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Best approximation of trigonometric series with coefficients satisfying some regularity conditions

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Dedicated to Professor Zoltán Daróczy on his 70th birthday

Abstract. Recently we have defined a new very large class of numerical sequences of monotone type, and now we prove two theorems utilizing the sequences of this class as the coefficients of the series $\sum b_n \sin nx$. The origin of these theorems goes back to A. A. Konyushkov.

1. Introduction

Recently several papers deal with extension of monotone decreasing sequences purposely that the results proved for monotone sequences should hold true for wider classes, too.

A short survey of extensions and their applications can be found e.g. in [4], [6] and [7].

Now we shall give two further applications utilizing one of the widest classes. In the present paper we shall recall only a few definitions of these classes. For the notations and notions, please, see the third section.

In [5] we proved the analogues of two essential theorems of Konyushkov [2] replacing his monotone and quasi-monotone coefficients by coefficients belonging to the $RBVS$-class. Namely in [3] we showed that the classes $CQMS$ and $RBVS$ are not comparable.

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Our results proved in [5] read as follows:

Theorem A. *Let*

$$g(x) := \sum_{n=1}^{\infty} b_n \sin nx, \tag{1.1}$$

where $\mathbf{b} := \{b_n\} \in RBVS$*. If* $1 < p < \infty$ *and*

$$\sum_{n=1}^{\infty} b_n^p\, n^{p-2} < \infty, \tag{1.2}$$

then

$$E_n(p,g) \ll b_{n+1}(n+1)^{1/p'} + \left(\sum_{k=n+1}^{\infty} b_k^p k^{p-2}\right)^{1/p}, \tag{1.3}$$

where $p' := p/(p-1)$.

Theorem B. *Let* $1 < p < \infty,\ \mathbf{b} \in RBVS$ *and*

$$\sum_{n=1}^{\infty} n^{-1/p} b_n < \infty. \tag{1.4}$$

Then the series

$$\sum_{n=1}^{\infty} b_n \sin nx$$

is the Fourier series of a function $g \in L^p_{2\pi}$ *and*

$$E_n(p,g) \ll n^{1/p'} b_{n+1} + \sum_{k=n+1}^{\infty} k^{-1/p} b_k. \tag{1.5}$$

2. New theorems

In the present paper we shall prove that the condition $\mathbf{b} \in RBVS$ in Theorems A and B can be replaced by the weaker assumption $\mathbf{b} \in MRBVS$, and the new results give analogous estimates for the best approximations which reduce to (1.3) and (1.5) if $\mathbf{b} \in RBVS$.

Theorem 1. *Under the assumptions of Theorem A with* $\mathbf{b} \in MRBVS$ *in place of* $\mathbf{b} \in RBVS$ *we have*

$$E_n(p,g) \ll n^{1/p'} \left(n^{-1} \sum_{k=n+1}^{2n+1} b_k\right) + \left(\sum_{k=n+1}^{\infty} b_k^p k^{p-2}\right)^{1/p}. \tag{2.1}$$

Theorem 2. *Under the assumptions of Theorem B with* $\mathbf{b} \in MRBVS$ *in place of* $\mathbf{b} \in RBVS$ *we have*

$$E_n(p,g) \ll n^{1/p'} \left(n^{-1} \sum_{k=n+1}^{2n+1} b_k \right) + \sum_{k=n+2}^{\infty} k^{-1/p} b_k. \tag{2.2}$$

We mention that the proofs will show that analogous results for cosine series are also valid.

3. Notations and notions

Denote by $s_n = s_n(x) = s_n(g;x)$ the nth partial sum of (1.1), furthermore by $E_n(p,g)$ the best approximation of g by trigonometric polynomials of order at most n in $L^p_{2\pi}$-space.

We use the notation $L \ll R$ ($L \gg R$) at inequalities if there exists a positive constant K such that $L \le KR$ ($KL \ge R$) holds.

The classical quasi-monotone sequences are defined by the inequalities

$$0 \le b_{n+1} \le b_n \left(1 + \frac{\alpha}{n}\right), \quad \alpha > 0 \quad \text{and} \quad n \ge n_0(\alpha),$$

in symbol: $\mathbf{b} \in CQMS$.

Let $\gamma := \{\gamma_n\}$ be a given positive sequence. A *null-sequence* $\mathbf{c} := \{c_n\}$ ($c_n \to 0$) of *real numbers* satisfying the inequalities

$$\sum_{n=m}^{\infty} |\Delta c_n| \le K(\mathbf{c})\gamma_m \qquad (\Delta c_n := c_n - c_{n+1}),\ m = 1, 2, \ldots \tag{3.1}$$

with a positive constant $K(\mathbf{c})$ is said to be a *sequence of* γ *rest bounded variation*, in symbol $\mathbf{c} \in \gamma RBVS$.

