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Polynomial hypergroup structures and applications to probability theory

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Abstract. A new trend in enlarging the repertoire of concrete hypergroups is the construction of polynomial hypergroup structures on higher-dimensional Euclidean spaces. It turns out that stochastic processes taking values in such structures and their duals reveal surprising phenomena. In the present exposition recent progress in the theory achieved by Koornwinder, Connett and Schwartz, and by the Tunisian School will be discussed and made accessible also to the non-specialized reader.

1. Introduction

There are two significant aspects motivating an up-to-date exposition on progress in the theory of commutative hypergroups with base space a compact subset K of k-dimensional euclidean space and whose convolution in the set $M(K)$ of bounded measures on K is defined via sequences of k-variable polynomials on K. Firstly, this survey shows how new hypergroup structures can be introduced for geometric configurations in $\mathbb{R}^2$ such as the unit square, the unit disk, the parabolic bi-angle and the simplex. Secondly, it is of interest to study stochastic processes with independent increments in these configurations, their structure and their long-term behaviour.

For general compact commutative hypergroups an elaborate harmonic analysis is available based on the generalized translation operation in $M(K)$. Basic results have been obtained in analogy to but remarkably distinct from the classical framework of a compact abelian group. It is worth recalling that in general there

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is no dual hypergroup attached to the given hypergroup K, that the Plancherel measure rarely has full support, and that positive definite functions can be unbounded. However there is a Haar measure for K, corresponding L^p-spaces, and an extended Fourier–Stieltjes theory. For details of the general theory of hypergroups reaching beyond the compact and the commutative cases the reader is referred to the authors' monograph [2].

The present exposition can be considered as a supplement to [7] which deals exclusively with the disk hypergroup, an example of a 2-dimensional structure; this booklet did not treat in any detail the more general structure. In the meantime analytic and probabilistic results have been proved for even more interesting 2-dimensional configurations by Connett and Schwartz, and Koornwinder in the first case, and Mokni and Trimèche, and Mili in the second. The analytic contributions published between 1975 and 1996 are mainly concerned with the proofs that the configurations K carry polynomial hypergroup structures. The probabilistic results achieved around 1999 are grounded on the cone-embedding of the compact set K.

In this paper we first review the fundamental notions from the theory of hypergroups (Section 2), and then specialize the discussion to k-variable polynomial hypergroups in Section 3. Section 4, which is central to the paper, is concerned with the highly sophisticated constructions leading to hypergroup structures for some cone-embedded compact subsets K. Sections 5 and 6 will cover applications of this theory to probability theory on these spaces, where the canonical decomposition of generalized Laplacians and a Central Limit Theorem for appropriately stopped random variables in K give an insight into the probabilistic significance of the underlying hypergroup structure.

2. Compact commutative hypergroups

We commence with a convolution structure arising from a pair (G, H) consisting of a locally compact group G and a compact subgroup H of G admitting a normalized *Haar measure* ω_H. We observe that the set

$$K := G /\!\!/ H := \{HxH : x \in G\}$$

of H-double cosets of G can be made into a locally compact space by furnishing it with the natural quotient topology, but not in general into a group or semigroup. Still the Banach space $M^b(K)$ of bounded measures on K carries a convolution structure inherited from that of $M^b(G)$, Indeed for $x, y \in G$ the convolution product of the Dirac measures at the corresponding double cosets is defined by

$$\varepsilon_{HxH} * \varepsilon_{HyH} := \int_H \varepsilon_{HxhyH}\, \omega_H(dh).$$

Evidently this convolution of Dirac measures yields a probability measure with compact but not necessarily singleton support. It turns out that $*$ can be extended to all measures in $M^b(K)$ in such a way that $\big(M^b(K), *\big)$ becomes a Banach $*$-algebra with unit element HeH and involution $HxH \mapsto Hx^{-1}H$. Moreover, $(K, *)$ is a *hypergroup* in the sense of Dunkl, Jewett and Spector, satisfying the following axioms.

Given an arbitrary locally compact space K there is a convolution $*$ in $M^b(K)$ such that

K1 $(M^b(K), *)$ is a Banach $*$-algebra.

K2 The mapping $(\mu, \nu) \mapsto \mu * \nu$ from $M^b(K) \times M^b(K)$ into $M^b(K)$ is continuous with respect to the weak topology τ_w in $M^b(K)$.

K3 For $x, y \in K$ the convolution $\varepsilon_x * \varepsilon_y$ belongs to the set $M_c^1(K$ of probability measures on K with compact support.

K4 There exists a unit element ε_e (for some distinguished $e \in K$) and an involution $\mu \mapsto \mu^-$ in $M^b(K)$ satisfying $e \in \operatorname{supp}(\varepsilon_x * \varepsilon_y) \iff x = y^-$ for all $x, y \in K$.

K5 The mapping $(x, y) \mapsto \operatorname{supp}(\varepsilon_x * \varepsilon_y)$ from $K \times K$ into the space $\mathcal{C}\,(K)$ of compact subsets of K furnished with the Michael topology τ_M is continuous. (For a definition of τ_M and a useful equivalent description in the metrisable case see [10].)

We refer to the hypergroup $(K, *)$ as being *commutative* if $\varepsilon_x * \varepsilon_y = \varepsilon_y * \varepsilon_x$ for all $x, y \in K$, and *hermitian* if the involution is the identity mapping.

Clearly every locally compact group is a hypergroup, and G'elfand pairs (G, H) yield commutative double coset hypergroups. The following two G'elfand pairs lead to the class of hypergroups that will be central to our discussion throughout the remaining part of this paper.

2.1. *Example* (Jacobi pair). Let $G := SO(d)$ and $H := SO(d-1)$ for $d \geq 3$. Then G/H can be identified with the d-dimensional unit sphere $\mathbb{S}^{d-1}$ and $G /\!\!/ H$ with the unit interval $\mathbb{I} := [-1, 1]$. The convolution $*$ in $M^b(\mathbb{I})$ induced from G is described by the sequence $\big(Q_n^{\alpha,\beta}\big)_{n \geq 0}$ of Jacobi polynomials for $\alpha = \beta = \frac{d-3}{2}$.

2.2. *Example* (Disk pair). Let $G := \mathbb{U}(d)$ and $H := \mathbb{U}(d-1)$ for $d \geq 3$. Then $G/H \cong \mathbb{S}^{2d-1}$ and $G/\!/H \cong \mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$. The convolution for measures on the unit disk $\mathbb{D}$ is induced from the group G; it is described by the sequence $\left(Q^{\alpha}_{m,n}\right)_{m,n\geq 0}$ of disk polynomials for $\alpha = d-2$.

Both examples give compact commutative hypergroups with polynomial convolution structures over $\mathbb{I}$ and $\mathbb{D}$ respectively. The construction of the convolution suggests that $\mathbb{I}$ and $\mathbb{D}$ admit hypergroup structures beyond “integer dimensions”.

We now recall some fundamental notions from the theory of a commutative hypergroup $(K, *)$.

2.3. The *generalized translation operator* T^x for $x \in K$ is defined by

$$T^x f(y) := \int_K f \, d\left(\varepsilon_x * \varepsilon_y\right)$$

for $y \in K$ whenever f is appropriately integrable on K.

2.4. There exists a non-zero translation invariant measure $\omega_K \in M_+(K)$ (the set of non-negative measures on K) which is unique to within a positive multiplicative constant. This *Haar measure* gives rise to an elaborate analysis of the spaces $L^p(K, \omega_K)$, in particular of the hypergroup algebra $L^1(K, \omega_K)$.

2.5. *Characters* of K are introduced as continuous hermitian homomorphisms $\chi : K \to \mathbb{D}$, where the latter property is given by

$$\chi(x)\chi(y) = T^x\chi(y)$$

for all $x, y \in K$. The set $K^\wedge$ of non-vanishing characters of K together with the compact open topology τ_{co} is a locally compact space, called the *dual* of K. It is only very rarely that $K^\wedge$ itself carries a hypergroup structure.

2.6. There exists an injective *Fourier–Stieltjes transform* $\mu \mapsto \overset{\wedge}{\mu}$ from $M^b(K)$ into the space $C^b\left(K^\wedge\right)$ of bounded continuous functions on $K^\wedge$ given by

$$\overset{\wedge}{\mu}(\chi) := \int_K \bar{\chi} d\mu$$

for all $\chi \in K^\wedge$. Moreover, the Fourier–Stieltjes transform admits the Lévy continuity property. The Fourier transform

$$f \mapsto \hat{f} := \left(f\omega_K\right)^\wedge$$

gives an isometric isomorphism from $L^2(K,\omega_K)$ into $L^2(K^\wedge,\pi_K)$, where $\pi_K \in M_+(K^\wedge)$ denotes the *Plancherel measure* of K associated with the Haar measure ω_K. It is a crucial deviation from classical harmonic analysis that in general the Plancherel measure does not have full support. Indeed, the representation theory even in the commutative case is far from complete, hardly surprising as commutative hypergroups encompass G'elfand pairs; this should not constitute a drawback but rather an encouragement for further research.

