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On spectra of abelian group rings

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Dedicated to Professor Adalbert Bovdi on the occasion of his seventieth birthday

Abstract. In this paper we study the spectrum of integral group rings of finitely generated abelian groups G from the scheme-theoretic viewpoint. We prove that the (closed) singular points of Spec $\mathbb{Z}[G]$, the (closed) intersection points of the irreducible components of Spec $\mathbb{Z}[G]$ and the (closed) points over the prime divisors of $|t(G)|$ coincide. We also determine the formal completion of Spec $\mathbb{Z}[G]$ at a singular point.

1. Introduction

Despite the great interest in the algebraic study of group rings, their units and modules (see [5], [6], [7], [13], [14], [17], [18]), explicit questions from the point of view of algebraic geometry do not seem to be discussed so far. Let G be a finitely generated abelian group and $\mathbb{Z}[G]$ its integral group ring. Then $\mathbb{Z}[G]$ is a noetherian commutative ring and it is natural to study the affine scheme associated to $\mathbb{Z}[G]$, i.e. the spectrum $\operatorname{Spec}\mathbb{Z}[G]$, that is, the set of all prime ideals in $\mathbb{Z}[G]$, considered as a topological space with the Zariski (spectral) topology (also denoted by $\operatorname{Spec}\mathbb{Z}[G]$), endowed with the structural sheaf. It is well-known (see [19, Sec. 2.2]) that $\mathbb{Z}[G]$ is a Hopf algebra representing a diagonalizable group scheme over $\mathbb{Z}$, i.e. an affine group scheme that is a finite product of copies of $\mathbf{G}_m$ and various $\boldsymbol{\mu}_n$. More generally, this holds over any commutative base

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ring k. If k is a field, the topological structure of $\operatorname{Spec}(k[G])$ is simpler than that of $\operatorname{Spec}\mathbb{Z}[G]$, since the decomposition into irreducible components is actually one into connected components. If, moreover, G is finite then each component consists of a single point.

However, greater complications arise when k is a ring such as the ring of integers of an algebraic number field that does not contain the primitive roots of unity of order $|t(G)|$, where $t(G)$ is the torsion part of G. Evidently $\mathbb{Z}$ is the most natural ring to start with. In this case even the topological structure of $\operatorname{Spec}(\mathbb{Z}[G])$ can be nontrivial in the sense that its irreducible components may intersect at several points (or even at higher dimensional subschemes). The intersections reflect the decomposition of certain ideals in some rings of cyclotomic integers, and show the importance of the arithmetic of such rings to the topology of $\operatorname{Spec}(\mathbb{Z}[G])$. When we pass to the level of sheaves, we obtain singularities, and the dimensions of the tangent spaces at these points are directly linked to the rank of the free part of G and to the minimal number of generators of the Sylow p-subgroup of G (p being a prime divisor of $|t(G)|$, lying below a given singularity). It is proven that the singular points, the intersection points and the points over the prime divisors of $|t(G)|$ are one and the same set (see Theorem 3.1).

The simplest example is that of C_p, the group of prime order p. In this case $\operatorname{Spec}\mathbb{Z}[C_p]$, consists of two irreducible components, one of which is isomorphic to $\operatorname{Spec}\mathbb{Z}$ and the other to $\operatorname{Spec}\mathbb{Z}[\zeta_p]$, where ζ_p is a primitive p-th root of unity. The components intersect at a single point, which is the only singular point of $\operatorname{Spec}\mathbb{Z}[C_p]$ (see Example 3.6). That the intersection consists of a single point is a consequence of the fact that there is only one prime in $\mathbb{Z}[\zeta_p]$ lying over p, thereby showing how the arithmetic of $\mathbb{Z}[\zeta_p]$ influences the topological structure (see examples 3.4 and 3.5).

The topological description of $\operatorname{Spec}\mathbb{Z}[G]$ with finite abelian G also follows from M. Atiyah's paper [2], in which, in particular, the topology of the spectrum of the representation ring for finite G was studied, and, more generally, from G. Segal's article [16] in which G is a compact Lie group. It can also be obtained using results by J. E. Roseblade [15] on prime ideals in group rings of polycyclic groups. There is, however, a short elementary argument for this description, which we gave in [8] and is generalized here (see (i) of Theorem 3.1).

In this paper we deal with a finitely generated abelian G. The case of a finite abelian G was previously considered in [8] where we described the decomposition of $\operatorname{Spec}\mathbb{Z}[G]$ into irreducible components, the intersection points and the singularities. The latter were determined by using the Kähler differentials, which also permit to find easily the ramification points of $\operatorname{Spec}\mathbb{Z}[G]$ over $\operatorname{Spec}\mathbb{Z}$.

The present paper is organized as follows. In Section 2 we set up some notation and calculate the modules of Kähler differentials of group rings, recalling previously some basic facts about them. As in the case of a finite G this is used to explain the ramifications of the morphism $\operatorname{Spec}\mathbb{Z}[G] \to \operatorname{Spec}\mathbb{Z}$ (see Proposition 2.10) and, moreover, is essential for the determination of the singularities. Our first result in Section 3 is Theorem 3.1, which generalizes the corresponding result from [8]. The proof permits to determine the dimensions of the Zariski tangent spaces at the singular points (see Corollary 3.3). A more detailed study of singularities is made by the determination of the formal completion of $\operatorname{Spec}\mathbb{Z}[G]$ at a given singular point, which is done in Theorem 3.7. The case of a finite (abelian) p-group G is simpler and is described in Corollary 3.8.

We shall use the following notations. For an ideal I in A we denote by $V(I)$ the set of prime ideals in A containing I. By $\dim A$ we mean the Krull dimension of A. For a commutative ring R and an ideal I of R, we denote by $\widehat{R_I}$ the I-adic completion $\varprojlim R/I^n$ of R. The symbol $\mathbb{F}_p$ stands for the field with p elements. As it is usual in Algebraic Geometry, given a scheme X and a point $x \in X$, the symbol $\mathcal{O}_x = \mathcal{O}_{X,x}$ denotes the local ring of x and $\mathfrak{m}_x$ its maximal ideal. For an abelian group G we shall denote by $t(G)$ its torsion part, G_p its Sylow p-subgroup for some prime p and $G_{p'}$ the p'-part of $t(G)$. Given a homomorphism $A \to B$ of commutative rings, the B-module of Kähler differentials over A is denoted by $\Omega_{B/A}$.

