Publ. Math. Debrecen
72/3-4 (2008), 257–267

On algebras that are sums of two subalgebras satisfying certain polynomial identities

By MAREK KĘPCZYK (Białystok)

Abstract. We study an associative algebra A over an arbitrary field that is a sum of two subalgebras B and C (i.e. $A = B + C$). We prove that if B and C have commutative ideals of finite codimension then A/I, for some nilpotent ideal I of A, has a commutative ideal of finite codimension. Similar statements are shown for nilpotent and nil of bounded index ideals.

1. Introduction

Let R be an associative ring and R_1, R_2 its subrings such that $R = R_1 + R_2$, i.e. for every $r \in R$ there are $r_1 \in R_1$ and $r_2 \in R_2$ such that $r = r_1 + r_2$. In [4] K. I. Beidar and A. V. Mikhalev stated the following problem: if both R_i satisfy polynomial identities (shortly, are PI rings),whether then also R is a PI ring. The problem for particular identities was studied in many papers (cf. [2], [5], [6], [7], [8], [9], [11]). Before the problem was raised, three results (important for this work) related to this problem were obtained. Kegel [5] proved that if R_i are nilpotent, then so is R. In [6] it was shown that if R_i are nil of bounded index (i.e. they satisfy identity $x^{n_i} = 0$), then so is R. In [2] Bahturin and Giambruno proved that if both R_i are commutative, then R satisfies the identity $[x_1, y_1][x_2, y_2] = 0$, where as usual $[x, y] = xy - yx$. In [11] Petravchuk considered certain generalization of the two cited results for algebras over an arbitrary field that have commutative ideals of finite codimension

---

Mathematics Subject Classification: 16D70.
Key words and phrases: commutative ideal, nilpotent subalgebra, nil of bounded index subalgebra.
Supported by Technical University of Bialystok Grant No. W/IMF/2/05.

(such algebras are called almost commutative) and algebras that have a nilpotent ideals of finite codimension (such algebras are called almost nilpotent). He proves that if both R_i are almost commutative subalgebras, then R contains a nilpotent ideal I such that R/I is almost commutative. Moreover he shows that if both R_i are almost nilpotent then so is R.

However, his proof contains a mistake. Namely Corollary 2, which plays a key role in the proof, is false.

It states: *Let H be an algebra and I be a right (left) almost nilpotent ideal of H. Then I is contained in some almost nilpotent ideal of the algebra H.*

We shall give a counterexample. Let H be the algebra of all infinite matrices over K that have only finitely many non-zero rows. Let us consider the subset I of H consisting of all the matrices having nonzero entries only in the first column and let J be the subset of all matrices in I whose first row is zero. It is clear that I is a left ideal of H, J is a ideal of I and $J^2 = 0$. Since in addition $\dim_K I/J = 1$, I is not almost nilpotent. Obviously H is simple infinite dimensional algebra over K. Hence, the only ideal of H containing I is H, but H is not almost nilpotent. Subset of all matrices in H that have nonzero entries only in the top row is a right almost nilpotent ideal of H, which shows that above lemma is not true.

In this paper we give correct proofs of Petravchuk's results. Our proofs are partially based on some of Petravchuk's ideas, but contain also some substantial new reasoning.

We also show in this paper that a sum of two almost nil of bounded index algebras is almost nil of bounded index.

2. The main results

We consider associative algebras over a fixed field K, which are not assumed to have an identity. If I is an ideal (left ideal, right ideal) of a ring (of an algebra) A, we write $I \lhd A$ ($I <_l A$, $I <_r A$).

By $\mathcal{F}$, $\mathcal{N}$, $\mathcal{B}$ and $\mathcal{C}$ we denote the class of all finite dimensional algebras, nilpotent algebras, nil of bounded index algebras and commutative algebras, respectively.

Let us consider two arbitrary classes of algebras $\mathbb{S}$ and $\mathbb{T}$, for which $0 \in \mathbb{S}$ and $0 \in \mathbb{T}$. Let $\mathbb{S}\mathbb{T} = \{A \mid \exists I \lhd A, I \in \mathbb{S} : A/I \in \mathbb{T}\}$. Obviously $\mathbb{S} \subseteq \mathbb{S}\mathbb{T}$ and $\mathbb{T} \subseteq \mathbb{S}\mathbb{T}$. Thus $\mathcal{C}\mathcal{F}$ denotes the class of almost commutative algebras; $\mathcal{N}\mathcal{F}$ the class of almost nilpotent algebras; $\mathcal{B}\mathcal{F}$ the class of almost nil of bounded index algebras. It is well known that if $J \lhd I \lhd A$ and J is nilpotent then J lies in some nilpotent ideal J_A of the algebra A and $J_A \subseteq I$. Thus $(\mathcal{N}\mathcal{R})\mathcal{S} = \mathcal{N}(\mathcal{R}\mathcal{S})$ for arbitrary classes of algebras $\mathcal{R}$ and $\mathcal{S}$. Clearly $\mathcal{N}\mathcal{N} = \mathcal{N}$ and $\mathcal{N}\mathcal{B} = \mathcal{B}$.