In order to illustrate the above fundamental notions we consider our main example.

2.7. *Example.* The disk hypergroup $(\mathbb{D}, *_\alpha) := (\mathbb{D}, *(Q^\alpha_{m,n}))$ where the convolution $*_\alpha$ is defined via the sequence $\left(Q^\alpha_{m,n}\right)$ of disk polynomials of order $\alpha > 0$ given by

$$Q^\alpha_{m,n}(z) := Q^{\alpha,|m-n|}_{m\wedge n}\left(2|z|^2-1\right)|z|^{|m-n|}$$

for all $z \in \mathbb{D}$ and $m,n \in \mathbb{Z}_+$, where $\left(Q^{\alpha,\beta}_n\right)_{n\geq 0}$ denotes the sequence of Jacobi polynomials with parameters $\alpha \geq \beta > -1$ and either $\beta > -\frac{1}{2}$ or $\alpha+\beta \geq 0$. As carried out in detail in [7] we employ the well-known *product formula for disk polynomials* in order to introduce the convolution $*_\alpha$ in $M^b(\mathbb{D})$. In fact for $z,w \in \mathbb{D}$ there exists a probability measure $\mu_{x,y}$ on $\mathbb{D}$ such that

$$Q^\alpha_{m,n}(z)Q^\alpha_{m,n}(w) = \int_{\mathbb{D}} Q^\alpha_{m,n}\, d\mu_{z,w}$$

for all $m,n \in \mathbb{Z}_+$. With

$$\varepsilon_z *_\alpha \varepsilon_w := \mu_{z,w}$$

we obtain a hypergroup structure in $\mathbb{D}$. The hypergroup $(\mathbb{D}, *_\alpha)$ is non-hermitian (with involution $z \longmapsto \bar{z}$) having $\mathbb{1} := (1,0) \in \mathbb{C}$ as its unit. The normed Haar measure ω_α of $\mathbb{D}$ can be computed as

$$\omega_\alpha(d(x,y)) = \frac{\alpha+1}{\pi}\left(1-x^2-y^2\right)^\alpha dxdy$$

so that the convolution of Dirac measures takes the form

$$\varepsilon_z * \varepsilon_w(f) = \frac{\alpha}{\alpha+1}\int_{\mathbb{D}} f\left(zw+\sqrt{1-z^2}\sqrt{1-w^2}\; v\right)\frac{1}{1-|v|^2}\omega_\alpha(dv)$$

for all $f \in C^b(\mathbb{D})$. The characters of $(\mathbb{D}, *_\alpha)$ are given by

$$\chi_{m,n}(z) := Q^\alpha_{m,n}(z)$$

so that the dual $\mathbb{D}^\wedge \cong \mathbb{Z}_+^2$, which also turns out to be a hypergroup. In fact $\left(\mathbb{Z}_+^2, \overset{\wedge}{*}_\alpha\right)$ is a discrete commutative hypergroup with convolution given by

$$\varepsilon_{m,n} \overset{\wedge}{*}_\alpha \varepsilon_{k,l} \left(\{(r,s)\}\right) := h(m,n;k,l;r,s)$$

where the $h(m,n;k,l;r,s)$ are the coefficients given in the non-negative linearization

$$Q^\alpha_{m,n}(z) Q^\alpha_{k,l}(z) = \sum_{r,s \geq 0} Q^\alpha_{r,s}(z) h(m,n;k,l;r,s)$$

valid for all $(m,n),(k,l) \in \mathbb{Z}_+^2$ and $z \in \mathbb{D}$. The Fourier–Stieltjes transform of $\mu \in M^b(\mathbb{D})$ is given by

$$\overset{\wedge}{\mu}(m,n) := \int_{\mathbb{D}} \overline{Q}^\alpha_{m,n} d\mu.$$

Finally we note that $(\mathbb{D}, *_\alpha)$ has the *Pontryagin property* in the sense that $(\mathbb{D}^\wedge)^\wedge \cong \mathbb{D}$.

Starting from the disk hypergroup $\mathbb{D}$ we can easily derive the hypergroup structure of the half-disk

$$\mathbb{D}_+ := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ y \geq 0\right\}$$

which appears as the dual $\mathbb{E}^\wedge$ of a hypergroup $\mathbb{E}$. The hypergroup structure of $\mathbb{E}$ was established via half-disk polynomials in [11] (the latter can be viewed as special Koornwinder type III polynomials; see Section 5 below). Independent of this rather technical approach we can show that $\mathbb{D}_+$ and $\mathbb{E}$ are hypergroups as follows: $\mathbb{D}_+$ is the quotient hypergroup $\mathbb{D}/\{-1,1\}$ where $\{-1,1\}$ is a subgroup of $\mathbb{D}$, and consequently $\mathbb{E} \cong \mathbb{D}_+^\wedge$ becomes a subhypergroup of $\mathbb{D}^\wedge \cong \mathbb{Z}_+^2$.

Examples 2.1, 2.2 and 2.7 suggest a more general constructive approach to compact commutative hypergroups. Given an arbitrary compact subset K of $\mathbb{R}^k$ $(k \geq 1)$ the following can be stressed as a challenging research programme (see [4]).

2.8. Polynomial hypergroup problem (PHP)**.** *Find a suitable family $\mathcal{P}$ of k-variable polynomials on K that provide a convolution in $M^b(K)$ such that $(K, *_{\mathcal{P}})$ becomes a (compact commutative) hypergroup.*

An ambitious far-reaching solution to this problem would be the identification of all hypergroups $(K,*)$ with compact Euclidean base space K such that $K^\wedge$ contains a "rich" family $\mathcal{P}$ of polynomials on K. We shall return to attempts to reach such a solution in the next section.

3. *k*-variable polynomial hypergroups

We are given a compact subset K of $\mathbb{R}^k$ $(k \geq 1)$ and a measure $\omega \in M_+(K)$. Let $\mathcal{P} \subset \mathbb{C}\,[X_1, X_2, \dots, X_k]$ be a family of polynomials on K that are orthogonal with respect to ω in the sense that

I. for all $P, Q \in \mathcal{P}$ with $P \neq Q$

$$\int_K P\overline{Q}\, d\omega = 0$$

and

II. for each $\ell \geq 1$ the set $\mathcal{P}_{\leq \ell}$ of polynomials in $\mathcal{P}$ with degree not exceeding ℓ spans $\mathbb{C}\,[X_1, X_2, \dots, X_k]_{\leq \ell}$.

Moreover we assume that

III. there exists $e \in K$ such that $P(e) = 1$ for all $P \in \mathcal{P}$ and that

IV. for $x, y \in K$ there exists $\mu_{x,y} \in M^1(K)$ satisfying the product formula

$$P(x)P(y) = \int_K P\, d\mu_{x,y}$$

whenever $P \in \mathcal{P}$.

Clearly from III and IV it follows that

$$\sup_{x \in K} |P(x)| = P(e) = 1$$

for all $P \in \mathcal{P}$. Now introducing the convolution product

$$\varepsilon_x *_{\mathcal{P}} \varepsilon_y := \mu_{x,y}$$

for $x, y \in K$ and extending it to all of $M^b(K)$ we obtain a commutative Banach algebra $\big(M^b(K), *\big)$ with unit ε_e.

In order that $(K, *_{\mathcal{P}})$ becomes a hypergroup two additional properties need to be satisfied.

V. There exists an involution $\mu \mapsto \mu^-$ in $M^b(K)$ such that

$$e \in \operatorname{supp}\left(\varepsilon_x *_{\mathcal{P}} \varepsilon_y\right) \Longleftrightarrow x = y^-$$

for all $x, y \in K$, and

VI. the mapping

$$(x, y) \mapsto \operatorname{supp}\left(\varepsilon_x *_{\mathcal{P}} \varepsilon_y\right)$$

from $K \times K$ into $\left(\mathcal{C}\left(K\right), \tau_M\right)$ is continuous.

3.1. *Definition.* A hypergroup $(K, *_\mathcal{P})$ constructed with Properties I–VI is referred to as a *k-variable polynomial hypergroup* (with defining family $\mathcal{P}$).

For arbitrary compact hypergroups K the dual $K^\wedge$ is always discrete. In the special case of a k-variable polynomial hypergroup $(K, *_\mathcal{P})$ the dual is even countable. In general $K^\wedge$ does not carry a hypergroup structure, a 2-dimensional exception being the disk hypergroup $(\mathbb{D}, *_\alpha)$ as was pointed out in Example 2.7.