2. Kähler differentials of group rings

In this section we shall make some remarks on Kähler differentials of group rings, the basic facts on differentials can be found, for example, in [1, Chapters 5, 6, 7], [12, Chapter 1]. For readers conveniens we remind the following well-known properties of the module $\Omega_{B/A}$, which will be useful for us. Given ring maps $A \to B \to C$, we have the exact sequence of C-modules:

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0. \tag{1}$$

If now $C = B/\mathfrak{K}$, where $\mathfrak{K}$ is any ideal of B, we have another exact sequence of C-modules:

$$\mathfrak{K}/\mathfrak{K}^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0, \tag{2}$$

the first map in which being $k + \mathfrak{K}^2 \mapsto d_{B/A}(k) \otimes \bar{1}$. Furthermore, the module of differentials is compatible with localisation, i.e., if $S \subset A$ is a multiplicative set being mapped into a multiplicative set $T \subset B$,

$$\Omega_{T^{-1}B/S^{-1}A} = T^{-1}\Omega_{B/A}. \tag{3}$$

The modules of differentials of group rings with finite abelian G were computed in [8], which can be easily extended to the case of a finitely generated abelian G. For we recall first that given a finite abelian G, one has the formula

$$\Omega_{\mathbb{Z}[G]/\mathbb{Z}} = \bigoplus (\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})[C_{p_i^{\alpha_i}}] \otimes_{\mathbb{Z}[C_{p_i^{\alpha_i}}]} \mathbb{Z}[G], \tag{4}$$

in which the direct sum is taken over the cyclic p-groups in the decomposition of G (so that the p_i's are not necessarily distinct).

Remark 2.1. The integral group ring of the r-generated torsion-free abelian group G is isomorphic to $\mathbb{Z}[T_1, T_1^{-1}, \dots, T_r, T_r^{-1}]$. Since the latter ring is the localisation of $\mathbb{Z}[T_1, \dots, T_r]$ at the element $T_1 \cdot T_2 \cdot \dots \cdot T_r$, the proposition [1, Proposition 1.20] shows that $\Omega_{\mathbb{Z}[G]/\mathbb{Z}} = (\mathbb{Z}[G])^{\oplus r}$ is free of rank r over $\mathbb{Z}[G]$. We also observe that $\operatorname{Spec} \mathbb{Z}[G]$ is regular, being the principal open subscheme $D(T_1 \cdot \ldots \cdot T_r)$ of $\operatorname{Spec} \mathbb{Z}[T_1, \dots, T_r]$, which is regular by [1, Chapter 7, Theorem 4.5].

Thus in view of [1, Proposition 1.12], (4) and Remark 2.1, we have the following:

Proposition 2.2. *Let G be a finitely generated abelian group whose free part has rank r. Then*

$$\Omega_{\mathbb{Z}[G]/\mathbb{Z}} = \bigoplus (\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})[C_{p_i^{\alpha_i}}] \otimes_{\mathbb{Z}[C_{p_i^{\alpha_i}}]} \mathbb{Z}[G] \bigoplus (\mathbb{Z}[G])^{\oplus r}.$$

Now, by Proposition 2.2 and [1, Proposition 1.18] we obtain at once:

Corollary 2.3. *Let G be as in Proposition 2.2 and suppose that k is a field of characteristic p. Then*

$$\Omega_{k[G]/k} = (k[G])^{\oplus(l_p + r)}, \tag{5}$$

where l_p is the number of p-cyclic factors of G.

The question of regularity for some schemes over fields can be decided by looking at the Kähler differentials. The next result is a generalization of Theorem 8.8 of [11, Chapter 2].

Theorem 2.4. *Let $(A, \mathfrak{m})$ be a local ring contaning a perfect field k such that A is a localisation of a finitely generated k-algebra. Then A is regular if and only if the module $\Omega_{A/k}$ is free over A of rank $\dim(A) + \operatorname{tr.d.} \kappa/k$, where $\kappa = A/\mathfrak{m}$ and $\operatorname{tr.d.} \kappa/k$ stands for the transcendence degree of κ over k.*

Proof. See [11, Chapter 2, Example 8.1]. □

Now we are ready to prove the following:

Proposition 2.5. *Let G be a finitely generated abelian group. Then the tangent space at all closed points of* $\operatorname{Spec}(\mathbb{F}_p[G])$ *has dimension $l_p + r$. In particular,* $\operatorname{Spec}(\mathbb{F}_p[G])$ *is singular if and only if $G_p \neq 1$.*

Proof. Suppose first that $G_p = 1$. Then $\mathbb{F}_p[t(G)]$ is a direct product of fields, and consequently $\mathbb{F}_p[G]$ is a direct product of group algebras of a free abelian group over field extensions of $\mathbb{F}_p$. Clearly any of these factors is a regular ring, since its spectrum is a principal open subscheme of an affine space over a field (see Remark 2.1). It follows that $\mathbb{F}_p[G]$ is regular.

Suppose now that $G_p \neq 1$ and let $\mathfrak{p}$ be any maximal ideal of $\mathbb{F}_p[G]$. The local ring $(\mathbb{F}_p[G])_{\mathfrak{p}}$ evidently satisfies the conditions of Theorem 2.4. By the version of Nullstellensatz in [3, Example 18, p. 70], the residue field of $(\mathbb{F}_p[G])_{\mathfrak{p}}$ is a finite extension of $\mathbb{F}_p$. On the other hand, by ([1, Proposition 1.20]) and Corollary 2.3 the module $\Omega_{(\mathbb{F}_p[G])_{\mathfrak{p}}/\mathbb{F}_p}$ is free of rank $l_p + r$, while $\dim(\mathbb{F}_p[G])_{\mathfrak{p}} \leq \dim(\mathbb{F}_p[G])$. The latter equals r by [3, Example 7, p. 126]. Consequently, $\mathfrak{p}$ is a singular point of $\operatorname{Spec}\mathbb{F}_p[G]$. □

We have proved in fact the next:

Corollary 2.6. *The tangent space of* $\operatorname{Spec}\mathbb{F}_p[G]$ *has dimension $l_p + r$ at all (closed) points of* $\operatorname{Spec}\mathbb{F}_p[G]$.

Proof. This is an immediate consequence of Corollary 2.3 and [11, Chapter 2, Example 8.1(a)]. □

Corollary 2.7. *If G is a finitely generated abelian group with $G_p \neq 1$, then each (closed) point of* $\operatorname{Spec}\mathbb{F}_p[G]$ *is singular.*

Remark 2.8. Let G be a finitely generated abelian group whose free part has rank r. Evidently $\mathbb{F}_p[G_{p'}] \cong \prod K_i$, a finite product of finite extensions of $\mathbb{F}_p$. Then the decomposition of $\operatorname{Spec}\mathbb{F}_p[G]$ into irreducible components is the disjoint sum of spectra of the localizations of the polynomial rings $(K_i[G_p])[T_1, \dots T_r]$ at the element $f = T_1 \cdot T_2 \cdot \dots T_r$:

$$\operatorname{Spec}\mathbb{F}_p[G] \cong \coprod \operatorname{Spec}((K_i[G_p])[T_1, \dots T_r])_{(f)}.$$

In fact, the radical of $K_i[G_p]$ (which is the augmentation ideal $\Delta(G_p)$ of $K_i[G_p]$) generates that of $(K_i[G_p])[T_1, \dots T_r]$ and since the latter is prime, it follows that $\operatorname{Spec}(K_i[G_p])[T_1, \dots T_r]$ is irreducible. Consequently, so is its principal open subscheme $\operatorname{Spec}((K_i[G_p])[T_1, \dots T_r])_{(f)}$.