Throughout the paper A is an algebra over K, B and C are subalgebras of A such that $A = B + C$. Moreover, let $B_0 \lhd B$ and $C_0 \lhd C$, where $\dim_K B/B_0 < \infty$ and $\dim_K C/C_0 < \infty$.

Using the above notation, one can state the main results of this paper as follows:

Theorem 1. *If $B \in \mathcal{N}\mathcal{F}$ and $C \in \mathcal{N}\mathcal{F}$, then $A \in \mathcal{N}\mathcal{F}$.*

Theorem 2. *If $B \in \mathcal{B}\mathcal{F}$ and $C \in \mathcal{B}\mathcal{F}$, then $A \in \mathcal{B}\mathcal{F}$.*

Theorem 3. *If $B \in \mathcal{C}\mathcal{F}$ and $C \in \mathcal{C}\mathcal{F}$, then $A \in \mathcal{N}\mathcal{C}\mathcal{F}$.*

3. Preliminary material

The centre of an algebra H is denoted by $Z(H)$. For a given subset S of an algebra H, by $l_H(S)$ and $r_H(S)$ we will denote the left and right annihilators of S in H, respectively.

We shall need the following

Lemma 4 ([10])**.** *Let H be an algebra over an arbitrary field and P a subalgebra of H such that $\dim H/P < \infty$. Then P contains an ideal I of H such that $\dim H/I < \infty$.*

We will use the following modification of Petravchuk's Lemma 7 from [11] (cf. also [13]). We include its short proof for completeness.

Lemma 5. *Let P_1 and P_2 be subalgebras of an algebra H and let I be an ideal of H such that $I \subseteq P_1 + P_2$. Then there exist subalgebras $Q_1 \subseteq P_1$ and $Q_2 \subseteq P_2$ of H such that $Q_1 + Q_2$ is subalgebra of H and $I \subseteq Q_1 + Q_2$.*

Proof. It is enough to take $Q_1 = \{p_1 \in P_1 \mid p_1 + p_2 \in I \text{ for some } p_2 \in P_2\}$ and $Q_2 = \{p_2 \in P_2 \mid p_1 + p_2 \in I \text{ for some } p_1 \in P_1\}$. Let $a, b \in P_1$, $c, d \in P_2$ and $a + c \in I$, $b + d \in I$. Then $ab - cd = (a + c)(b + d) - (a + c)d - c(b + d)$. Hence it is easy to notice that Q_1 and Q_2 are subalgebras of H, which also imply $Q_1 + I \subseteq Q_1 + Q_2$ and $Q_2 + I \subseteq Q_1 + Q_2$. Take any $q_1 \in Q_1$ and $q_2 \in Q_2$. From the definition of Q_1 there exists $p_2 \in P_2$ such that $(q_1 + p_2)q_2 \in I$. Clearly $p_2 \in Q_2$, so $p_2q_2 \in Q_2$. Hence, $q_1q_2 \in Q_2 + I \subseteq Q_1 + Q_2$. Similarly one can show that $q_2q_1 \in Q_1 + Q_2$. Hence, we get that $Q_1Q_2 \subseteq Q_1 + Q_2$ and $Q_2Q_1 \subseteq Q_1 + Q_2$. Then $Q_1 + Q_2$ is a subalgebra of H. Obviously $I \subseteq Q_1 + Q_2$. $\square$

We shall need some information about the classes $\mathcal{B}$. Obviously every algebra from $\mathcal{B}$ is a nil PI algebra. Let as denote by $W(H)$ the sum of all nilpotent ideals of an algebra H. Clearly $a \in W(H)$ if and only if the right (left) ideal aH (Ha) of H is nilpotent. It implies that if $I \lhd H$ then $W(I) \subseteq W(H)$. Indeed, if $i \in W(I)$ then there exists a natural number n such that $(iI)^n = 0$. Hence $(iH)^{2n} \subseteq (iHiH)^n \subseteq (iI)^n = 0$, so $i \in W(H)$.

Proposition 6 ([1])**.** *For every nil PI algebra H there exists a natural number n such that $H^n \subseteq W(H)$.*

Let us consider the class $\mathcal{NRF}$, where $\mathcal{R}$ is one of the class $\mathcal{C}$, $\mathcal{N}$ or $\mathcal{B}$. Now we are ready to make some generalization of [11, Proposition 1].