For general k-variable polynomial hypergroups $(K, *_\mathcal{P})$ we observe

3.2. *Remark.* The Fourier–Stieltjes transform $\overset{\wedge}{\mu}$ of $\mu \in M^b(K)$ is given by

$$\overset{\wedge}{\mu}(P) = \int_K \overline{P}\, d\mu$$

for all $P \in \mathcal{P}$, and the mappings $\mu \mapsto \overset{\wedge}{\mu}$ and $f \mapsto \hat{f}$ are injective on $M^b(K)$ and $L^1(K, \omega_K)$ respectively.

3.3. *Remark.* Identifying the Plancherel measure π_K of K with the measure

$$\pi(P) := \|P\|_2^{-2}, P \in \mathcal{P}$$

on $\mathcal{P} \cong K^\wedge$, Plancherel's formula provides an isometric isomorphism $f \mapsto \hat{f}$ from $L^2(K, \omega_K)$ onto $\ell^2(\{\pi(P) : P \in \mathcal{P}\})$.

We now investigate the special role played by the disk hypergroup within all 2-variable polynomial hypergroups; any result in this direction would support the hope for a complete classification of 2-variable polynomial hypergroups.

We first briefly report on some work contained in [5].

3.4. *Definition.* Two hypergroups $(K, *)$, $(L, \circ)$ with respective compact base spaces $K, L \subset \mathbb{C} \cong \mathbb{R}^2$ are said to be *linearly equivalent* if $(L, \circ)$ is the image of $(K, *)$ under an affine-linear homeomorphism from K onto L.

In the following we restrict ourselves to non-hermitian hypergroups $(K, *)$ with compact base space.

3.5. Theorem. *If* $(K, *)$ *admits exactly two distinct non-constant polynomial characters of degree* 1 *then* $(K, *)$ *is equivalent to a* canonical hypergroup $(L, \circ)$ *defined by the following properties.*

3.5.1. The unit of $(L, \circ)$ is $1 \in \mathbb{C}$.

3.5.2. The involution of $(L, \circ)$ is complex conjugation.

3.5.3. $L \subset \mathbb{D}$.

3.5.4. The polynomial characters of L of degree 1 are the functions

$$z \longmapsto z \text{ and } z \longmapsto \overline{z}.$$

3.6. Theorem. *Let $(L, \circ)$ be a canonical hypergroup. The following statements are equivalent.*

3.6.1. $(L, \circ) = (D, *_\alpha)$ for some $\alpha > 0$.

3.6.2. 1 is an accumulation point of L and the only $w \in \mathbb{C}$ satisfying

$$\langle w, z-1 \rangle = o(\|z-1\|) \quad \text{for } z \to z_0$$

with $z \in L$ is $w = 0$, and $\{z \in L : |z| = 1\}$ contains at least one point not a $4^{th}, 5^{th}$ or 6^{th} root of unity.

3.7. Corollary. *If $(\mathbb{D}, *)$ (as a non-hermitian hypergroup with an arbitrary convolution $*$) is canonical then $(\mathbb{D}, *) = (\mathbb{D}, *_\alpha)$ for some $\alpha > 0$.*

4. Cone-embedded hypergroups

The new types of 2-variable polynomial hypergroups to be studied in the sequel are constructed for the parabolic triangle $\mathbb{B}$ and for the triangle $\mathbb{T}$ as base spaces. In the spirit of PHP quoted in Example 2.8 families $\mathcal{P}$ of 2-variable polynomials on these spaces K have to be found such that the Properties I-VI are fulfilled. Then a convolution $*_\mathcal{P}$ turns the space K into a (compact commutative) hypergroup $(K, *_\mathcal{P})$. Once the family $\mathcal{P}$ is chosen to be orthogonal (Property I) and rich (Property II) the main task in establishing the hypergroup structure will be the verification of the product formula (Property IV) and the continuity of the support (Property VI).

The groundwork for the constructions has been laid out by Koornwinder [9]; detailed proofs of Properties IV and VI can be found in Koornwinder and Schwartz [10].

Before we enter a discussion of the construction of hypergroup structures for $\mathbb{B}$ and $\mathbb{T}$ we note that following Mokni and Trimèche [13] there is a class of 2-dimensional compact spaces containing

$$\mathbb{B} := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_2^2 \leq x_1 \leq 1\}, \ \mathbb{T} := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_2 \leq x_1 \leq 1\}$$

and the square $\mathbb{Q} := \mathbb{I}^2$ as distinguished examples.

4.1. *Definition.* A compact subspace D of $\mathbb{R}^k$ $(k \geq 2)$ is said to be *cone-embedded* if

(a) $\overline{D^0} = D$.

(b) $e := (1, 1, \ldots, 1) \in \partial D$.

(c) For every $\ell \in \{1, 2, \ldots, k\}$ there is a C^1-curve $\gamma_\ell : [0, 1] \to \partial D$ such that

  (c1) $\gamma_\ell(0) = e$ for all $\ell \in \{1, 2, \ldots, k\}$,

  (c2) $\{\gamma_\ell'(0) : 1 \leq \ell \leq k\}$ is a basis for $\mathbb{R}^k$,

  (c3) whenever C is the convex cone generated by $\{\mathbb{R}_+\gamma_\ell'(0) : 1 \leq \ell \leq k\}$ then D is a subset of the cone $e + C$.

For later application we note that $e + C$ is a proper cone in $\mathbb{R}^k$ in the sense that

$$(e + C) \cap (e - C) = \{e\}$$

and hence there exists a normal vector $\mathbf{n}$ for C^0 (not necessarily unique) such that

$$(e + C)\backslash\{e\} \subset \{x \in \mathbb{R}^k : h(x) > 0\}$$

where

$$h(x) := \langle \mathbf{n} \mid x - e \rangle$$

for all $x \in \mathbb{R}^k$.

The following general scheme directs the approach to verifying Properties IV–VI for the compact spaces $\mathbb{B}$ and $\mathbb{T}$.

4.2. *Special case.* For $K := \mathbb{B}$ or $\mathbb{T}$ we write $K^\times := K\backslash\{e\}$. Then for Property IV we establish separately the product formulae

IV.1. For $x, y \in K$

$$P(x)P(y) = \int_\Delta P(T(x, y; w))\, \mu(dw),$$

IV.2. For $x \in K$

$$P(x)P(e) = \int_{\Delta_0} P(T(x; w))\, \mu_0(dw)$$

where

$$(x, y; w) \longmapsto T(x, y; w)$$

and

$$(x; w) \mapsto T_0(x; w)$$

are *generalized translations* on $(K^\times)^2 \times \Delta$ and $K^\times \times \Delta_0$ with compact subsets Δ, Δ_0 in Euclidean spaces of appropriate dimensions, and μ, μ_0 are measures in $M^1((K^\times)^2 \times \Delta), M^1(K^\times \times \Delta_0)$ respectively.

In accordance with previous formulations of Property IV we have

$$T(x,y;\Delta) = \operatorname{supp}\mu_{x,y} = \operatorname{supp}(\varepsilon_x *_{\mathcal{P}} \varepsilon_y)$$

and

$$T_0(x;\Delta_0) = \operatorname{supp}\mu_{x,0} = \operatorname{supp}(\varepsilon_x *_{\mathcal{P}} \varepsilon_0).$$

We next demonstrate the equivalences:

V.1. $e \in T(x,y;\Delta) \Longleftrightarrow x = y \qquad (x,y \in K^\times)$;

V.2. $e \in T_0(x;\Delta_0) \Longleftrightarrow x = 0 \qquad (x \in K)$
which imply Property V.

And in order to obtain Property VI it remains to prove the limit relation

VI.1. $$\lim_{(x,y)\to(z,0),x,y\in K^\times} T(x,y;\Delta) = T_0(x;\Delta_0).$$

We note that

$$T(x,e;w) = x \Longleftrightarrow \varepsilon_x * \varepsilon_e = \varepsilon_x$$

whenever $w \in \Delta$.

To gain a better understanding of the very complicated generalized translation T we reformulate the well-known example of the square hypergroup.

4.3. *Example.* The *square hypergroup* is given by $(K, *_{\mathcal{P}})$ with $K := \mathbb{S} := \mathbb{I}^2$ where $\mathcal{P}$ is the family of square polynomials in K [9]. Applying the notion of product hypergroup we obtain the product formula for $(\mathbb{S}, *_{\mathcal{P}})$ directly from that of $(\mathbb{I}, *_{\mathcal{P}})$, where the sequence $\mathcal{P}$ of (normalized) Jacobi polynomials is given for $\alpha > \beta > -\frac{1}{2}$ by

$$\{Q_n^{(\alpha,\beta)} : n \in \mathbb{Z}_+\}$$

with

$$Q_n^{(\alpha,\beta)}(x) = F\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right)$$

for all $x \in \mathbb{I}$, where F denotes the Gaussian hyperbolic function. It is known that the family $\mathcal{P}$ satisfies Property I with the orthogonality (Haar) measure

$$\omega_{\alpha,\beta}(dx) := c_{\alpha,\beta}(1-x)^\alpha(1+x)^\beta dx \in M^1(\mathbb{I}),$$

and clearly also Properties II and III. From [8] we obtain the product formula in the form

$$Q_n^{(\alpha,\beta)}(x)Q_n^{(\alpha,\beta)}(y) = \int_{I\times J} Q_n^{(\alpha,\beta)}(T(x,y;r,\psi))\,\nu_{\alpha,\beta}(d(r,\psi))$$

with

$$\nu_{\alpha,\beta}(d(r,\psi)) := \frac{2\Gamma(\alpha+1)}{\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})\Gamma(\frac{1}{2})}(1-r^2)^{\alpha-\beta-1}2^{2\beta+1}\sin^{2\beta}\psi\, dr d\psi \in M^1(I\times J)$$

where $I := [0,1]$ and $J := [0,\pi]$.