We examine next the ramification question. The appropriate base ring in this context is $\mathbb{Z}[F]$, where F is the free part of G, i.e. $G = t(G) \times F$. We observe first the next

Remark 2.9. Let G be a finitely generated abelian group. The fiber of

$$\mathrm{Spec}(\mathbb{Z}[G]) \to \mathrm{Spec}(\mathbb{Z}[F])$$

over a prime $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z}[F])$ is $\mathrm{Spec}(\kappa(\mathfrak{p})[t(G)])$, where $\kappa(\mathfrak{p})$ is the residue field of $\mathfrak{p}$, i.e. the field of fractions of $\mathbb{Z}[F]/\mathfrak{p}$. In particular, over the generic point it is $\mathrm{Spec}(\mathbb{Q}(T_1, \dots T_r)[t(G)])$, where $\mathbb{Q}(T_1, \dots, T_r)$ stands for the field of rational functions over $\mathbb{Q}$ in r variables. Indeed, the fibre $X_\mathfrak{p}$ of $X = \mathrm{Spec}(\mathbb{Z}[G])$ over a $\mathfrak{p}$ is, by definition, $X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\kappa(\mathfrak{p}))$. Thus we have that $X_\mathfrak{p} = \mathrm{Spec}(\mathbb{Z}[G] \otimes_{\mathbb{Z}F} \kappa(\mathfrak{p})) \cong \mathrm{Spec}(\mathbb{Z}[t(G)] \otimes_{\mathbb{Z}} \kappa(\mathfrak{p})) \cong \mathrm{Spec}(\kappa(\mathfrak{p})[t(G)])$. The case of the generic point follows immediately.

Proposition 2.10. *Let G be a finitely generated abelian group, $G{=}t(G){\times}F$.*

(i) *The morphism $X_\mathfrak{p} \to \mathrm{Spec}(\kappa(\mathfrak{p}))$, for a prime $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z}[F])$ lying over $p\mathbb{Z}$, is ramified if and only if p divides the order of $t(G)$.*

(ii) *The ramification points of $\mathrm{Spec}(\mathbb{Z}[G]) \to \mathrm{Spec}(\mathbb{Z}[F])$ are precisely those primes lying over the prime divisors of $|t(G)|$, in particular, $\mathrm{Spec}(\mathbb{Z}[G]) \to \mathrm{Spec}(\mathbb{Z}[F])$ is not étale.*

Proof. (i) We know that a morphism of noetherian schemes $X \to S$, which is locally of finite type, is unramified at $x \in X$ if and only if the stalk $(\Omega_{X/S})_x$ is 0 (see [1, Chapter 6, Proposition 3.3]). Thus, taking $X = \mathrm{Spec}(\kappa(\mathfrak{p})[t(G)])$ and $S = \mathrm{Spec}(\kappa(\mathfrak{p}))$, (ii) follows immediately from Corollary 2.3.

By [12, Chapter 1, Proposition 3.2] a morphism of noetherian schemes $\phi : X \to S$, which is locally of finite type, is unramified at $x \in X$ if and only if the fibre over $\phi(x)$ is unramified over the residue field of $\phi(x)$. Thus (i) implies (ii) (taking $X = \mathrm{Spec}(\mathbb{Z}[G])$ and $S = \mathrm{Spec}(\mathbb{Z}[F])$). □

Remark 2.11. It obviously follows from the proof of (i) that if $\mathfrak{p}$ lies over a prime divisor of $|t(G)|$, then $X_\mathfrak{p} \to \mathrm{Spec}(\kappa(\mathfrak{p}))$ is ramified at all points.

Remark 2.12. The dimension of $\mathbb{Z}[G]$ can be obtained using [15], however we give below a short argument, which does not use any deep result on primes of group rings.

Since $\mathbb{Z}[G]$ is flat over $\mathbb{Z}$, if G is a finitely generated abelian group and r the rank of its free part, then the fibre of $\mathrm{Spec}(\mathbb{Z}[G]) \to \mathrm{Spec}(\mathbb{Z})$ over $p\mathbb{Z}$ is

$\operatorname{Spec}(\mathbb{F}_p[G])$, which is a principal open subscheme in the spectrum of the polynomial ring $(\mathbb{F}_p[t(G)])[T_1, \ldots, T_r]$ with coefficients in the group ring $\mathbb{F}_p[t(G)]$. By [3, Example 7, p. 126], the dimension of $(\mathbb{F}_p[t(G)])[T_1, \ldots, T_r]$ is r, and consequently $\dim \mathbb{F}_p[G] = r$. Using [12, Chapter 1, Remark 2.6] we conclude that $\dim \mathbb{Z}[G] = r + 1$.

3. Geometric properties of abelian integral group rings

Let G be an arbitrary finitely generated abelian group and write $G = t(G) \times F$, where F is the free part of G, i.e. a finitely generated free abelian group, whose rank we denote by r. Given a finite subgroup H of G, denote by h its order. For a prime $\mathfrak{p}$ of $\mathbb{Z}[G]$ write $\mathfrak{m}_{\mathfrak{p}}$ for the maximal ideal of $(\mathbb{Z}[G])_{\mathfrak{p}}$ and $\kappa(\mathfrak{p}) = (\mathbb{Z}[G])_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$, the residue field of $\mathfrak{p}$.

Theorem 3.1. *Using the above notation we have:*

(i) *There is a one-to-one correspondence between the finite cyclic subgroups H of G and the minimal prime ideals $\mathfrak{a}_H$ (generic points of the components) of $\mathbb{Z}[G]$ such that*

$$\operatorname{Spec} \mathbb{Z}[G] = \bigcup_H V(\mathfrak{a}_H), \tag{6}$$

is the decomposition of $\operatorname{Spec} \mathbb{Z}[G]$ *into irreducible components. Moreover,*

$$V(\mathfrak{a}_H) \cong \operatorname{Spec}(\mathbb{Z}[\zeta_h])[F], \tag{7}$$

where ζ_h is a primitive h-th root of unity, and

$$\mathbb{Q}[t(G)] \cong \prod_H \mathbb{Q}(\zeta_h) \tag{8}$$

is the Wedderburn decomposition of $\mathbb{Q}[t(G)]$. For a fixed h dividing the exponent of G, the number of irreducible components satisfying (7) *is equal to the number of cyclic subgroups of G of order h. If $\mathfrak{p}$ is an intersection point, i.e., belongs to at least two components, then $\mathfrak{p}$ lies over a prime divisor of $|t(G)|$.*

(i) *If $\mathfrak{q} \in \operatorname{Spec} \mathbb{Z}[G]$ belongs to only one component of $\operatorname{Spec} \mathbb{Z}[G]$, say $V \cong \operatorname{Spec}(\mathbb{Z}[\zeta_h])[F]$, then $(\mathbb{Z}[G])_{\mathfrak{q}} \cong ((\mathbb{Z}[\zeta_h])[F])_{\tilde{\mathfrak{q}}}$, where $\tilde{\mathfrak{q}} \in \operatorname{Spec}(\mathbb{Z}[\zeta_h])[F]$ is the prime corresponding to $\mathfrak{q}$. In particular, $\mathfrak{q}$ is regular.*