If (3.1) holds with $\gamma_m = c_m$ then $\mathbf{c}$ is called a *sequence of rest bounded variation*, in brief $\mathbf{c} \in RBVS$.

If for all n $c_n \ge 0$ and $\gamma_m = m^{-1} \sum_{\nu=m}^{2m-1} c_\nu$ (> 0), then this sequence $\mathbf{c}$ is said to be a *sequence of mean rest bounded variation*, in symbol: $\mathbf{c} \in MRBVS$.

It is easy to see that if $\mathbf{c} \in RBVS$ then it is also *almost monotonic*, that is, for all $n \ge m$

$$(0 \le) c_n \le K(\mathbf{c}) c_m,$$

but not if $\mathbf{c} \in MRBVS$. A sequence $\mathbf{c} \in MRBVS$ may have a lot of zero terms as well.

4. Lemmas

We require the following lemmas.

Lemma 1. *If* $\lambda_k \ge 0$, *then*

$$\left|\sum_{k=m}^{n} \lambda_k \sin kx \sin \frac{x}{2}\right| \le \frac{1}{2}\left(\lambda_m + \sum_{k=m}^{n-1} |\lambda_k - \lambda_{k+1}| + \lambda_n\right).$$

This is clear by Abel rearrangement.

Lemma 2. *If the series*

$$\sum_{n=1}^{\infty} n^{-1} b_n \sin nx$$

is the Fourier series of some function $h_1(x) \in L_{2\pi}^p$, $1 < p < \infty$ *and*

$$\sum_{n=1}^{\infty} E_n(p, h_1) < \infty,$$

then the series

$$\sum_{n=1}^{\infty} b_n \sin nx$$

is the Fourier series of some function $h(x) \in L_{2\pi}^p$ *and*

$$E_n(p, h) \ll n E_n(p, h_1) + \sum_{k=n+1}^{\infty} E_k(p, h_1).$$

This lemma is a special case of [2, Theorem 1].

Lemma 3 (see [1])**.** *If* $d_n \ge 0$, $c > 1$ *and* $p > 1$, *then*

$$\sum_{n=1}^{\infty} n^{-c} \left(\sum_{k=1}^{n} d_k\right)^p \le K(p, c) \sum_{n=1}^{\infty} n^{-c} (n d_n)^p.$$

Lemma 4. *If* $\{\gamma_n\} \in MRBVS$ *and the inequality*

$$\left|\sum_{n=1}^{m} a_n\right| \le A$$

holds for any $m \ge 1$, *then the series* $\sum_{n=1}^{\infty} a_n \gamma_n$ *converges and*

$$\left|\sum_{n=m+1}^{\infty} a_n \gamma_n\right| \ll A m^{-1} \sum_{k=m+1}^{2m+1} \gamma_k.$$

PROOF. Using the notation

$$\alpha_n := \sum_{k=1}^{n} a_k$$

and the assumptions of Lemma 4, we get that

$$\begin{aligned}\left|\sum_{n=m+1}^{q} a_n\gamma_n\right| &= \left|\sum_{n=m+1}^{q-1} \alpha_n(\gamma_n-\gamma_{n+1}) + \alpha_q\gamma_q - \alpha_m\gamma_{m+1}\right| \\ &\le A\left(\sum_{n=m+1}^{q-1} |\Delta\gamma_n| + \gamma_q + \gamma_{m+1}\right) \le 2A\left(\sum_{n=m+1}^{q-1} |\Delta\gamma_n| + \gamma_q\right) \\ &\ll A\left(\gamma_q + m^{-1}\sum_{k=m+1}^{2m+1}\gamma_k\right),\end{aligned}$$

which proves the assertions of Lemma 4. □

5. Proofs

PROOF OF THEOREM 1. Let $0 < x < \pi$. Lemmas 1 and 4, furthermore $\mathbf{b} \in MRBVS$ imply that

$$|g(x) - s_n(x)| = \left|\sum_{k=n+1}^{\infty} b_k \sin kx\right| \ll x^{-1}n^{-1}\sum_{k=n+1}^{2n+1} b_k.$$

Thus, if $\pi/(m+1) < x < \pi$, then

$$|g(x) - s_n(x)| \ll mn^{-1}\sum_{k=n+1}^{2n+1} b_k,$$

whence for any $n \ge m$, we obtain that

$$\begin{aligned}\int\limits_{\pi/(n+1)}^{\pi} |g-s_n|^p dx &= \sum_{m=1}^{n}\int_{\pi/(m+1)}^{\pi/m} |g-s_n|^p dx \ll \sum_{m=1}^{n} m^p\left(n^{-1}\sum_{k=n+1}^{2n+1} b_k\right)^p m^{-2} \\ &\ll \left(n^{-1}\sum_{k=n+1}^{2n+1} b_k\right)^p n^{p-1} \ll n^{-1}\left(\sum_{k=n+1}^{2n+1} b_k\right)^p. \qquad (5.1)\end{aligned}$$