With this more convenient form of the generalized translation T we then have the product formula

$$Q_n^{(\alpha,\beta)}(2x^2-1)Q_n^{(\alpha,\beta)}(2y^2-1) = \int_{I\times J} Q_n^{(\alpha,\beta)}(T'(x,y;r,\psi))\,\nu_{\alpha,\beta}(d(r,\psi)).$$

Here $T' := 2E^2 - 1$ where

$$E(x,y;r,\psi) := (x^2y^2 + (1-x^2)(1-y^2)r^2 + 2xy(1-x^2)^{1/2}(1-y^2)^{1/2}r\cos\psi)^{1/2}$$

for all $|x|, |y|, r \in I$ and $\psi \in J$.

Related transforms that will be used in the sequel are

$$D(x,y;r,\psi) := xy + (1-x^2)^{1/2}(1-y^2)^{1/2}r\cos\psi,$$

$$C(x,y;r,\psi) := \frac{D(x,y;r,\psi)}{E(x,y;r,\psi)},$$

$$G(x_1,x_2,y_1,y_2;r_1,\psi_1,\psi_2,\psi_3) = D\left(C, D\left(\frac{x_2}{x_1},\frac{y_2}{y_1};1,\psi_2\right),\psi_3\right), \qquad \text{and}$$

$$H(x_1,x_2,y_1,y_2;r_1,r_2,r_3,r_4,\psi_1,\psi_2,\psi_3) = E\left(\left[(1-r_2)C^2+r_2\right]^{1/2}, E\left(\frac{x_2}{x_1},\frac{y_2}{y_1};r_3,\psi_2\right);r_4,\psi_3\right).$$

Where there is no possible ambiguity we will suppress the variables and use the symbols E, D, C, G and H accordingly.

5. Koornwinder type III polynomials

These are given for $\alpha, \beta > 1$ and $n, k \in \mathbb{Z}_+$ with $n \geq k \geq 0$ by

$$Q_{n,k}^{(\alpha,\beta)}(x_1,x_2) := Q_{n-k}^{(\alpha,\beta+k+\frac{1}{2})}(2x_1-1)x_2^{k/2}Q_k^{(\beta,\beta)}\left(\frac{x_2}{\sqrt{x_1}}\right).$$

For fixed α, β the sequence

$$\left\{Q_{n,k}^{(\alpha,\beta)} : n \geq k \geq 0\right\} \subset \mathbb{C}[X_1,X_2]_{\leq n}$$

is orthogonal with respect to the measure

$$\omega\left(d(x_1,x_2)\right) := (1-x_1)^{\alpha}\left(x_1-x_2^2\right)^{\beta}dx_1dx_2$$

on the parabolic bi-angle $\mathbb{B}$.

5.1. *Special case.* For the choices $(\alpha,\beta) := \left(2d-3,-\frac{1}{2}\right), d \geq 2$ and $(\alpha,\beta) := \left(3,\frac{5}{2}\right)$ the polynomials $Q_{n,k}^{(\alpha,\beta)}$ are spherical functions of the compact G'elfand pairs (G,H) with

$$G := \mathrm{Sp}(d) \times \mathrm{Sp}(1), H := \mathrm{Sp}(d-1) \times \mathrm{diag}(\mathrm{Sp}(1) \times \mathrm{Sp}(1))$$

where $G/H := \mathbb{S}^{4d-1}$ and $G/H := \mathbb{S}^{15}$ respectively.

The example of the first mentioned G'elfand pair (G,H) also shows that the class $\mathfrak{P}$ of Pontryagin hypergroups K defined by the property $K^{\wedge\wedge} \cong K$ does not admit an induction principle. In fact, if L is a subhypergroup of K, and K/L and L both belong to $\mathfrak{P}$, then K does not necessarily belong to $\mathfrak{P}$ even if $K^{\wedge}$ is a hypergroup. From [17] we deduce the following details.

At first we note that the double coset hypergroup $K := G/\!/H$ is isomorphic to the commutative hypergroup $\mathbb{B}$ associated with the Koornwinder polynomials $Q_{n,k}^{(\alpha,\beta)}$ of Type III. The set

$$L := \{(1,x_2) : x_2 \in \mathbb{I}\}$$

is a compact subhypergroup of K whose characters are given as mappings

$$(1,x_2) \mapsto Q_k^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x_2)$$

on K for all $k \in \mathbb{Z}_+$. In other words, L is isomorphic to the compact hypergroup $\mathbb{I} \in \mathfrak{P}$ whose characters are the Chebychev polynomials of the first kind. Moreover

$$K^{\wedge} \cong \left\{(n,k) \in \mathbb{Z}_+^2 : n \geq k\right\}$$

admits a dual hypergroup structure (as the dual of a double coset hypergroup arising from a compact G'elfand pair), and

$$(K/L)^{\wedge} \cong A\left(K^{\wedge}, L\right) \cong \{(k,0) : k \in \mathbb{Z}_+\}$$

where $A\left(K^{\wedge}, L\right)$ denotes the annihilator of L in $K^{\wedge}$. This shows that $(K/L)^{\wedge}$ is isomorphic to the Jacobi polynomial hypergroup $\mathbb{Z}_+ \in \mathfrak{P}$ associated with the Jacobi polynomials $\left(Q_m^{(2d-3,0)}\right)$.

Altogether we obtain that K/L and L belong to $\mathfrak{P}$, and $K^{\wedge}$ is a hypergroup but, by [17], K doesn't belong to $\mathfrak{P}$.

5.2. Theorem. *The parabolic bi-angle* $\mathbb{B}$ *can be made into a 2-variable polynomial hypergroup* $(\mathbb{B}, *_{\alpha,\beta})$ *with convolution* $*_{\alpha,\beta}$ *induced by the family*

$$\left\{Q_{n,k}^{(\alpha,\beta)} : n \geq k \geq 0\right\}$$

of Koornwinder type III polynomials satisfying for $\alpha \geq \beta + \frac{1}{2} \geq 0$ *the product formulae (IV.1 and IV.2 of Case 4.2)*

$$Q_{n,k}^{(\alpha,\beta)}(x_1^2, x_2)Q_{n,k}^{(\alpha,\beta)}(y_1^2, y_2) = \int_{I\times J^3} Q_{n,k}^{(\alpha,\beta)}\left(E^2, EG\right)\,\rho_{\alpha,\beta}\left(d(r_1,\psi_1,\psi_2,\psi_3)\right)$$

whenever $(x_1, x_2), (y_1, y_2) \in \mathbb{B}\backslash\{(0,0)\}$, *and*

$$Q_{n,k}^{(\alpha,\beta)}(x_1^2, x_2)Q_{n,k}^{(\alpha,\beta)}(0,0) = \int_{I\times J} Q_{n,k}^{(\alpha,\beta)}(E^2, D)\,\nu_{\alpha,\beta+\frac{1}{2}}(d(r_1,\psi_1))$$

whenever $(x_1, x_2) \in \mathbb{B}$ *and* $(y_1, y_2) = (0,0)$. *Here* $\rho_{\alpha,\beta} \in M^1(I \times J^3)$ *is given by*

$$\rho_{\alpha,\beta}(d(r_1,\psi_1,\psi_2,\psi_3)) = \nu_{\beta-\frac{1}{2}}(d\psi_3)\nu_{\beta-\frac{1}{2}}(d\psi_2)\nu_{\alpha,\beta+\frac{1}{2}}(r_1,\psi_1)$$

where

$$\nu_\gamma(d\psi) := \frac{\Gamma(\beta+\frac{3}{2})}{\Gamma\left(\frac{1}{2}\right)\Gamma(\beta+1)}\sin^{2\gamma+1}\psi\,(d\psi) \in M^1(J).$$

In the highly technical proof of the theorem most of the work has been done in order to establish for $\alpha > \beta + \frac{1}{2} > -\frac{1}{2}$ the product formula

$$\begin{aligned} &Q_{n,k}^{(\alpha,\beta)}(x_1^2, x_2)Q_{n,k}^{(\alpha,\beta)}(y_1^2, y_2) \\ &\quad= \int_{I\times J} Q_k^{(\beta,\beta)}(C)E^k Q_{n-k}^{\left(\alpha,\beta+k+\frac{1}{2}\right)}(2E^2-1)\,\nu_{\alpha,\beta+\frac{1}{2}}(d(r_1,\psi_1)) \\ &\quad= \int_{I\times J} Q_{n,k}^{(\alpha,\beta)}(E^2, D)\,\nu_{\alpha,\beta+\frac{1}{2}}(d(r_1,\psi_1)). \end{aligned}$$