(iii) *The (closed) singular points of* $\operatorname{Spec}\mathbb{Z}[G]$ *are exactly those maximal ideals* $\mathfrak{p} \in \operatorname{Spec}\mathbb{Z}[G]$ *that lie over the prime divisors of* $|t(G)|$. *In particular, each (closed) singular point is an intersection point, i.e. it belongs to at least two components.*

Proof. (i) By [14, p. 148], the Wedderburn decomposition of $\mathbb{Q}[t(G)]$ is given by (8), and corresponding to it we have the standard epimorphisms $\psi_0 : t(G) \to \langle \zeta_h \rangle$, $\psi_1 : \mathbb{Z}[G] \to (\mathbb{Z}[\zeta_h])[F]$ and $\psi_2 : \mathbb{Q}[G] \to \mathbb{Q}(\zeta_h)[F]$. Defining $\mathfrak{a}_H = \operatorname{Ker}(\psi_1)$, we have the exact sequence

$$0 \to \mathfrak{a}_H \to \mathbb{Z}[G] \to (\mathbb{Z}[\zeta_h])[F] \to 0.$$

On tensoring it with $\mathbb{Q}$ (over $\mathbb{Z}$), we obtain the exact sequence

$$0 \to \mathfrak{a}_H \otimes \mathbb{Q} \to \mathbb{Q}[G] \to \mathbb{Q}(\zeta_h)[F] \to 0,$$

because $\mathbb{Q}$ is flat over $\mathbb{Z}$. Since

$$\mathbb{Q}[G] \cong \prod_H \mathbb{Q}(\zeta_h)[F],$$

we have that

$$\operatorname{Spec}\mathbb{Q}[G] = \coprod_H \operatorname{Spec}\mathbb{Q}(\zeta_h)[F]$$

is the decomposition into irreducible (and connected) components. Thus we have the bijection $\mathfrak{a}_H \longleftrightarrow \mathfrak{a}_H \otimes \mathbb{Q}$ between the $\mathfrak{a}_H$'s and the minimal primes of $\operatorname{Spec}\mathbb{Q}[G]$ (the generic points of its components). Extending the coefficients from $\mathbb{Z}$ to $\mathbb{Q}$ is a localisation, which gives a one-to-one correspondence between the primes of $\mathbb{Z}[G]$ lying over 0 and the primes of $\mathbb{Q}[G]$. This evidently preserves the heights of the primes and their inclusions, and consequently, the $\mathfrak{a}_H$'s are the minimal primes of $\mathbb{Z}[G]$.

Let $\mathfrak{q} \in \operatorname{Spec}\mathbb{Z}[G]$ be such that $\mathfrak{q} \cap \mathbb{Z} = (q)$ and q does not divide $|t(G)|$. Let further f be the product of all prime divisors of $|t(G)|$. All primitive idempotents of $\mathbb{Q}[t(G)]$ lie in the group ring $\mathbb{Z}[\frac{1}{f}][t(G)]$ and so we have

$$\operatorname{Spec}\mathbb{Z}[1/f][G] = \coprod_H \operatorname{Spec}(\mathbb{Z}[1/f, \zeta_h])[F].$$

Since $(\mathbb{Z}[1/f, \zeta_h])[F]$ has no non-trivial idempotents, the above is the decomposition into connected components. We see that no intersection point of $\operatorname{Spec}\mathbb{Z}[G]$ belongs to the principal open subset $D(f) = \operatorname{Spec}(\mathbb{Z}[1/f])[G]$ of $\operatorname{Spec}\mathbb{Z}[G]$. However, $\mathfrak{q} \in D(f)$ as q does not divide f. This completes the proof of (i).

(ii) Evidently, since $\mathfrak{q}$ is not an intersection point, removing from $\operatorname{Spec} A$ all components but V we obtain an open neighbourhood in $\operatorname{Spec}(\mathbb{Z}[\zeta_h])[F]$. The latter scheme is regular, being a principal open subscheme in $\operatorname{Spec}(\mathbb{Z}[\zeta_h])[T_1, \dots, T_r]$ (see Remark 2.1). Because the local ring of a point x is determined in any open neighbourhood of x, we have that $\mathfrak{q}$ is regular and $(\mathbb{Z}[G])_{\mathfrak{q}} \cong ((\mathbb{Z}[\zeta_h])[F])_{\tilde{\mathfrak{q}}}$.

(iii) Suppose that $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[G]$ is maximal and lies over the prime divisor p of $|t(G)|$. Write $\mathfrak{m} = \mathfrak{m}_{\mathfrak{p}}$ and $\kappa = \kappa(\mathfrak{p})$. We shall derive the singularity of $\mathfrak{p}$ from the exact sequence (2), which in our case becomes:

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{(\mathbb{Z}[G])_{\mathfrak{p}}/\mathbb{Z}} \otimes_{(\mathbb{Z}[G])_{\mathfrak{p}}} \kappa \to \Omega_{\kappa/\mathbb{Z}} \to 0. \tag{9}$$

Because $\mathfrak{p}$ lies over p and δ annihilates the image of $\mathbb{Z} \cap \mathfrak{p}$ in $\mathfrak{m}/\mathfrak{m}^2$, we see that $p \in \operatorname{Ker}(\delta)$. It is easily seen that $p \notin \mathfrak{p}^2$. For the image of $\mathfrak{p}$ under the augmentation homomorphism $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ is a prime ideal in $\mathbb{Z}$, which must be $p\mathbb{Z}$, as $p \in \mathfrak{p}$. Hence $\varepsilon(\mathfrak{p}^2) = p^2\mathbb{Z}$ and thus p cannot be in $\mathfrak{p}^2$. Since $\mathfrak{p}$ is maximal, one has a canonical isomorphism $\mathfrak{p}/\mathfrak{p}^2 \cong \mathfrak{m}/\mathfrak{m}^2$ given by $x + \mathfrak{p}^2 \mapsto x/1 + \mathfrak{m}^2, x \in \mathfrak{p}$ (see [4, Chapter II, §3.3, Proposition 9]). Consequently, $p/1 \notin \mathfrak{m}^2$, and thus $\operatorname{Ker}(\delta) \neq 0$.