Proposition 7. *For the class $\mathcal{NRF}$, where $\mathcal{R} = \mathcal{C}, \mathcal{N}$ or $\mathcal{B}$ the following statements hold:*

(i) *every subalgebra and every quotient algebra of an algebra from $\mathcal{NRF}$ belongs to $\mathcal{NRF}$.*

(ii) *if $P, Q \in \mathcal{NRF}$ then the direct product $P \times Q \in \mathcal{NRF}$.*

(iii) *if $I \lhd H$, $H/I \in \mathcal{NRF}$ then $H \in \mathcal{NRF}$.*

Proof. The statements (i) and (ii) are obvious. We show that (iii) holds.
For $\mathcal{R} = \mathcal{C}$ see [11, Proposition 1].
For $\mathcal{R} = \mathcal{N}$ see [11, Corolary 3].
Let $\mathcal{R} = \mathcal{B}$, $I \lhd H$, $H/I \in \mathcal{BF}$ and $I \in \mathcal{BF}$. Hence in particular there exists an ideal J of I such that $J \in \mathcal{B}$ and $I/J \in \mathcal{F}$. Since $J \in \mathcal{B}$ and $J \lhd I \lhd H$, then there exists a natural number n such that $J^n \subseteq W(J)$ and $W(J) \subseteq W(I) \subseteq W(H)$. Let $W(I)_H$ be the ideal of H generated by $W(I)$. It is not hard to check than $W(I)_H \subseteq W(I)$. Thus $(W(I)_H + J)/J$ is nil. Additionally $(W(I)_H + J)/J \in \mathcal{F}$. Hence $(W(I)_H + J)/J$ is nilpotent, so $W(I)_H \in \mathcal{B}$. Clearly we can assume that $W(I)_H = 0$, which implies $J^n = 0$. Now we show that $H \in \mathcal{BF}$. Since J is nilpotent and $J \lhd I \lhd H$ then J is contained in some nilpotent ideal of the algebra H, so we can assume that $I \in \mathcal{F}$. Let $S/I \in \mathcal{B}$ and S/I be an ideal of H/I such that $H/S \in \mathcal{F}$ and let $G = r_S(I)$. Obviously $G \lhd H$ and $S/G \in \mathcal{F}$, which implies $H/G \in \mathcal{F}$. Since $(G \cap I)^2 = 0$ and $G/(G \cap I) \approx (G + I)/I \in \mathcal{B}$, then $G \in \mathcal{B}$. Hence $H \in \mathcal{BF}$ and the proof is complete. □

Definition 8. An algebra $A = B + C$ over an arbitrary field K is called $\mathcal{R}$-counter-example, where $\mathcal{R} = \mathcal{C}, \mathcal{N}$ or $\mathcal{B}$, if A satisfies the following conditions:

(1) $A \notin \mathcal{NRF}$;

(2) the subalgebras B and C have ideals $B_0 \lhd B$ and $C_0 \lhd C$ such that $B_0, C_0 \in \mathcal{R}$ and the number $\dim A/(B_0+C_0)$ is the smallest one;

(3) the algebra A does not have not nonzero ideals that lie in K-subspace B_0+C_0 from condition (2).

Suppose that $A = B + C$ is an algebra satisfying (1) and (2) from the above definition. Let T be the sum of all ideals of A that are contained in $B_0 + C_0$. By Lemma 5, $T \subseteq Q_1 + Q_2$, where $Q_1 + Q_2$ is a subalgebra of A and Q_1, $Q_2 \in \mathcal{R}$. So $(Q_1 + Q_2) \in \mathcal{NR}$ and $T \in \mathcal{NR}$. Additionally Proposition 7 gives $A/T \notin \mathcal{NRF}$. Clearly $A/T = (B + T)/T + (C + T)/T$. Now it is easy to see that A/T is an $\mathcal{R}$-counter-example.

Lemma 9. *Let A be an $\mathcal{R}$-counter-example, where $\mathcal{R} = \mathcal{C}, \mathcal{N}$ or $\mathcal{B}$. Then*

(i) *for every $0 \neq I \lhd A$, $A/I \in \mathcal{NRF}$;*

(ii) *the algebra A has no nonzero ideals from $\mathcal{NRF}$;*

(iii) *A is a prime algebra.*

Proof. Let A be an $\mathcal{R}$-counter-example and $0 \neq I \lhd A$. Denote $\overline{A} = A/I$, $\overline{B} = (B+I)/I$, $\overline{C} = (C+I)/I$. Moreover $\overline{B_0} = (B_0+I)/I$ and $\overline{C_0} = (C_0+I)/I$. Clearly $\overline{A} = \overline{B} + \overline{C}$ and $\overline{B}, \overline{C} \in \mathcal{NR}$. By Definition 8, $I \nsubseteq B_0 + C_0$, so $\dim \overline{A}/(\overline{B_0} + \overline{C_0}) < \dim A/(B_0 + C_0)$. So $\overline{A} \in \mathcal{NRF}$, which gives (i).