From this it follows that

$$\begin{aligned} &Q_{n,k}^{(\alpha,\beta)}(x_1^2, x_2)Q_{n,k}^{(\alpha,\beta)}(y_1^2, y_2) \\ &\quad= \int_{I\times J} E^k Q_{n-k}^{\left(\alpha,\beta+k+\frac{1}{2}\right)}(2E^2-1)Q_k^{(\beta,\beta)}(C)\,\nu_{\alpha,\beta+\frac{1}{2}}(d(r_1,\psi_1)) \\ &\qquad\times \int_J Q_k^{(\beta,\beta)}\left(D\left(\frac{x_2}{x_1}, \frac{y_2}{y_1}; 1, \psi_2\right)\right)\nu_{\beta-\frac{1}{2}}(d\psi_2) \\ &\quad= \int_{I\times J^3} E^k Q_{n-k}^{\left(\alpha,\beta+k+\frac{1}{2}\right)}(2E^2-1) \\ &\qquad\times Q_k^{(\beta,\beta)}\left(D\left(\frac{D}{E}, D\left(\frac{x_2}{x_1}, \frac{y_2}{y_1}; 1, \psi_3\right)\right)\right)\,\nu_{\beta-\frac{1}{2}}(d\psi_3)\nu_{\beta-\frac{1}{2}}(d\psi_2)\nu_{\alpha,\beta+\frac{1}{2}}(r,\psi_1) \\ &\quad= \int_{I\times J^3} Q_{n,k}^{(\alpha,\beta)}(E^2, EG)\,\rho_{\alpha,\beta}(d(r_1,\psi_1,\psi_2,\psi_3)). \end{aligned}$$

6. Koornwinder type IV polynomials

These are defined for $\alpha, \beta, \gamma > -1$ and $n, k \in \mathbb{Z}_+$ with $n \geq k \geq 0$ by

$$Q_{n,k}^{(\alpha,\beta,\gamma)}(x_1, x_2) := Q_{n-k}^{(\alpha,\beta+\gamma+2k+1)}(2x_1 - 1)\, Q_k^{(\beta,\gamma)}\left(\frac{2x_2}{x_1} - 1\right).$$

For fixed α, β, γ the sequence

$$\left\{Q_{n,k}^{(\alpha,\beta,\gamma)} : \; n \geq k \geq 0\right\} \subset \mathbb{C}\left[X_1, X_2\right]_{\leq n}$$

is orthogonal with respect to the measure

$$\omega\left(d\left(x_1, x_2\right)\right) := \left(1 - x_1\right)^{\alpha}\left(x_1 - x_2\right)^{\beta} x_2^{\gamma}\, dx_1 dx_2$$

on the triangle $\mathbb{T}$.

6.1. Theorem. *The triangle $\mathbb{T}$ can be made into a 2-variable polynomial hypergroup $(\mathbb{T}, *_{\alpha,\beta,\gamma})$ with convolution $*_{\alpha,\beta,\gamma}$ induced by the family of Koornwinder type IV polynomials satisfying for $\alpha \geq \beta + \gamma + 1$ and $\beta \geq \gamma \geq -\frac{1}{2}$ the product formulae (IV.1 and IV.2 of 4.2)*

$$\begin{aligned} Q_{n,k}^{(\alpha,\beta,\gamma)}&(x_1^2, x_2^2) Q_{n,k}^{(\alpha,\beta,\gamma)}(y_1^2, y_2^2) \\ &= \int_{I^4 \times J^3} Q_{n,k}^{(\alpha,\beta,\gamma)}(E^2, E^2 H^2)\, \rho_{\alpha,\beta,\gamma}(d(r_1, r_2, r_3, r_4, \psi_1, \psi_2, \psi_3)) \end{aligned}$$

whenever $(x_1, x_2), (y_1, y_2) \in \mathbb{T} \backslash \{(0,0)\}$, and

$$\begin{aligned} Q_{n,k}^{(\alpha,\beta,\gamma)}&(x_1^2, x_2^2) Q_{n,k}^{(\alpha,\beta,\gamma)}(0,0) \\ &= \int_{I^2 \times J} Q_{n,k}^{(\alpha,\beta,\gamma)}(E^2, E^2((1 - r_2)C^2 + r_2))\, \sigma_{\alpha,\beta,\gamma}(d(r_1, r_2, \psi_1)) \end{aligned}$$

whenever $(x_1, x_2) \in \mathbb{T}$ and $(y_1, y_2) = (0,0)$. The representing measures $\rho_{\alpha,\beta,\gamma} \in M^1(I^4 \times J^3)$ and $\sigma_{\alpha,\beta,\gamma} \in M^1(I^2 \times J)$ are given by

$$\begin{aligned} \rho_{\alpha,\beta,\gamma}&(d(r_1, r_2, r_3, r_4, \psi_1, \psi_2, \psi_3)) \\ &= \nu_{\beta,\gamma}(d(r_4, \psi_3)) \nu_{\beta,\gamma}(d(r_3, \psi_2)) \tau_{\beta,\gamma-\frac{1}{2}}(dr_2) \nu_{\alpha,\beta+\gamma+1}(r_1, \psi_1) \end{aligned}$$

with

$$\tau_{\xi,\zeta}(dr) := \frac{\Gamma(\xi + \zeta + 2)}{\Gamma(\xi + 1)\Gamma(\zeta + 1)}(1 - r)^{\xi} r^{\zeta} dr \in M^1(I)$$

and

$$\sigma_{\alpha,\beta,\gamma}(d(r_1,r_2,\psi)) := \tau_{\beta,\gamma-\frac{1}{2}}(dr_2)\nu_{\alpha,\beta+\gamma+1}(r_1,\psi) \in M^1(I^2\times J)$$

respectively.

For the proof of the product formulae we first need to establish that

$$Q_{n,k}^{(\alpha,\beta,-\frac{1}{2})}(x_1^2,x_2^2)Q_{n,k}^{(\alpha,\beta,-\frac{1}{2})}(y_1^2,y_2^2) = \int_{I\times J} Q_{n,k}^{(\alpha,\beta,-\frac{1}{2})}(E^2,D^2)\,\upsilon_{\alpha,\beta+\frac{1}{2}}(d(r_1,\psi_1))$$

for $\alpha > \beta - \frac{1}{2} > -\frac{1}{2}$, then with β replaced by $\beta+\gamma+\frac{1}{2}$

$$Q_{n,k}^{(\alpha,\beta,\gamma)}(x_1^2,x_2^2)Q_{n,k}^{(\alpha,\beta,\gamma)}(y_1^2,y_2^2) = \int_{I^2\times J} Q_{n,k}^{(\alpha,\beta,\gamma)}(E^2,E^2((1-r_2)C^2+r_2))\tau_{\beta,\gamma-\frac{1}{2}}(dr_2)\nu_{\alpha,\beta+\gamma+1}(r_1,\psi_1)$$

and finally we obtain for $\alpha \geq \beta+\gamma+1$ and $\beta\geq\gamma\geq-\frac{1}{2}$ the desired formulae

$$\begin{aligned}
Q_{n,k}^{(\alpha,\beta,\gamma)}&(x_1^2,x_2^2)Q_{n,k}^{(\alpha,\beta,\gamma)}(y_1^2,y_2^2)\\
&= \int_{I^2\times J} Q_{n-k}^{(\alpha,\beta+\gamma+2k+1)}(2E^2-1)E^{2k}Q_k^{(\beta,\gamma)}(2(1-r_2)C^2+2r_2-1)\\
&\quad\times \tau_{\beta,\gamma-\frac{1}{2}}(dr_2)\nu_{\alpha,\beta+\gamma+1}(r_1,\psi_1)\\
&\quad\times \int_{I\times J} Q_k^{(\beta,\gamma)}\left(2E^2\left(\frac{x_2}{x_1},\frac{y_2}{y_1};r_3,\psi_2\right)-1\right)\nu_{\beta,\gamma}(d(r_3,\psi_2))\\
&= \int_{I^4\times J^3} Q_{n-k}^{(\alpha,\beta+\gamma+2k+1)}(2E^2-1)E^{2k}\\
&\quad\times Q_k^{(\beta,\gamma)}\left(2E^2\left(((1-r_2)C^2+r_2)^{1/2},E\left(\frac{x_2}{x_1},\frac{y_2}{y_1};r_3,\psi_2\right);r_4,\psi_3\right)-1\right)\\
&\quad\times \nu_{\beta,\gamma}(d(r_4,\psi_3))\nu_{\beta,\gamma}(d(r_3,\psi_2))\tau_{\beta,\gamma-\frac{1}{2}}(dr_2)\nu_{\alpha,\beta+\gamma+1}(d(r_1,\psi_1))\\
&= \int_{I^4\times J^3} Q_{n,k}^{(\alpha,\beta,\gamma)}(E^2,E^2H^2)\,\rho_{\alpha,\beta,\gamma}(d(r_1,r_2,r_3,r_4,\psi_1,\psi_2,\psi_3)).
\end{aligned}$$

6.2. *Remark.* In [10] the approach leading to a hypergroup structure on $\mathbb{T}$ has been extended to the more general k-simplex

$$\mathbb{S}^{(k)} := \left\{(x_1,x_2,\ldots,x_k)\in R^k : 0\leq x_k\leq\ldots\leq x_1\leq 1\right\}.$$

The proof of the analogue of Theorem 6.1 is performed via induction with respect to k, where the statement of Theorem 6.1 serves as the start $k=2$ of the induction.