Applying (1) to $\mathbb{Z} \to \mathbb{F}_p \to \kappa$, we have the exact sequence

$$\Omega_{\mathbb{F}_p/\mathbb{Z}} \otimes_{\mathbb{F}_p} \kappa \to \Omega_{\kappa/\mathbb{Z}} \to \Omega_{\kappa/\mathbb{F}_p} \to 0. \tag{10}$$

Since $\mathfrak{p}$ is maximal, $\kappa \cong \mathbb{Z}[G]/\mathfrak{p}$, hence κ is a finitely generated algebra over $\mathbb{F}_p$. An appropriate version of the Nullstellensatz (see [3, Example 18, p. 70]) implies that $\kappa \supseteq \mathbb{F}_p$ is a finite algebraic extension. Since $\mathbb{F}_p$ is perfect, it is also separable. It follows that the module $\Omega_{\kappa/\mathbb{F}_p}$ is 0, and so is $\Omega_{\mathbb{F}_p/\mathbb{Z}}$, in view of the epimorphism $\mathbb{Z} \to \mathbb{F}_p$. Hence $\Omega_{\kappa/\mathbb{Z}} = 0$ and (9) becomes

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{(\mathbb{Z}[G])_{\mathfrak{p}}/\mathbb{Z}} \otimes_{(\mathbb{Z}[G])_{\mathfrak{p}}} \kappa \to 0. \tag{11}$$

By Proposition 2.2,

$$\Omega_{\mathbb{Z}[G]/\mathbb{Z}} = \bigoplus_{\substack{q \\ \sigma \in \Sigma_q}} (\mathbb{Z}/o(\sigma)\mathbb{Z}) \langle \sigma \rangle \otimes_{\mathbb{Z}\langle \sigma \rangle} \mathbb{Z}[G] \bigoplus (\mathbb{Z}[G])^{\oplus r}, \tag{12}$$

where q runs over the prime divisors of $|t(G)|$, and Σ_q is a minimal set of generators of the Sylow q-subgroup of G. Taking $S = \{1\}$ and $T = \mathbb{Z}[G] \setminus \mathfrak{p}$ in (3), one has

$$\Omega_{(\mathbb{Z}[G])_{\mathfrak{p}}/\mathbb{Z}} = (\Omega_{\mathbb{Z}[G]/\mathbb{Z}})_{\mathfrak{p}} = \Omega_{\mathbb{Z}[G]/\mathbb{Z}} \otimes_{\mathbb{Z}[G]} (\mathbb{Z}[G])_{\mathfrak{p}}.$$

It follows from (12) that

$$\Omega_{(\mathbb{Z}[G])_{\mathfrak{p}}/\mathbb{Z}} = \bigoplus_{\sigma\in\Sigma_p} (\mathbb{Z}/o(\sigma)\mathbb{Z})\ \langle\sigma\rangle \otimes_{\mathbb{Z}\langle\sigma\rangle} (\mathbb{Z}[G])_{\mathfrak{p}} \bigoplus (\mathbb{Z}[G])_{\mathfrak{p}}^{\oplus r},$$

where $(\mathbb{Z}[G])_{\mathfrak{p}}$ is a $\mathbb{Z}\langle\sigma\rangle$-module via $\mathbb{Z}\langle\sigma\rangle \to \mathbb{Z}[G] \to (\mathbb{Z}[G])_{\mathfrak{p}}$. Consequently,

$$\begin{aligned}\Omega_{(\mathbb{Z}[G])_{\mathfrak{p}}/\mathbb{Z}} \otimes_{(\mathbb{Z}[G])_{\mathfrak{p}}} \kappa &= \left(\bigoplus_{\sigma\in\Sigma_p} (\mathbb{Z}/o(\sigma)\mathbb{Z})\langle\sigma\rangle \otimes_{\mathbb{Z}\langle\sigma\rangle} (\mathbb{Z}[G])_{\mathfrak{p}} \bigoplus (\mathbb{Z}[G])_{\mathfrak{p}}^{\oplus r}\right) \otimes_{(\mathbb{Z}[G])_{\mathfrak{p}}} \kappa \\ &= \bigoplus_{\sigma\in\Sigma_p} (\mathbb{Z}/o(\sigma)\mathbb{Z})\ \langle\sigma\rangle \otimes_{\mathbb{Z}\langle\sigma\rangle} \kappa \bigoplus \kappa^{\oplus r},\end{aligned}$$

where κ is a $\mathbb{Z}\langle\sigma\rangle$-module by means of $\mathbb{Z}\langle\sigma\rangle \to \mathbb{Z}[G] \to (\mathbb{Z}[G])_{\mathfrak{p}} \to (\mathbb{Z}[G])_{\mathfrak{p}}/\mathfrak{m} = \kappa$. Since σ is a p-element, $o(\sigma)$ is zero in κ and thus $o(\sigma)\mathbb{Z}\cdot\kappa = 0$ and

$$(\mathbb{Z}/o(\sigma)\mathbb{Z})\ \langle\sigma\rangle \otimes_{\mathbb{Z}\langle\sigma\rangle} \kappa \cong \mathbb{Z}\ \langle\sigma\rangle/(o(\sigma)\mathbb{Z}\langle\sigma\rangle) \otimes_{\mathbb{Z}\langle\sigma\rangle} \kappa \cong \kappa/(o(\sigma)\mathbb{Z}\langle\sigma\rangle)\cdot\kappa = \kappa,$$

using the canonical isomorphism $A/I \otimes_A M \cong M/I\cdot M$ with $A = \mathbb{Z}\langle\sigma\rangle$, $I = o(\sigma)\mathbb{Z}\langle\sigma\rangle$ and $M = \kappa$. Hence

$$\Omega_{(\mathbb{Z}[G])_{\mathfrak{p}}/\mathbb{Z}} \otimes_{(\mathbb{Z}[G])_{\mathfrak{p}}} \kappa \cong \kappa^{\oplus(l_p+r)}, \qquad (13)$$

where l_p is the number of elements in Σ_p. Therefore, the exact sequence (11) becomes

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \kappa^{\oplus(l_p+r)} \to 0.$$

As $\mathrm{Ker}(\delta) \neq 0$, we conclude that

$$\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) \geq l_p + r + 1 = l_p + \dim \mathbb{Z}[G] > \dim \mathbb{Z}[G] \geq \dim\ (\mathbb{Z}[G])_{\mathfrak{p}}. \qquad (14)$$

This means, in particular, that $\mathfrak{p}$ is singular.

Obviously by (ii) $\mathfrak{p}$ must be an intersection point.

□

Remark 3.2. Item (i) of the above theorem establishes a one-to-one correspondence between the irreducible components of $\operatorname{Spec}\mathbb{Z}[G]$ and the connected components (which are also irreducible) of $\operatorname{Spec}\mathbb{Q}[G]$. We observe here that the map $\mathfrak{p} \mapsto \mathbb{Q}\otimes\mathfrak{p}$, with $\mathfrak{p}\cap\mathbb{Z} = 0$, gives a homeomorphism between the fiber of $\operatorname{Spec}\mathbb{Z}[G]$ over 0 and $\operatorname{Spec}\mathbb{Q}[G]$.

Given a morphism of schemes $f : X \to Y$, the Zariski tangent space of X/Y at $x \in X$ is the vector $\kappa(x)$-space

$$T_{X/Y}(x) := \mathrm{Hom}_{\kappa(x)}(\Omega_{X/Y}(x), \kappa(x)),$$

where $\Omega_{X/Y}(x) := \Omega_{X/Y} \otimes_{\mathcal{O}_X} \kappa(x)$.