By (i) and Proposition 7, (ii) becomes obvious.

Let us prove the statement (iii). Suppose that A is not a prime algebra. Hence there exist nonzero ideals I and J of A such that $IJ = 0$. Since $(I \cap J)^2 = 0$, then in view of a part (ii), $I \cap J = 0$. Hence A can be embedded into the product $A/I \times A/J$. Therefore applying statements (i) and (ii) of Lemma 5 we obtain that $A \in \mathcal{NRF}$. However A is an $\mathcal{R}$-counter-example, so $A \notin \mathcal{NRF}$, a contradiction. $\square$

Lemma 10. *Assume that A is a prime algebra, $l_{B_0}(B_0) \neq 0$ and $r_{C_0}(C_0) \neq 0$. If $U_1 = B_0 + B_0A$ and $U_2 = C_0 + C_0A$ are PI algebras, then A is finite dimensional.*

Proof. It is clear that $U_1 + U_2$ is a subalgebra of A, $U_1 <_r A$ and $U_2 <_r A$. By Corollary 4 in [8], $U_1 + U_2$ is a PI algebra. But $\dim A/(U_1 + U_2) < \infty$, so according to Lemma 4, there exists $J \lhd A$ such that $J \subseteq U_1 + U_2$ and $\dim A/J < \infty$. Consequently A ia a PI algebra.

We show now that $Z(A)$ is finite dimensional over K. This will be proved by shoving that $Z(A) \cap (B_0 + C_0) = 0$. Suppose, contrary to our claim, that $Z(A) \cap (B_0 + C_0) \neq 0$, so there exists $0 \neq z = b_0 + c_0$, where $b_0 \in B_0$, $c_0 \in C_0$ and $z \in Z(A)$. Of course $l_{B_0}(B_0) z r_{C_0}(C_0) = 0$. Since A is a prime algebra and $z \in$

$Z(A)$, it follows that $l_{B_0}(B_0)r_{C_0}(C_0)=0$. But $l_{B_0}(B_0)\lhd B$ and $r_{C_0}(C_0)\lhd C$, so $l_{B_0}(B_0)Ar_{C_0}(C_0)\subseteq l_{B_0}(B_0)Br_{C_0}(C_0)+l_{B_0}(B_0)Cr_{C_0}(C_0)\subseteq l_{B_0}(B_0)r_{C_0}(C_0)=0$, contrary to primeness of A. Hence $Z(A)\cap(B_0+C_0)=0$. Therefore $\dim_K Z(A)<\infty$. Since A is a prime algebra, $Z(A)$ is a commutative finite dimensional domain, so $Z(A)$ is a field. Hence the central localization $Z(A)^{-1}A$ of A is equal to A. We showed that A is a PI algebra, so by Posner's Theorem [12], A is finite dimensional over $Z(A)$. Consequently $\dim_K A<\infty$. □

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. Suppose the assertion of the theorem is false. It follows easily that there exists $\mathcal{N}$-counter-example. So we can assume that A is a $\mathcal{N}$-counter-example. Let B_0 and C_0 be nilpotent. We proceed by induction with respect to $n=n_1+n_2$, where n_1 and n_2 are natural numbers such that $B_0^{n_1}=0$, $C_0^{n_2}=0$. For $n=2$, $\dim_K A<\infty$ and we have a contradiction. If $n_1=1$ then $\dim A/B_0<\infty$, so by Lemma 4, $A\in\mathcal{NF}$. Similarly if $n_2=1$ then $A\in\mathcal{NF}$. Assume that $n>2$ and the result holds for smaller integers.

By Lemma 9, A is a prime algebra. Consider $A_1=B+B_0A$. It is clear that A_1 is a subalgebra of A and since $B\subseteq A_1$, $A_1=A_1\cap(B+C)=B+A_1\cap C$. Since $B_0^{n_1}=0$, then $B_0^{n_1-1}<_l A_1$. From this, there exists a nilpotent ideal I of A_1 such that $B_0^{n_1-1}\subseteq I$. Of course $A_1/I=(B+I)/I+((A_1\cap C)+I)/I$, where $(B+I)/I\in\mathcal{NF}$, $((A_1\cap C)+I)/I\in\mathcal{NF}$ and $(B_0+I)/I$, $((A_1\cap C_0)+I)/I$ are nilpotent ideals of $(B+I)/I$ and $((A_1\cap C)+I)/I$, respectively, of finite codimension. Moreover $((B_0+I)/I)^{n_1-1}=0$ and $(((A_1\cap C_0)+I)/I)^{n_2}=0$, so the induction assumption gives $A_1/I\in\mathcal{NF}$ and, since I is nilpotent, we have $A_1\in\mathcal{NF}$. Let $U_1=B_0+B_0A$. Since $U_1\subseteq A_1$, $U_1\in\mathcal{NF}$. Similarly we can show that $U_2=C_0+C_0A\in\mathcal{NF}$. It is obvious that $U_1<_r A$, $U_2<_r A$ and in particular they are PI algebras. Our assumption that $A\notin\mathcal{NF}$ gives that the indices of nilpotency of B_0 and C_0 are bigger than one. Hence $l_{B_0}(B_0)\neq 0$ and $r_{C_0}(C_0)\neq 0$. Now we can use Lemma 10, getting that $\dim_K A<\infty$. Thus A is not an $\mathcal{N}$-counter-example, a contradiction. □