6.3. Summary. *In this section we have seen that the square $\mathbb{Q}$, the parabolic bi-angle $\mathbb{B}$ and the k-simplex $\mathbb{S}^{(k)}$ for $k \geq 2$ are examples of cone-embedded hypergroups. It is this class of k-variable polynomial hypergroups for which we intend to discuss some properties of Lévy processes and random walks.*

7. Lévy–Khintchine representations

We begin by describing Faraut's approach [6] to canonical (Lévy–Khintchine type) representations of generalized Laplacians on subsets K of $\mathbb{R}^k, k \geq 1$ (containing a distinguished element e). For any such set K we consider the space

$$C^\infty(K) := \mathrm{Res}\,_K C^\infty(\mathbb{R}^k)$$

of test functions on K furnished with the Fréchet topology, and the convex cone

$$\mathcal{M} := \left\{ f \in C^\infty(K) : \sup_{x \in K} f(x) = f(e) \geq 0 \right\}$$

in $C^\infty(K)$.

7.1. *Definition.* A (real) linear functional L on $C^\infty(K)$ is said to be a *generalized Laplacian* on K if $\langle L, f \rangle \leq 0$ for all $f \in \mathcal{M}$. We denote the dual (convex) cone of $\mathcal{M}$ by $\mathcal{L}(K)$. We note that

7.1.1. given $L \in \mathcal{L}(K)$ the restriction $\mathrm{Res}_{K^\times} L$ of L to $K^\times := K \backslash \{e\}$ is a measure in $M_+(K^\times)$ and that

7.1.2. $-\varepsilon_e \in \mathcal{L}(K)$.

7.2. *Special case.* Consider $K := \mathbb{R}^k$ with $e = 0$. Faraut [6] showed

7.2.1. that the linear space generated by the cone $\mathcal{M}$ is given by

$$\langle \mathcal{M} \rangle = \left\{ f \in C^\infty\left(\mathbb{R}^k\right) : \frac{\partial f}{\partial x_i}(0) = 0 \text{ for } i = 1, 2, \ldots, k \right\}$$

and that

7.2.2. for any linear functional L on $C^\infty\left(\mathbb{R}^k\right)$ the following statements are equivalent.

(i) $L \in \mathcal{L}(K)$ is supported by $\{0\}$.

(ii) There exist a constant $a \geq 0$, a vector $(b_1, b_2, \ldots, b_k) \in \mathbb{R}^k$ and a quadratic form

$$(\xi_1, \xi_2, \ldots, \xi_n) \longmapsto \sum_{i,j=1}^{k} c_{ij} \xi_i \xi_j \geq 0$$

on $\mathbb{R}^k$ such that for every $f \in C^\infty\left(\mathbb{R}^k\right)$

$$\langle L, f\rangle = -af(0) + \sum_{i=1}^{k} b_i \frac{\partial f}{\partial x_i}(0) + \sum_{i,j=1}^{k} c_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(0).$$

7.3. Theorem (Lévy–Khintchine)**.** *For any linear functional L on $C^\infty\left(\mathbb{R}^k\right)$ the following statements are equivalent.*

(i) $L \in \mathcal{L}\left(\mathbb{R}^k\right)$.

(ii) *There exist $L_0 \in \mathcal{L}\left(\mathbb{R}^k\right)$ supported by $\{0\}$, $\eta \in M_+\left(\left(\mathbb{R}^k\right)^\times\right)$ with*

$$\eta\left(\left\{x \in \mathbb{R}^k : |x| > 1\right\}\right) < \infty$$

satisfying

$$\int_{|x|<1} |x|^2 \eta(dx) < \infty$$

and $u \in C^\infty\left(\mathbb{R}^k\right)$ with $0 \leq u \leq 1$ and $u(V) = 1$ for some neighbourhood V of 0 such that for all $f \in C^\infty\left(\mathbb{R}^k\right)$

$$\langle L, f\rangle = \langle L_0, f\rangle + \int_{\left(\mathbb{R}^k\right)^\times} \left(f(x) - f(0) + \sum_{i=1}^{k} x_i \frac{\partial f}{\partial x_i}(0)\, u(x)\right) \eta(dx).$$

7.4. *Special case.* Consider the disk $\mathbb{D}$ with $e := (1, 0)$, which was treated by Annabi and Trimèche [1]. Analogous to Special Case 7.2 we have

7.4.1.

$$\langle \mathcal{M}\rangle = \left\{f \in C^\infty(\mathbb{D}) : \frac{\partial f}{\partial y}(1, 0) = 0\right\}.$$

7.4.2. For any linear functional L on $C^\infty(\mathbb{D})$ the following statements are equivalent.

(i) $L \in \mathcal{L}(\mathbb{D})$ is supported by $\{(1, 0)\}$.

(ii) There exist constants $a, b, c, d \in \mathbb{R}, a, c, d \geq 0$ such that for all $f \in C^\infty(\mathbb{D})$

$$\langle L, f\rangle = -af(1, 0) + b\frac{\partial f}{\partial y}(1, 0) + c\frac{\partial^2 f}{\partial y^2}(1, 0) + d\frac{\partial f}{\partial x}(1, 0).$$

7.5. Theorem (Lévy–Khintchine)**.** *For any linear functional L on $C^\infty(\mathbb{D})$ the following statements are equivalent.*

(i) $L \in \mathcal{L}(\mathbb{D})$.

(ii) *There exist constants $a, b, c, d \in \mathbb{R}, a, c, d \geq 0$ and $\eta \in M_+(\mathbb{D}^\times)$ with*

$$\int_{\mathbb{D}^\times} (1-x)\,\eta(d(x,y)) < \infty$$

such that for all $f \in C^\infty(\mathbb{D})$

$$\begin{aligned}\langle L, f\rangle = &-af(1,0) + b\frac{\partial f}{\partial y}(1,0) + c\left(\frac{\partial^2 f}{\partial y^2}(1,0) - \frac{\partial f}{\partial x}(1,0)\right) - d\frac{\partial f}{\partial x}(1,0)\\ &+ \int_{\mathbb{D}^\times}\left(f(x,y) - f(1,0) - y\frac{\partial f}{\partial y}(1,0)\right)\eta(d(x,y)).\end{aligned}$$

7.6. *Special case.* Consider a cone-embedded space $K \subset \mathbb{R}^k$ with $e := (1,\ldots,1) \in \mathbb{R}^k$. This special case has been discussed by Mokni and Trimèche [13].

It is easily seen that

7.6.1. for each $\ell = 1, 2, \ldots, k$ the linear functional

$$L_\ell := -\left(\gamma'_\ell(0)\;\middle|\;\frac{\partial}{\partial x}\right)\varepsilon_e$$

on $C^\infty(K)$ where $\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_k}\right)$ is a generalized Laplacian on K.

The next step is to show

7.6.2. For any linear functional L on $C^\infty(K)$ the following statements are equivalent.

(i) $L \in \mathcal{L}(K)$ is supported by $\{e\}$.

(ii) There exist constants $a, b_1, b_2, \ldots, b_k \in \mathbb{R}$ such that for all $f \in C^\infty(K)$

$$\langle L, f\rangle = af(e) + \sum_{\ell=1}^{k} b_\ell\left\langle\frac{\partial}{\partial x_\ell}\varepsilon_e, f\right\rangle.$$

And finally we have

7.7. Theorem (Lévy–Khintchine)**.** *For any linear functional L on $C^\infty(K)$ the following statements are equivalent.*

(i) $L \in \mathcal{L}(K)$.

(ii) *There exist constants $a, b_1, b_2, \ldots, b_k \in \mathbb{R}$ and $\eta \in M_+(K^\times)$ such that for all $f \in C^\infty(K)$*

$$\langle L, f\rangle = -af(e) + \sum_{\ell=1}^{k} b_\ell \langle L_\ell, f\rangle + \int_{K^\times} (f - f(e))\, d\eta$$

with L_ℓ as in 7.6.1 for $\ell = 1, 2, \ldots, k$.