Corollary 3.3. *With the above notation, the Zariski tangent space of* $\mathrm{Spec}(\mathbb{Z}[G]) \to \mathrm{Spec}(\mathbb{Z})$ *at* $\mathfrak{p}$ *has dimension* $l_p + r$.

PROOF. This follows directly from (13), as $\Omega_{\mathbb{Z}[G]/\mathbb{Z}} \otimes_{\mathbb{Z}[G]} \kappa \cong (\Omega_{\mathbb{Z}[G]/\mathbb{Z}} \otimes_{\mathbb{Z}[G]} (\mathbb{Z}[G])_{\mathfrak{p}}) \otimes_{(\mathbb{Z}[G])_{\mathfrak{p}}} \kappa \cong \Omega_{(\mathbb{Z}[G])_{\mathfrak{p}}/\mathbb{Z}} \otimes_{(\mathbb{Z}[G])_{\mathfrak{p}}} \kappa$, by ([1, Proposition 1.20]). $\square$

Using the well-known formula

$$V(I) \cap V(J) \cong \mathrm{Spec}(A/I \otimes_A A/J),$$

where I and J are some ideals in a ring A, one can eventually determine the intersection of irreducible components of $\mathrm{Spec}(\mathbb{Z}[G])$. The arithmetic of the cyclotomic integers influences the intersections as illustrated in the examples below.

Example 3.4. Let G be a cyclic group of order 12, generated by σ. The irreducible components $V_1 := \mathrm{Spec}(\mathbb{Z}[i])$ and $V_2 := \mathrm{Spec}(\mathbb{Z}[\zeta_{12}])$ intersect at a single point. Indeed, $\Phi_{12}(x) = x^4 - x^2 + 1$ and by the formula for the intersection given above, we have $V_1 \cap V_2 \cong \mathrm{Spec}(\mathbb{Z}[i]/(\Phi_{12}(i))) = \mathrm{Spec}(\mathbb{Z}[i]/(3)) \cong \mathrm{Spec}(\mathbb{F}_9)$, since 3 remains prime in $\mathbb{Z}[i]$.

Example 3.5. For an example of an intersection with two points, take for instance the scheme $X = \mathrm{Spec}(\mathbb{Z}[G])$, where $G = \langle\sigma\rangle$ is the cyclic group of order 20. The components $V_1 := V(\Phi_{20}(\sigma)) = \mathrm{Spec}(\mathbb{Z}[\zeta_{20}])$ and $V_2 := V(1+\sigma^2) = \mathrm{Spec}(\mathbb{Z}[i])$ intersect at two points. In fact, $\Phi_{20}(x) = x^8 - x^6 + x^4 - x^2 + 1$ and $V_1 \cap V_2 \cong \mathrm{Spec}(\mathbb{Z}[i]/(\Phi_{20}(i))) \cong \mathrm{Spec}(\mathbb{Z}[i]/(5))$, and since $(5) = (2+i)(2-i)$, we have $V_1 \cap V_2 = \mathrm{Spec}(\mathbb{F}_5) \coprod \mathrm{Spec}(\mathbb{F}_5)$.

The case of a finite abelian p-group admits a simpler description, and moreover can be obtained by a direct inspection, as given in the next example.

Example 3.6. Let G be a finite abelian p-group, $G = \langle \sigma_1, \ldots, \sigma_m : \sigma_i^{p^{n_i}} = 1, i = 1, \ldots, m\rangle$. Then any two irreducible components of $\mathrm{Spec}\,\mathbb{Z}[G]$ intersect at the single point $\mathfrak{p} = (1-\sigma_1, \ldots, 1-\sigma_m, p)$; $\mathfrak{p}$ is the only singular point of $\mathrm{Spec}\,\mathbb{Z}[G]$, at which the tangent space $(\mathfrak{m}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2)^\vee$ has dimension $m+1$.

Indeed, $\mathfrak{p} = (1-\sigma_1, \ldots, 1-\sigma_m, p)$ is a prime of $\mathbb{Z}[G]$ over $p\mathbb{Z}$, with residue field $\kappa(\mathfrak{p}) = \mathbb{F}_p$. By Proposition 2.10 the fibre of $\mathrm{Spec}\,\mathbb{Z}[G]$ over $p\mathbb{Z}$ is $\mathrm{Spec}\,\mathbb{F}_p[G]$, which is a local ring of dimension 0, and thus $\operatorname{Spec}\mathbb{Z}[G]$ has exactly one prime over $p\mathbb{Z}$, which must be $\mathfrak{p}$.

Using the notation from the proof of item (i) of Theorem 3.1, observe that each $\mathfrak{a}_H$ is of the form $(\Delta(\operatorname{Ker}\psi_0), \Phi_h(\sigma))$ for some $\sigma \in G$, where $\Phi_h(T)$ is the h-th cyclotomic polynomial. Since h is a power of p the augmentation of $\Phi_h(\sigma)$ is p. Evidently,

$$V(\mathfrak{a}_{H_1}) \cap V(\mathfrak{a}_{H_2}) \cong \operatorname{Spec}\mathbb{Z}[G]/(\mathfrak{a}_{H_1} + \mathfrak{a}_{H_2}).$$

We have that the augmentation of any element from $\mathfrak{a}_{H_1} + \mathfrak{a}_{H_2}$ belongs to $p\mathbb{Z}$ and consequently $\mathfrak{a}_{H_1} + \mathfrak{a}_{H_2} \neq \mathbb{Z}[G]$. Thus any two irreducible components of $\operatorname{Spec}\mathbb{Z}[G]$ have a non-empty intersection, which by the above consists of the single point $\mathfrak{p}$.

Evidently, the images of $1-\sigma_1, \dots, 1-\sigma_m, p$ are generators of the $\mathbb{F}_p$-space $\mathfrak{p}/\mathfrak{p}^2$ and thus $\dim_{\mathbb{F}_p}(\mathfrak{p}/\mathfrak{p}^2) \leq m+1$. But by (14) $\dim_{\mathbb{F}_p}(\mathfrak{p}/\mathfrak{p}^2) \geq m+1$ and hence $\dim_{\mathbb{F}_p}(\mathfrak{m}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2)^\vee = \dim_{\mathbb{F}_p}(\mathfrak{p}/\mathfrak{p}^2) = m+1$, as claimed.

More complete information about a (closed) singular point x in a noetherian scheme X is contained in the formal completion $\hat{X}$ of X at $\{x\}$. The latter consists of a one point space $\{x\}$ with the ring $\varprojlim \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ as its structural sheaf (see [11, p. 195]). We determine next $\varprojlim (\mathbb{Z}[G])_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^n$, with G finitely generated abelian and $\mathfrak{p}$ lying over a prime divisor p of the order of $t(G)$.