We shall need the following lemma, the proof of which is based on a well known idea used by Amitsur in his proof of Levitzki–Amitsur Theorem (for example [14, Theorem 1.6.36]). We sketch its proof for completeness.

Lemma 11. *Let R be semiprime ring and T be PI subring of R of degree d. Moreover let I be a nilpotent ideal of R and n be a natural number such that $I^n = 0$ and $I^{n-1} \neq 0$. If for all $1 \leq i \leq n-1$, $A_i = I^{n-i}RI^i \subseteq T$, then $n \leq d$.*

Proof. Suppose that $n > d$. The subring T is PI of degree d, so it satisfies the identity

$$x_1 x_2 \dots x_d = \sum_{id \neq \pi \in S_d} \alpha_\pi x_{\pi(1)} x_{\pi(2)} \dots x_{\pi(d)},$$

where S_d is the set of permutations of the set $\{1, 2, \dots d\}$ and α_π are some integers. Therefore

$$(I^{m-1}R)^d I^d = A_1 A_2 \dots A_d = \sum_{id \neq \pi \in S_d} \alpha_\pi A_{\pi(1)} A_{\pi(2)} \dots A_{\pi(d)} = 0,$$

so $(I^{m-1}R)^{d+1} = 0$. Since R is semiprime, it follows that $I^{m-1} = 0$, a contradiction. Thus $n \leq d$. □

Proof of Theorem 2. Suppose that there exists $A \notin \mathcal{BF}$. Without loss of generality, we can assume that A is a $\mathcal{B}$-counter-example. Moreover B_0 and C_0 are nil of bounded index and such that $\dim A/(B_0 + C_0)$ is the smallest number for which $A \notin \mathcal{BF}$. Applying Lemma 9 we have that A is a prime algebra. By Proposition 6, there exist a natural number $n > 0$ such that $B_0^n \subseteq W(B_0)$. Observe that if L, K are ideals of B_0 such that $LK = 0$, then $\overline{B_0} = B_0 + KAL$ is a subalgebra of A, $KAL \lhd \overline{B_0}$ and $(KAL)^2 = 0$. It follows that $\overline{B_0}$ is nil of bounded index. Since $\overline{B_0}/(KAL)$ is a homomorphic image of B_0, then $\overline{B_0}$ is nil of bounded index and $(\overline{B_0})^n \subseteq W(\overline{B_0})$. Consider $\overline{B} = B + KAL$. It is clear that $\overline{B}$ ia a subalgebra of A, $\overline{B_0} \lhd \overline{B}$ and $\dim \overline{B}/\overline{B_0} < \infty$. Moreover $A = \overline{B} + C$. If $a \in \overline{B_0} \cap C_0$, then $aA = a\overline{B} + aC$ and $a\overline{B}$, aC are nil subalgebras of bounded index of aA. By Theorem 2 [6], $aA \in \beta$, where β is the prime radical. Since A is a prime algebra, $a = 0$. Hence $\overline{B_0} \cap C_0 = 0$. Thus if $B_0 \subsetneq \overline{B_0}$, $\dim A/(\overline{B_0} + C_0) < \dim A/(B_0 + C_0)$, which is in contradiction with the choice of B_0 and C_0. Hence $\overline{B_0} = B_0$. It follows that $KAL \subseteq B_0$. In particular if $I \lhd B_0$ and $I^m = 0$, $I^{m-1} \neq 0$ for a natural number m, then for every $1 \leqq i \leqq m-1$, $A_i = I^{m-i}AI^i \subseteq B_0$. Since $B_0 \in \mathcal{B}$, B_0 is a PI algebra of degree, say, d. By Lemma 11, $m \leq d$ and consequently for every nilpotent ideal J of B_0, $J^d = 0$. Hence $(W(B_0))^d = 0$. But $B_0^n \subseteq W(B_0)$, so B_0 is a nilpotent ideal of B. Therefore $B \in \mathcal{NF}$. In a similar way we show that $C \in \mathcal{NF}$. By Theorem 1, $A \in \mathcal{NF}$. It is clear that $\mathcal{NF} \subseteq \mathcal{BF}$. This contradicts our assumption that $A \notin \mathcal{BF}$, and completes the proof. □