The crucial step in the proof of Theorem 7.7 modified from the proofs of Theorems 7.3 and 7.5 is that given $L \in \mathcal{L}(K)$ and $f \in \mathcal{M}$ there exist $A(f) \in \mathbb{R}_+$ and $\eta \in M_+(K^\times)$ coinciding with $\mathrm{Res}_{K^\times} L$ such that

$$(f(e) - f)\, L = A(f)\, \varepsilon_e + (f(e) - f)\, \eta.$$

Once $f \mapsto A(f)$ has been extended from a positively homogeneous mapping on $\mathcal{M}$ to a linear functional A on $C^\infty(K) = \mathcal{M} - \mathcal{M}$ it is easy to see that

$$-A = \sum_{\ell=1}^{k} b_\ell L_\ell$$

and the equality introducing A implies that

$$\langle L, f\rangle = f(e)\langle L, 1\rangle - A(f) + \int_{K^\times} (f - f(e))\, d\eta$$

whenever $f \in C^\infty(K)$. In order to establish the inclusion $C^\infty(K) \subset \mathcal{M} - \mathcal{M}$ we use the fact that for every $f \in C^\infty(K)$ there exists $\alpha > 0$ such that $|f| \le \alpha h$ with h as defined in Definition 4.1.

For the remainder of this section we are working with a cone-embedded hypergroup $(K, *_\mathcal{P})$ with defining family $\mathcal{P}$ of k-variable polynomials on K.

7.8. *Definition.* The family $(\lambda_P)_{P\in\mathcal{P}}$ is said to be *negative definite (with respect to $\mathcal{P}$)* if

(a) $\lambda_{\mathbb{1}} \ge 0$.

(b) For each $(c_P)_{P\in\mathcal{P}} \in \mathbb{C}^\mathcal{P}$ such that $c_P = 0$ for all but finitely many $P \in \mathcal{P}$ the two conditions

$$\sum_{P\in\mathcal{P}} c_P P(x) \ge 0 \text{ for all } x \in K \text{ and } \sum_{P\in\mathcal{P}} c_P = 0$$

imply that $\sum_{P\in\mathcal{P}} c_P \lambda_P \le 0$.

Obviously the totality $\mathcal{N}(\mathcal{P})$ of negative definite families forms a closed cone in $\mathbb{C}^\mathcal{P}$ containing all positive constant families.

7.9. Theorem. *For any family* $(\lambda_P)_{P\in\mathcal{P}} \in \mathbb{C}^{\mathcal{P}}$ *the following statements are equivalent.*

(i) $(\lambda_P)_{P\in\mathcal{P}} \in \mathcal{N}(\mathcal{P})$.

(ii) *There exists* $L \in \mathcal{L}(K)$ *such that* $\lambda_P = -\langle T, P\rangle$ *for all* $P \in \mathcal{P}$.

While the implication $(ii) \Rightarrow (i)$ is straightforward, the other implication requires the fact that a linear functional L defined on $\mathbb{R}[X_1, X_2, \dots, X_k]$ by

$$\langle L, f\rangle = -\sum_{P\in\mathcal{P}} c_P \lambda_P$$

for all f of the form

$$f := \sum_{P\in\mathcal{P}} c_P P$$

can be extended to a generalized Laplacian on K, the extension relying on the equality

$$\overline{\mathcal{M} \cap \mathbb{R}[X_1, X_2, \dots, X_k]} = C^{\infty}(K)$$

where the closure is taken in the topology of $C^{\infty}(K)$.

7.10. *Example.* For $(\mathbb{D}, *_\alpha)$ (which is **not** a cone-embedded hypergroup) the above equivalence reads as follows. The double sequence $(\lambda_{m,n})_{(m,n)\in\mathbb{Z}_+^2}$ belongs to $\mathcal{N}\left(\left\{Q_{m,n}^{\alpha} : (m,n) \in \mathbb{Z}_+^2\right\}\right)$ if and only if there are constants $a, b, c, d \in \mathbb{R}, a, c, d \geq 0$ and $\eta \in M_+(\mathbb{D}^{\times})$ with

$$\int_{\mathbb{D}^{\times}} (1-x)\, \eta(d(x,y)) < \infty$$

such that for all $(m,n) \in \mathbb{Z}_+^2$

$$\begin{aligned}\lambda_{m,n} = a - ib(m-n) + c(m-n)^2 + d\left(m + n + \frac{2mn}{\alpha+1}\right) \\ + \int_{\mathbb{D}^{\times}} \left(1 - Q_{m,n}^{\alpha}(x,y) + iy(m-n)\right) \eta(d(x,y)).\end{aligned}$$

7.11. *Definition.* A family $(\mu_t)_{t\geq 0} \subset M_+(K)$ is called a *(continuous) convolution semigroup* on K if

(a) $\|\mu_t\| \leq 1$ for all $t \geq 0$,

(b) $\mu_t * \mu_s = \mu_{t+s}$ for all $t, s \geq 0$,

(c) $\mu_0 = \varepsilon_e$,

(d) The mapping $t \mapsto \mu_t$ from $\mathbb{R}_+$ into $M_+(K)$ is τ_w-continuous.

7.12. *Remark.* There is a one-to-one correspondence between convolution semigroups $(\mu_t)_{t\geq 0}$ on K and stationary independent increment processes $(X_t)_{t\geq 0}$ (on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$) with values in K such that

$$\mathbb{P}\left(X_t \in B \,|\, X_s = x\right) = \left(\mu_{t-s} * \varepsilon_x\right)(B)$$

for all $0 \leq s \leq t$, Borel subsets B of K, and $x \in K$.

7.13. Theorem (Schoenberg correspondence)**.** *For any family* $(\lambda_P)_{P\in\mathcal{P}} \in \mathbb{C}^{\mathcal{P}}$ *the following statements are equivalent.*

(i) $(\lambda_P)_{P\in\mathcal{P}} \in \mathcal{N}(\mathcal{P})$.

(ii) *There exists a convolution semigroup* $(\mu_t)_{t\geq 0}$ *on* K *satisfying*

$$\overset{\wedge}{\mu_t}(P) = \exp\left(-t\lambda_P\right)$$

for all $P \in \mathcal{P}$.

The proof of this theorem given in [13] is based on Theorem 7.9. While again the implication (ii)$\Rightarrow$(i) is obvious, (i)$\Rightarrow$(ii) requires an additional argument that follows from Theorem 7.7. In fact for any generalized Laplacian L on a cone-embedded space K there exist $(\nu_s)_{s\geq 0} \subset M_+(K)$ with $\|\nu_s\| \leq 1$ for all $s \geq 0$ and $(\alpha_s)_{s\geq 0} \subset \mathbb{R}_+$ such that

$$\langle L, f\rangle = \lim_{s\downarrow 0} \alpha_s \left(\int_{\mathbb{D}} f \, d\nu_s - f(e)\right)$$

whenever $f \in C^\infty(K)$.

7.14. *Remark.* It should be noted that Theorem 7.13 can be proved in a much wider framework for arbitrary commutative hypergroups K employing the notion of strongly negative definite functions on the dual $K^\wedge$ of K; see [2], Theorem 5.2.15. Since for the compact k-variable polynomial hypergroup $(K, *_{\mathcal{P}})$ the family $\mathcal{P}$ constitutes the countable dual $K^\wedge$, the set $\mathcal{N}(\mathcal{P})$ corresponds to the set $SN(K^\wedge)$ of all strongly negative definite functions on $K^\wedge$.

What we do not have in general is the correspondence

$$(\lambda_P)_{P\in\mathcal{P}} \leftrightarrow L$$

between $\mathcal{N}(\mathcal{P})$ and $\mathcal{L}(K)$ given in Theorem 7.9. For Lévy–Khintchine representations of negative definite functions with additional properties on an arbitrary commutative hypergroup K or its dual $K^\wedge$; see [2], Section 4.5.

8. Central Limit results for random walks

We first discuss the central limiting behaviour of random walks on $(\mathbb{D}, *_\alpha)$ within the setting of Example 2.7 and then progress to the analogous behaviour for the 2-variable polynomial hypergroups $(\mathbb{B}, *_{\alpha,\beta})$ and $(\mathbb{T}, *_{\alpha,\beta,\gamma})$. In all cases under consideration the limit parameter will be α.

8.1. Let $(K, *)$ be an arbitrary hypergroup. Given $\mu \in M^1(K)$ we consider random walks $(X_\ell(\mu))_{\ell\in\mathbb{Z}_+}$ on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ with values in K as Markov chains in K having a transition kernel N of the form

$$N\,(x, B) := (\mu * \varepsilon_x)\,(B)$$

for all $x \in K$ and all Borel subsets B of K. More precisely, *random walks* $(X_\ell(\mu))_{\ell\in\mathbb{Z}_+}$ in K with *transition measure* μ are characterized by the properties

(a) $X_0(\mu) = 1$ $\mathbb{P}$-almost surely.