Write $t(G) = G_p \times G_{p'}$ (with p dividing $|t(G)|$). The coefficients of the primitive idempotents of $\mathbb{Q}[G_{p'}]$ lie in $\mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at $p\mathbb{Z}$, so that the Wedderburn decomposition $\mathbb{Q}[G_{p'}] \cong \prod_H \mathbb{Q}(\zeta_h)$ gives rise to the decomposition $\mathbb{Z}_{(p)}[G_{p'}] \cong \prod_H \mathbb{Z}_{(p)}[\zeta_h]$. Then

$$\mathbb{Z}_{(p)}[G] \cong \prod_H \mathbb{Z}_{(p)}[\zeta_h][G_p \times F],$$

and

$$\operatorname{Spec}\mathbb{Z}_{(p)}[G] \cong \coprod_H \operatorname{Spec}(\mathbb{Z}_{(p)}[\zeta_h][G_p \times F]), \tag{15}$$

with H running over the cyclic subgroups of $G_{p'}$. We have that (15) is the decomposition of $\operatorname{Spec}\mathbb{Z}_{(p)}[G]$ into its connected components. For if $e \in \mathbb{Z}_{(p)}[\zeta_h][G_p \times F]$ is an idempotent, then by A. E. Zalesski's Theorem [20], the trace $\operatorname{tr} e$ of e must be in $\mathbb{Q}$, so one can write $\operatorname{tr} e = s/t$, $s, t \in \mathbb{Z}$, $\gcd(s,t) = 1$. If $e \neq 0, 1$ then by I. Kaplansky's Theorem [13, Chapter 2, Theorem 1.8], $0 < tr\ e < 1$, and hence $t \neq 1$. By a result of G. Cliff and S. K. Sehgal (see [9] or [17, Chapter I., Theorem 2.15]), for each prime divisor q of t, there exists a q-element in $G_p \times F$, hence $q = p$ (see also Problem 2 in [18, Chapter 7]). Since p is not invertible in

$\mathbb{Z}_{(p)}[\zeta_h]$, it follows that the ring $\mathbb{Z}_{(p)}[\zeta_h][G_p \times F]$ has no non-trivial idempotents. Hence it is indecomposable, as claimed.

Let $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[G]$ be a maximal prime lying over p and $\mathfrak{p}_{(p)}$ the (maximal) prime in $\operatorname{Spec} \mathbb{Z}_{(p)}[G]$ corresponding to it, i.e. $\mathfrak{p}_{(p)} \cap \mathbb{Z}[G] = \mathfrak{p}$. Then $\mathfrak{p}_{(p)}$ belongs to exactly one of the components in (15), i.e., we have that $\mathfrak{p}_{(p)} \in \operatorname{Spec}(\mathbb{Z}_{(p)}[\zeta][G_p \times F])$ for some $\zeta = \zeta_h$. We have that $\operatorname{Spec}(\mathbb{Z}_{(p)}[\zeta_h][G_p \times F])$ is an open subscheme of the affine scheme of the group ring $R[G_p]$ of G_p over the polynomial ring $R = \mathbb{Z}_{(p)}[\zeta][T_1, \dots, T_r]$ of r variables with coefficients in $\mathbb{Z}_{(p)}[\zeta]$. Indeed, $\mathbb{Z}_{(p)}[\zeta_h][G_p \times F]$ is the localization of $R[G_p]$ at the element $T_1 \cdot T_2 \cdot \ldots \cdot T_r$, and thus $\operatorname{Spec}(\mathbb{Z}_{(p)}[\zeta][G_p \times F]) \cong D(T_1 \cdot T_2 \cdot \ldots \cdot T_r)$, the principal open subscheme of $\operatorname{Spec} R[G_p]$ determined by the element $T_1 \cdot T_2 \cdot \ldots \cdot T_r$. The prime of $R[G_p]$ corresponding to x shall be denoted by $\mathfrak{p}'$.

Now we are ready to state the next result.

Theorem 3.7. *With the above notation the completion* $\varprojlim(\mathbb{Z}[G])_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^n$ *of* $\mathbb{Z}[G]$ *at the (closed) singular point* $\{\mathfrak{p}\}$ *is isomorphic to the group ring* $\widehat{R}_{\bar{\mathfrak{p}}}\,[G_p]$ *of* G_p *over the* $\bar{\mathfrak{p}}$*-adic completion of* R, *where* $\bar{\mathfrak{p}} = R \cap \mathfrak{p}'$. *If* $\mathfrak{q}$ *is a nonsingular closed point from* $V(\mathfrak{a}_H) \cong \operatorname{Spec} \mathbb{Z}[\zeta_h][F]$ *then* $\varprojlim(\mathbb{Z}[G])_{\mathfrak{q}}/\mathfrak{m}_{\mathfrak{q}}^n$ *is isomorphic to the* $\mathfrak{q}'$*-adic completion of* $\mathbb{Z}[\zeta_h][T_1, \dots, T_r]$, *where* $\mathfrak{q}'$ *is the prime corresponding to* $\mathfrak{q}$, *when* $\operatorname{Spec} \mathbb{Z}[\zeta_h][F]$ *is viewed as a principal open subscheme of* $\operatorname{Spec} \mathbb{Z}[\zeta_h][T_1, \dots, T_r]$.

Proof. The canonical homomorphism $\mathbb{Z}[G] \to (\mathbb{Z}[G])_{\mathfrak{p}}$ factors through the embedding $\mathbb{Z}[G] \hookrightarrow \mathbb{Z}_{(p)}[G]$ and the canonical map $\mathbb{Z}_{(p)}[G] \to (\mathbb{Z}[G])_{\mathfrak{p}}$. Clearly $\mathfrak{p}_{(p)}$ is the inverse image of the maximal ideal $\mathfrak{m}_{\mathfrak{p}}$ of $(\mathbb{Z}[G])_{\mathfrak{p}}$ with respect to the latter map. Since $\mathfrak{p}$ is maximal, so is $\mathfrak{p}_{(p)}$ and $(\mathbb{Z}[G])_{\mathfrak{p}}$ is isomorphic to the localization of $\mathbb{Z}_{(p)}[G]$ at $\mathfrak{p}_{(p)}$. Thus $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}_{(p)}[G],\, \mathfrak{p}_{(p)}} \cong \mathcal{O}_{\operatorname{Spec} \mathbb{Z}[G],\, \mathfrak{p}}$.