5. Proof of Theorem 3

We now present several facts which will be used in the proof of Theorem 3. Let R be a ring and $I \lhd R$. Applying the identity $[xy,t]=x[y,t]+[x,t]y$, it is easy to show that

(i) $I[R,R] \subseteq [I,R]R^*$;

(ii) $I[I,R] \subseteq [I,I]R^*$,

where R^* denotes the ring R with an identity adjoined. Hence we obtain that if I is a commutative ideal of the ring R, then $[I,R] \subseteq r_R(I)$ (similar arguments give, $[I,R] \subseteq l_R(I)$). Moreover if $r_I(I)=0$ or $l_I(I)$=0, then $I \subseteq Z(R)$. If $r_R(I)=0$ or $l_R(I)=0$, then R is a commutative ring.

Lemma 12. *Let A be a prime algebra. Assume that $B_0 \subseteq Z(B)$ and $r_C(C_0) \neq 0$. Then $\dim(AC_0+C_0)/C_0 < \infty$.*

Proof. Let us denote $r_C(C_0)=I$. First we will prove that

(i) if $0 \neq a \in AC_0$ and $a=b_0+c_0$, where $b_0 \in B_0$, $c_0 \in C_0$, then $b_0=0$.

Since $0=aI=b_0I+c_0I$ and $c_0I=0$, then $b_0I=0$. Let us see, that $b_0AI \subseteq b_0BI+c_0CI \subseteq Bb_0I+c_0I=0$, since $B_0 \subseteq Z(B)$ and $I \lhd C$. Therefore $b_0=0$, as $I \neq 0$ and A is a prime algebra. This proves (i).

By (i) it is clear that $AC_0 \cap (B_0+C_0) = AC_0 \cap C_0$. Now $(AC_0+C_0)/C_0 \approx AC_0/AC_0 \cap C_0 = AC_0/AC_0 \cap (B_0+C_0) \approx (AC_0+(B_0+C_0))/(B_0+C_0)$. Thus, having $\dim A/(B_0+C_0) < \infty$, it follows that $\dim(AC_0+C_0)/C_0 < \infty$. This proves the lemma. □

Remark 1. If in the assumption of the above lemma we replace $r_C(C_0) \neq 0$ by $l_C(C_0) \neq 0$, then the similar proof gives $\dim(C_0A+C_0)/C_0 < \infty$.

Lemma 13. *Let R be a K-algebra and let S, T be finite dimensional K-subspaces of R. If M and P are K – subspaces of R such that $\dim(SMT+P)/P < \infty$, then $\dim M/N < \infty$, where $N=\{v \in M \mid SvT \subseteq P\}$.*

Proof. Let $e_1, e_2, \dots, e_m$ and $f_1, f_2, \dots f_n$ be K-bases of S and T, respectively. Define for every $1 \leq i \leq m$, $1 \leq j \leq n$, $\varphi_{ij} : M \to SMT$, by $\varphi_{ij}(x) = e_i x f_j$ and $\psi_{ij} = \eta \circ \varphi_{ij}$, where η is the canonical K-linear map of SMT onto $\dim(SMT+P)/P$. Clearly ψ_{ij} are K – linear. Since $\dim(SMT+P)/P < \infty$, $\dim M/\ker\psi_{ij} < \infty$ for $i=1,2,\dots,m$ and $j=1,2,\dots,n$. Consequently $\dim M/\ker\bigcap_{i,j}\psi_{ij} < \infty$ and $\ker\bigcap_{i,j}\psi_{ij} = \{x \in M \mid e_i x f_j \in P, i=1,2,\dots,m,$ $j=1,2,\dots,n\}$. □

Proposition 14. *Let A be a prime algebra. Assume that $B_0 \subseteq Z(B)$, $r_C(C_0) \neq 0$ and $l_C(C_0) \neq 0$. Then $\dim B_0 < \infty$ or $\dim C_0 < \infty$.*

Proof. Suppose, contrary to our claim, that $\dim B_0 = \infty$ and $\dim C_0 = \infty$. Adjoining, if necessary, an identity element we can, without loss of generality, assume that A is a K-algebra with unity.