(b) For all $\ell \in \mathbb{Z}_+, x \in K$ and all Borel subsets B of K

$$\mathbb{P}\left(X_\ell\,(\mu) \in B\,|X_{\ell-1}\,(\mu) = x\,\right) = (\mu * \varepsilon_x)\,(B)\,.$$

Clearly for distributions of the random variable $X_\ell(\mu)$ we have

$$\mathbb{P}_{X_\ell(\mu)} = \mu^{*\ell}$$

whenever $\ell \in \mathbb{Z}_+$.

8.2. *Special case.* Disk hypergroup

We assume given a sequence $(\alpha(p))_{p\in\mathbb{N}}$ (of dimensions) in $\mathbb{R}_+^\times$, a sequence of points $(r(p), \psi(p))_{p\in\mathbb{N}}$ in $\mathbb{D}$ (given in polar coordinates), and a sequence $(j(p))_{p\in\mathbb{N}}$ (of jumps) in $\mathbb{N}$. For each $p \in \mathbb{N}$ let

$$\left(X_j\left(\alpha\left(p\right), \left(r\left(p\right), \psi\left(p\right)\right)\right)\right)_{j\in\mathbb{Z}_+}$$

denote the random walk on $(\mathbb{D}, *_{\alpha(p)})$ starting at time $j = 0$ at $(1, 0) \in \mathbb{D}$ and having transition measure $\varepsilon_{(r(p),\psi(p))}$.

In [15] Voit studied the limiting behaviour for $p \to \infty$ of the sequence $(Y_p)_{p\in\mathbb{N}}$ of standard $\mathbb{R}^2$-valued random variables

$$Y_p := X_{j(p)}\left(\alpha\left(p\right), \left(r\left(p\right), \psi\left(p\right)\right)\right)$$

(on $(\Omega, \mathfrak{A}, \mathbb{P})$). The desired limiting result due to Voit requires the following assumptions.

(a) $\lim_{p\to\infty} \alpha(p) = \infty$.

(b) There exists a constant $\rho > 0$ such that

$$\lim_{p\to\infty} \alpha\left(p\right)^{\rho}\left(1 - r\left(p\right)\right) = 0.$$

(c) $\lim_{p\to\infty} j(p)\psi(p) = 0$.

(d) There exists a constant $c \in \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ such that

$$\lim_{p\to\infty} \left(j\left(p\right)\left(1 - r\left(p\right)^{2}\right) - \ln \alpha\left(p\right) \right) = c.$$

8.3. Theorem. *Under the assumptions* (a) *to* (d) *we have*

$$\tau_{w} - \lim_{p\to\infty} \mathbb{P}_{Y_{p}} = \nu\left(c\right)$$

where $\nu(c) := n(c)\lambda^{2} \in M^{1}(\mathbb{R}^{2})$ *denotes the bivariate normal distribution with* λ^{2}*-density* $n(c)$ *given by*

$$n\left(c\right)\left(x,y\right) := \frac{1}{\pi} e^{-y^{2} - \left(x - e^{c/2}\right)^{2}}$$

whenever $(x,y) \in \mathbb{R}^{2}$.

If $c := \infty$ then the resulting λ^{2}-density is the function

$$\left(x,y\right) \mapsto \frac{1}{\pi} e^{-y^{2} - x^{2}}$$

on $\mathbb{R}^{2}$.

8.4. *Special case.* $\alpha := d - 2$ $(d \geq 2)$ Here we have the double class hypergroup $U\left(d\right) /\!/ U(d-1)$ (of Example 2.2). We reach the limiting statement of Theorem 8.3 for the random walk

$$\left(X_{j(d)}\left(d, r\left(d\right)\right)\right)_{d\geq 2}$$

with transition measures $\varepsilon_{r(d)}$ supported by $r\left(d\right) \in \,]0,1[\, \subset \mathbb{D}$, where the random variables can be viewed as

$$\left(X_{j(d)}\left(d, r\left(d\right)\right)\right) = \mathrm{pr} \circ Y^{d}_{j(d)}$$

where $(Y^{d}_{j})_{j\geq 0}$ denotes the isotropic random walk on the homogeneous space $U(d)/U(d-1) \cong \mathbb{S}^{2d-1}$ and pr denotes the canonical projection from $U(d)/U(d-1)$ onto $U(d) /\!/ U(d-1)$. Obviously the random walk $(X_{j(d)}(d, r(d)))_{d\geq 2}$ arises from the isotropic random variables $(Y^{d}_{j})_{j\geq 0}$ obtained by stopping the d^{th} walk after $j(d)$ steps.

8.5. *Special cases.* Parabolic bi-angle $\mathbb{B}$ and triangle $\mathbb{T}$. These were treated by Mili in [12]. We keep the notation of the random walks

$$\left(X_j\left(\alpha\left(p\right),\left(r\left(p\right),\psi\left(p\right)\right)\right)\right)_{j\in\mathbb{Z}_+}$$

and $\left(Y_p\right)_{p\in\mathbb{N}}$ and also the formulation of the assumptions (a) to (d) above with the following modifications.

8.5.1. For $(\mathbb{B},*_{\alpha,\beta})$ we assume that $\beta>-\frac{1}{2}$ and $\alpha\left(p\right)\geq\beta+\frac{1}{2}\geq 0$ whenever $p\in\mathbb{N}$.

8.5.2. For $(\mathbb{T},*_{\alpha,\beta,\gamma})$ we assume that $\beta,\gamma>-1,\beta\geq\gamma>-\frac{1}{2}$ and $\alpha(p)\geq\beta+\gamma+1$ whenever $p\in\mathbb{N}$.

8.6. Theorem. *Given the* 2*-variable polynomial hypergroup* $K=\mathbb{B}$ *or* $\mathbb{T}$ *we have*

$$\tau_w-\lim_{p\to\infty}=\nu=n\lambda^2\in M^1\left(\mathbb{R}^2\right)$$

where either

8.6.1. n has the form n_β given as

$$n_\beta\left(x,y\right):=\frac{1}{\sqrt{\pi}\,\Gamma\left(\beta+1\right)}x^{-\frac{1}{2}(\beta+1)}\left(x-y^2\right)^{\beta}e^{-\left(x+2ye^{-c/2}+e^{c}\right)}$$

or

8.6.2. n has the form $n_{\beta,\gamma}$ given as

$$n_{\beta,\gamma}(x,y):=\frac{1}{\Gamma(\beta+1)\Gamma(\gamma+1)}x^{-\frac{c}{2}(\beta-\gamma+1)}e^{-(x+e^{-c})}xy^{\gamma/2}(x-y)^{\beta}I_\gamma(2\sqrt{y}e^{-c})$$

for all $(x,y)\in\mathbb{R}^2$. Here I_γ denotes the modified Bessel function of index γ.

The proof of the theorem is based on the classical fact that sequences $\left(\mu_p\right)_{p\geq 1}$ of distributions on $\mathbb{R}^k$ converge weakly to a distribution ν on $\mathbb{R}^k$ once their (multidimensional) moments converge. In the present context the hypergroups $K=\mathbb{B}$ or $\mathbb{T}$ are described in polar coordinates in the form

$$\left\{\left(r^2,r\cos\psi\right):r\in[0,1]\,,\ \psi\in[0,\pi]\right\}$$

and

$$\left\{\left(r,r\cos^2\psi\right):r\in[0,1]\,,\ \psi\in\left[0,\frac{\pi}{2}\right]\right\},$$

the measures μ_p are living on $\mathbb{R}_+ \times [0,\pi]$ and $\mathbb{R}_+ \times \left[0, \frac{\pi}{2}\right]$, and their (modified) moments $M^p_{n,k}$ are given by

$$\int_{\mathbb{R}_+ \times [0,\pi]} L_{n-k}^{\beta+k+\frac{1}{2}}\left(r^2\right) r^k Q_k^{(\beta,\beta)}(\cos\psi)\, \mu_p\left(d\left(r,\psi\right)\right)$$

and

$$\int_{\mathbb{R}_+ \times \left[0,\frac{\pi}{2}\right]} L_{n-k}^{\beta+\gamma+2k+1}(r)\, r^k Q_k^{(\beta,\gamma)}\left(2\cos^2\psi - 1\right) \mu_p\left(d\left(r,\psi\right)\right)$$

respectively. Here L_n^α denotes the Laguerre polynomial of degree n and index α. Since the moments $M_{n,k}$ of the bivariate normal distributions ν of 8.6.1 and 8.6.2 can be calculated explicitly, the remaining part of the proof consists in showing that for the moments $M^p_{n,k}$ of the distributions μ_p of Y_p the limit relationship

$$\lim_{p\to\infty} M^p_{n,k} = M_{n,k}$$

holds for all $n, k \in \mathbb{N}, n \geq k$.

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