Because the local ring of $x = \mathfrak{p}_{(p)} \in \operatorname{Spec} \mathbb{Z}_{(p)}[G]$ is determined in its open neighbourhood $\operatorname{Spec} B$, $B = \mathbb{Z}_{(p)}[\zeta][G_p \times F]$, we have that $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}_{(p)}[G],\, x} \cong \mathcal{O}_{\operatorname{Spec} B,\, x}$. On the other hand, as it was mentioned already, $\operatorname{Spec} B$ is an open subscheme of the affine scheme of the group ring $R[G_p]$. Hence $\mathcal{O}_{\operatorname{Spec} B,\, x} \cong \mathcal{O}_{\operatorname{Spec} R[G_p],\, x} = (R[G_p])_{\mathfrak{p}'}$. Since x is closed, $\mathfrak{p}'$ is maximal, and by [4, Chapter II, §3.3, Proposition 9] we have

$$(R[G_p])_{\mathfrak{p}'}/\mathfrak{m}_{\mathfrak{p}'}^n \cong (R[G_p])/(\mathfrak{p}')^n.$$

Consequently,

$$\varprojlim(R[G_p])_{\mathfrak{p}'}/\mathfrak{m}_{\mathfrak{p}'}^n \cong \varprojlim(R[G_p])/(\mathfrak{p}')^n.$$

Since $R[G_p]$ is a free R-module, the map $R \to R[G_p]$ is faithfully flat, and hence $\operatorname{Spec} R[G_p] \to \operatorname{Spec} R$ is surjective (see, for example, [19, p. 105]). It follows that $\bar{\mathfrak{p}} = R \cap \mathfrak{p}'$ is maximal. Since $\bar{\mathfrak{p}}$ lies over p, $R/\bar{\mathfrak{p}}$ is a field of characteristic p, and by a result of D. B. Coleman (see [14, p. 208]), the augmentation ideal of the modular group algebra of G_p over $R/\bar{\mathfrak{p}}$ is nilpotent. It coincides with the image of $\mathfrak{p}'$ modulo $\bar{\mathfrak{p}}$, as the unique prime of the local artinian ring $(R/\bar{\mathfrak{p}})[G_p]$ is its augmentation ideal. It follows that $(\mathfrak{p}')^f \subseteq \bar{\mathfrak{p}}\ R[G_p]$, for some $f > 0$.

Given an integer $n \geq 0$, write it in the form $n = sf + r$ with $0 \leq r \leq f - 1$ and let

$$\psi_n : R[G_p]/(\mathfrak{p}')^n \to (R/\bar{\mathfrak{p}}^s)[G_p]$$

be the map obtained from the natural homomorphism $R[G_p] \to (R/\bar{\mathfrak{p}}^s)[G_p]$ by going modulo $(\mathfrak{p}')^n \subseteq \bar{\mathfrak{p}}^s R[G_p]$. Observe that $\psi_o : 0 \to 0$ and for $1 \leq n \leq f - 1$ we have that $s = 0$ and thus the maps $\psi_0, \psi_1, \ldots, \psi_{f-1}$ are all zero. Set $K_n = \mathrm{Ker}(\psi_n) = \bar{\mathfrak{p}}^s R[G_p]/(\mathfrak{p}')^n$ and $R_n = (R/\bar{\mathfrak{p}}^s)[G_p]$ $(n \geq 0)$. Notice that each $(R/\bar{\mathfrak{p}}^s)[G_p]$ is repeated consecutively f times in the sequence $R_0, R_1, \ldots$. The sequences $\{K_n\}, \{R[G_p]/(\mathfrak{p}')^n\}$ and $\{R_n\}$ can be considered as inverse systems by taking the following maps. In $\{R[G_p]/(\mathfrak{p}')^n\}$ the maps are the natural ones. The map $R_{n+1} \to R_n$ is identity if $R_n = R_{n+1}$, and natural otherwise. As to $\{K_n\}$, for $\bar{\mathfrak{p}}^s R[G_p]/(\mathfrak{p}')^{n+1} \to \bar{\mathfrak{p}}^s R[G_p]/(\mathfrak{p}')^n$ we take the natural map, whereas for $\bar{\mathfrak{p}}^s R[G_p]/(\mathfrak{p}')^{n+1} \to \bar{\mathfrak{p}}^{s-1} R[G_p]/(\mathfrak{p}')^n$ we compose the embedding $\bar{\mathfrak{p}}^s R[G_p] \hookrightarrow \bar{\mathfrak{p}}^{s-1} R[G_p]$ with the natural homomorphism $R[G_p] \to R[G_p]/(\mathfrak{p}')^n$ and take it modulo $(\mathfrak{p}')^{n+1}$. The maps ψ_n together with the embeddings $K_n \hookrightarrow R[G_p]/(\mathfrak{p}')^n$ give rise to the short exact sequence of inverse systems

$$0 \to \{K_n\} \to \{R[G_p]/(\mathfrak{p}')^n\} \to \{R_n\} \to 0.$$

For each $n \geq 1$ we have that $K_{nf} = \bar{\mathfrak{p}}^n R[G_p]/(\mathfrak{p}')^{nf} \to \bar{\mathfrak{p}}^s R[G_p]/(\mathfrak{p}')^n = K_n$ is the zero map, because $\bar{\mathfrak{p}}^n R[G_p] \subseteq (\mathfrak{p}')^n$. Consequently $\varprojlim K_n = 0$ and moreover the inverse system $\{K_n\}$ satisfies the Mittag–Leffler condition (see [11, p. 192]). Therefore we come to the short exact sequence of inverse limits

$$0 \to \varprojlim K_n \to \varprojlim R[G_p]/(\mathfrak{p}')^n \to \varprojlim R_n \to 0$$

and thus $\varprojlim R[G_p]/(\mathfrak{p}')^n \cong \varprojlim R_n$.

Clearly in $\varprojlim R_n$ one can omit the repetitions, so that

$$\varprojlim R_n = \varprojlim (R/\bar{\mathfrak{p}}^n\ [G_p]) = (\varprojlim R/\bar{\mathfrak{p}}^n)[G_p],$$

which is the group ring $\widehat{R_{\bar{\mathfrak{p}}}}\ [G_p]$ of G_p over the $\bar{\mathfrak{p}}$-adic completion $\widehat{R_{\bar{\mathfrak{p}}}} = \varprojlim R/\bar{\mathfrak{p}}^n$.

Finally let $\mathfrak{q} \in V(\mathfrak{a}_H) \cong \mathrm{Spec}(\mathbb{Z}[\zeta_h])[F]$ be non-singular. Hence it is not an intersection point and removing all other components from $\mathrm{Spec}\,\mathbb{Z}[G]$, we have that the local ring $(\mathbb{Z}[G])_{\mathfrak{q}}$ is isomorphic to the corresponding local ring of $(\mathbb{Z}[\zeta_h])[F]$. Similarly as we have seen above, $\operatorname{Spec}(\mathbb{Z}[\zeta_h])[F]$ can be seen as a principal open subscheme of $\operatorname{Spec}\mathbb{Z}[\zeta_h][T_1,\dots,T_r]$, and so the completions are the same. $\square$

We obviously have the following

Corollary 3.8. *Let G be a finite abelian p-group. Then the formal completion $\varprojlim(\mathbb{Z}[G])_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^n$ of $\operatorname{Spec}\mathbb{Z}[G]$ at the singular point $\{\mathfrak{p}\}$ is isomorphic to the group ring $\mathbb{Z}_p[G]$ of G over the p-adic integers $\mathbb{Z}_p$. If $\mathfrak{q}\neq\mathfrak{p}$ is a closed point from the irreducible component $V\cong\operatorname{Spec}\mathbb{Z}[\zeta_{p^k}]$ then $\varprojlim(\mathbb{Z}[G])_{\mathfrak{q}}/\mathfrak{m}_{\mathfrak{q}}^n\cong\varprojlim\mathbb{Z}[\zeta_{p^k}]_{\mathfrak{q}'}/\mathfrak{m}_{\mathfrak{q}'}^n$.*

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