Note that, by Remark 1, $\dim(C_0A + C_0)/C_0 < \infty$. Hence there exists subspace $V \subseteq C_0A$ such that $V + C_0 = C_0A + C_0$ and $\dim V < \infty$. Setting $S = K$, $T = V$, $P = C_0$ and $M = C_0$ in Lemma 13, we obtain subspace $N \subseteq C_0$ such that $NV \subseteq C_0$ and $\dim C_0/N < \infty$. Consequently $NC_0A \subseteq C_0$ and $\dim C/N < \infty$. Let us note that since $\dim C_0 = \infty$, $\dim N = \infty$. Further $NC_0Ar_C(C_0) = 0$. As A is prime and $r_C(C_0) \neq 0$, we have $NC_0 = 0$, so $N \subseteq l_{C_0}(C_0) \lhd C$. Since $\dim C/N < \infty$ we can assume that $C_0^2 = 0$.

Fix any $0 \neq c \in C_0$ and consider the left ideal L of C generated by c. Since $C_0c = 0$ and $\dim C/C_0 < \infty$, $\dim L < \infty$. As $C_0^2 = 0$, $l_C(C_0) \neq 0$. By Lemma 12, $\dim(AC_0 + C_0)/C_0 < \infty$. Since $B_0L \subseteq AC_0$ we can apply Lemma 13 for $S_1 = K$, $T_1 = L$, $P_1 = C_0$ and $M_1 = B_0$. Hence there exists a K - subspace $N_1 \subseteq B_0$ such that $N_1L \subseteq C_0$ and $\dim B_0/N_1 < \infty$. Therefore, since $\dim B_0 = \infty$, $\dim N_1 = \infty$. It is clear that $VN_1 \subseteq C_0A$. Since $C_0A \subseteq V + C_0$, $\dim L < \infty$ and $C_0L = 0$, $\dim VN_1L < \infty$. Now let $S_2 = V$, $T_2 = L$, $P_2 = 0$ and $M_2 = N_1$. Again by Lemma 13, there exists $N_2 \subseteq N_1$ such that $VN_2L = 0$ and $\dim N_1/N_2 < \infty$. Since $\dim N_1 = \infty$, $\dim N_2 = \infty$. Let us note that $N_2L \subseteq N_1L \subseteq C_0$, $C_0^2 = 0$ and $C_0A \subseteq V + C_0$. Hence $C_0AN_2L \subseteq VN_2L + C_0N_2L = 0$. But $C_0 \neq 0$ and A is a prime algebra, so $N_2L = 0$. As $N_2 \subseteq N_1 \subseteq B_0 \subseteq Z(B)$ and $L <_l C$, we have $N_2AL \subseteq N_2BL + N_2CL \subseteq BN_2L + N_2L = 0$. Moreover $L \neq 0$ and as $\dim N_2 = \infty$, $N_2 \neq 0$. This contradicts the fact that A is a prime algebra, so $\dim B_0 < \infty$ or $\dim C_0 < \infty$. □

Corollary 15. *Let A be a prime algebra. Suppose that $B_0 \subseteq Z(B)$, C_0 is commutative and $r_C(C_0) \neq 0$. Then $A \in \mathcal{CF}$.*

Proof. Since $r_C(C_0) \neq 0$ and C_0 is commutative, $l_C(C_0) \neq 0$ (if $l_C(C_0) = 0$, C is commutative). By Proposition 14, $\dim B_0 < \infty$ or $\dim C_0 < \infty$. Assume first that $\dim B_0 < \infty$. Hence $\dim B < \infty$. Since $\dim C/C_0 < \infty$ and $A = B + C$, then $\dim A/C_0 < \infty$. From Lemma 4, there exists $I \lhd A$ such that $I \subseteq C_0$ and $\dim A/I < \infty$, so $A \in \mathcal{CF}$. Similar arguments can be applied to the case $\dim C_0 < \infty$. □

Corollary 16. *If an algebra A is a $\mathcal{C}$-counter-example, where B_0 and C_0 satisfy conditions of Definition 8, then $r_{B_0}(B_0) \neq 0$ and $r_{C_0}(C_0) \neq 0$.*

