phi_k'(\frac{1}{2^k})=\Phi_k'(\frac{1}{2^{k+1}})=1$.

One can easily see that

$$\Phi\left(\pm\frac{1}{2^k}r\right)=\pm\frac{1}{2^k}\Phi(r),\quad k=0,1,2\dots. \tag{4.9}$$

Since each $\gamma\in O(N)\times\big\langle\frac{1}{2}\mathcal{I}_N\big\rangle$ can be expressed as either $\gamma:(r,\theta)\mapsto\big(\frac{1}{2^k}r,\theta+\alpha\big)$ or $\gamma:(r,\theta)\mapsto\big(\frac{1}{2^k}r,-\theta\big)$, where k is a positive integer and α is a real number, we can check with (4.9) that F is equivariant under the action of the group $O(N)\times\big\langle\frac{1}{2}\mathcal{I}_N\big\rangle$. Thus conditions in Theorem 2 are fulfilled by this mapping F.

References

[1] D. Bessis, S. Marmi and G. Turchetti, On the singularities of divergent majorant series arising from normal form theory, *Rend. Math. Ser* VII **9** (1989), 645–659.

[2] J. G. Dhombres, Itération linéaire d'ordre deux, *Publ. Math. Debrecen* **24** (1977), 277–287.

[3] L. Fuchs, Infinite Abelian Groups, *Academic Press*, *New York*, 1970.

[4] M. Golubitsky aand D. G. Schaeffer, Singularities and Groups in Bifurcation Theory, vol. 1, Appl. Math. Sci. **51**, *Springer*, *New York*, 1985.

[5] M. Golubitsky, I. N. Stewart and D. G. Schaeffer, Singularities and Groups in Bifurcation Theory, vol. 2, Appl. Math. Sci. **69**, *Springer*, *New York*, 1988.

[6] M. Hall, The Theory of Groups, *Macmillan*, *New York*, 1959.

[7] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, *Clarendon Press*, *Oxford*, 1962.

[8] W. Jarczyk, On an equation of linear iteration, *Aequationes Math.* **51** (1996), 303–310.

[9] M. Kuczma, Functional Equations in a single variable, PWN-Polish Scientific Publishing, *Warszawa*, 1968.

[10] M. Kulczycki and J. Tabor, Iterative functional equations in the class of Lipschitz functions, *Aequationes Math.* **64** (2002), 24–33.

[11] J. H. Mai and X. H. Liu, Existence, uniqueness and stability of C^m solutions of iterative functional equations, *Science in China* **A43** (2000), 897–913.

[12] M. Malenica, On the solutions of the functional equations $\phi(x)+\phi^2(x)=F(x)$, *Mat. Vesnik* **6** (1982)), 301–305.

[13] J. Matkowski and W. Zhang, On the polynomial-like iterative functional equation, Functional Equations and Inequalities, Math. Its Appl. **518**, (T. M. Rassias, ed.), *Kluwer Academic*, *Dordrecht*, 2000, 145–170.

[14] P. J. McCarthy, Projective actions, invariant sigma-curves and quadratic functional equations, *Math. Proc. Camb. Phil. Soc.* **98** (1985), 195–212.

[15] A. Mukherjea and J. S. Ratti, On a functional equation involving iterates of a bijection on the unit interval, *Nonlinear Anal.* **7** (1983), 899–908; II, *Nonlinear Anal.* **31** (1998), 459–464.

[16] S. Nabeya, On the functional equation $f(p + qx + rf(x)) = a + bx + cf(x)$, *Aequationes Math.* **11** (1974), 199–211.

[17] C. T. Ng aand W. Zhang, Invariant curves for a planar mapping, *J. Difference Eqns. Appl.* **3** (1997), 147–168.

[18] P. J. Olver, Applications of Lie Groups to Differential Equations, *Springer-Verlag, New York*, 1993.

[19] J. Si and W. Zhang, Analytic solutions of a nonlinear iterative equation near neutral fixed points and poles, *J. Math. Anal. Appl.* **284** (2003), 373–388.

[20] J. Tabor and J. Tabor, On a linear iterative equation, *Results in Math.* **27** (1995), 412–421.

[21] D. Yang and W. Zhang, Characteristic solutions of polynomial-like iterative equations, *Aequationes Math.* **67** (2004), 80–105.

[22] J. Zhang, L. Yang and W. Zhang, Some advances in functional equations, *Adv. Math. Chin.* **24** (1995), 385–405.

[23] W. Zhang, Discussion on the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$, *Chin. Sci. Bull.* **32** (1987), 1444–1451.

[24] W. Zhang, Discussion on the differentiable solutions of the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$, *Nonlinear Anal.* **15** (1990), 387–398.

[25] W. Zhang, Solutions of equivariance for a polynomial-like iterative equation, *Proc. Royal Soc. Edinburgh* **130A** (2000), 1153–1163.

[26] W. Zhang, On existence for polynomial-like iterative equations, *Results in Math.* **45** (2004), 185–194.

[27] W. Zhang and J. A. Baker, Continuous solutions for a polynomial-like iterative equation with variable coefficients, *Ann. Polon. Math.* **73** (2000), 29–36.

[28] L. Zhao, A theorem concerning the existence and uniqueness of solutions of functional equation $\lambda_1 f(x) + \lambda_2 f^2(x) = F(x)$, *J. Univ. Sci. Tech. China* **32** (1983), 21–27, Special Issue (in *Chinese*).

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