Proof. Let $I = r_{B_0}(B_0)$ and $J = r_{C_0}(C_0)$. Suppose that $I = 0$. Then $B_0 \subseteq Z(B)$. Now if $r_C(C_0) \neq 0$, then one can apply Corollary 15. Hence $A \in \mathcal{CF}$ which contradicts the choice of A. Thus let $r_C(C_0) = 0$. Hence C is a commutative algebra. If $r_B(B_0) \neq 0$ then again by Corollary 15, $A \in \mathcal{CF}$, contradiction. Therefore it has to be $r_B(B_0) = 0$. But then B is commutative. So $B \in \mathcal{C}$ and $C \in \mathcal{C}$. Hence, $A \in \mathcal{NC}$, contrary to the choice of A. Consequently $I \neq 0$. One similarly obtains that $J \neq 0$. □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. Suppose the assertion of the theorem is false. Hence, without loss of generality, we can assume that $A = B + C$ is a $\mathcal{C}$-counter-example. Let B_0 and C_0 are commutative. By Lemma 9, A is a prime algebra. Consider $A_1 = B + B_0A$. It is clear that, since $B \subseteq A_1$, $A_1 = A_1 \cap (B + C) = B + A_1 \cap C$. We shall show that $A_1 \in \mathcal{NCF}$. Suppose that $A_1 \notin \mathcal{NCF}$. Let us note that $\dim A_1/(A_0 + (A_1 \cap C_0)) \leq \dim A/(B_0 + C_0)$, so since $A_1 \notin \mathcal{NCF}$, $\dim A_1/(A_0 + (A_1 \cap C_0)) = \dim A/(B_0 + C_0)$. Hence A_1/T is a $\mathcal{C}$-counter-example, where T is a sum of all ideals of A_1 that lie in the K-subspace $B_0 + (A_1 \cap C_0)$. It is obvious that $A_1/T = (B + T)/T + ((A_1 \cap C) + T)/T$. Moreover $l_{B_0}(B_0) <_r A_1$. Since $[B, B_0] \subseteq l_{B_0}(B_0)$, $(l_{B_0}(B_0))^2 = 0$ and A_1/T is a prime algebra, $[B, B_0] \subseteq T$. Hence $(B_0 + T)/T \subseteq Z((B + T)/T)$. By Corollary 16 and 15, $A_1/T \in \mathcal{CF}$, a contradiction. So indeed $A_1 \in \mathcal{NCF}$. Let $U_1 = B_0 + B_0A$. Since $U_1 \subseteq A_1$, $U_1 \in \mathcal{NCF}$. Similarly we can obtain that $U_2 = C_0 + C_0A \in \mathcal{NCF}$. In particular U_1 and U_2 are PI algebras. Corollary 16 shows $r_{B_0}(B_0) \neq 0$ and $r_{C_0}(C_0) \neq 0$. Of course $r_{B_0}(B_0) = l_{B_0}(B_0)$. Now we can apply Lemma 10. Hence $\dim_K A < \infty$, thus A is not a $\mathcal{C}$-counter-example, a contradiction. □

Acknowledgement. The author is grateful to Prof. E. R. Puczyłowski for suggesting the problem and for constant help during the preparation of this paper.

References

[1] S. A. Amitsur, Nil PI-rings, *Proc. Amer. Math. Soc.* **2** (1951), 538–540.

[2] Yu. Bahturin and A. Giambruno, Identities of sums of commutative subalgebras, *Rend. Mat. Palermo* **43** (1994), 250–258.

[3] Yu. Bahturin and O. H. Kegel, Universal sums of abelian subalgebras, *Comm. Algebra* **23**(8) (1995), 2975–2990.

[4] K. I. Beidar and A. V. Mikhalev, Generalized polynomial identities and rings which are sums of two subrings, *Algebra and Logic* **34** (1995), 3–11.

[5] O. H. Kegel, Zur Nilpotenz gewisser assoziatver Ringe, *Math. Ann.* **149** (1962/63), 258–260.

[6] M. Kępczyk and E. R. Puczyłowski, On radicals of rings which are sums of two subrings, *Arch. Math.* **66** (1996), 8–12.

[7] M. Kępczyk and E. R. Puczyłowski, Rings which are sums of two subrings, *J. Pure Appl. Algebra* **133** (1998), 151–162.

[8] M. Kępczyk and E. R. Puczyłowski, Rings which are sums of two subrings satisfying polynomial identities, *Comm. Algebra* **29** (2001), 2059–2065.

[9] A. A. Klein, The sum of nil one-sided ideals of bounded index of a ring, *Israel J. Math.* **88** (1994), 25–30.

[10] A. Mekey, On subalgebras of finite codimension, *Stud. Sci. Math. Hung.* **27** (1992), 119–123.

[11] A. Petravchuk, On associtive algebras which are sum of two almost commutative subalgebras, *Publ. Math. Debrecen* **53** (1998), 191–206.

[12] E. Posner, Prime rings satisfying a polinomial identity, *Proc. Amer. Math. Soc.* **11** (1960), 180–184.

[13] E. R. Puczyłowski, Some results and questions on radicals of rings which are sums of two subrings, in: Trends in Theory of Rings and Modules, (S. Tariq Rizvi and S. M. A. Zaidi, eds.), 2005, 125–138.

[14] L. H. Rowen, Polynomial identities in ring theory, *Academic Press*, 1980.

MAREK KĘPCZYK
INSTITUTE OF MATHEMATICS AND PHYSICS
TECHNICAL UNIVERSITY OF BIAŁYSTOK
15-333 BIAŁYSTOK, ZWIERZYNIECKA 14
POLAND

E-mail: kep@katmat.pb.bialystok.pl

(Received May 3, 2006; accepted April 5, 2007)