If $m > n+1$, then

$$\left|\sum_{k=n+1}^{\infty} b_k \sin kx\right| \ll \sum_{k=n+1}^{m} b_k + x^{-1}m^{-1}\sum_{k=m+1}^{2m+1} b_k,$$

whence we get that

$$\int_0^{\pi/(n+1)} |g - s_n|^p dx = \sum_{m=n+1}^{\infty} \int_{\pi/(m+1)}^{\pi/m} |g - s_n|^p dx$$
$$\ll \sum_{m=n+1}^{\infty} \left(\left(\sum_{k=n+1}^{m} b_k\right)^p m^{-2} + m^{-2}\left(\sum_{k=m+1}^{2m+1} b_k\right)^p\right). \quad (5.2)$$

Here the first sum can be estimated by Lemma 3, whence we get that

$$\sum_{m=n+1}^{\infty} m^{-2}\left(\sum_{k=n+1}^{m} b_k\right)^p \le K(p,2)\sum_{m=n+1}^{\infty} m^{-2}(mb_m)^p. \quad (5.3)$$

On the other hand, using Hölder inequality, we get that

$$\sum_{m=n+1}^{\infty} m^{-2}\left(\sum_{k=m+1}^{2m+1} b_k\right)^p \ll \sum_{m=n+1}^{\infty} m^{-2}m^{p-1}\sum_{k=m+1}^{2m+1} b_k^p \ll \sum_{k=n+2}^{\infty} k^{p-2}b_k^p. \quad (5.4)$$

The inequalities (5.1)–(5.4) clearly imply (2.1), and the proof is complete. □

Proof of Theorem 2. First we show that if $\{b_n\} \in MRBVS$, then $\{b_n/n\}$ also belongs to $MRBVS$. Namely

$$\sum_{k=n}^{\infty}\left|\frac{b_k}{k} - \frac{b_{k+1}}{k+1}\right| \le \sum_{k=n}^{\infty}\frac{1}{k}|b_k - b_{k+1}| + \sum_{k=n}^{\infty}\frac{1}{k^2}b_{k+1}$$
$$\ll \frac{1}{n}\cdot\frac{1}{n}\sum_{\nu=n}^{2n} b_\nu + \sum_{k=n}^{\infty}\frac{1}{k^2}\sum_{\nu=k+1}^{\infty}|\Delta b_\nu|$$
$$\ll n^{-2}\sum_{\nu=n}^{2n} b_\nu + n^{-1}\sum_{\nu=n+1}^{\infty}|\Delta b_\nu| \ll n^{-2}\sum_{\nu=n}^{2n} b_\nu \ll n^{-1}\sum_{\nu=n}^{2n}\frac{b_\nu}{\nu}.$$

Next we consider the series

$$g_1(x) \sim \sum_{n=1}^{\infty} n^{-1}b_n \sin nx.$$

The coefficients of g_1 satisfy the condition (1.2), we refer to condition (1.4) and the well-known inequality

$$\sum a_n^p \le \left(\sum a_n\right)^p, \quad p \ge 1 \quad \text{and} \quad a_n \ge 0,$$

therefore we can apply Theorem 1, which gives that

$$\begin{aligned}
E_n(p, g_1) &\ll n^{1/p'} \left(n^{-1} \sum_{k=n+1}^{2n+1} \frac{b_k}{k}\right) + \left(\sum_{k=n+1}^{\infty} \left(\frac{b_k}{k}\right)^p k^{p-2}\right)^{1/p} \\
&\ll n^{1/p'-1} \left(n^{-1} \sum_{k=n+1}^{2n+1} b_k\right) + \left(\sum_{k=n+1}^{\infty} \left(\sum_{\nu=k}^{\infty} |\Delta b_\nu|\right)^p k^{-2}\right)^{1/p} \\
&\ll n^{1/p'-1} \left(n^{-1} \sum_{k=n+1}^{2n+1} b_k\right) + \left(\sum_{\nu=n+1}^{\infty} |\Delta b_\nu|\right) n^{-1/p} \\
&\ll n^{1/p'-1} \left(n^{-1} \sum_{k=n+1}^{2n+1} b_k\right).
\end{aligned}$$

Using this and Lemma 2, we get that

$$\begin{aligned}
E_n(p, g) &\ll n E_n(p, g_1) + \sum_{k=n+1}^{\infty} E_k(p, g_1) \\
&\ll n^{1/p'} \left(n^{-1} \sum_{k=n+1}^{2n+1} b_k\right) + \sum_{k=n+1}^{\infty} k^{-1/p} \left(k^{-1} \sum_{\nu=k+1}^{2k+1} b_\nu\right) \\
&\ll n^{1/p'} \left(n^{-1} \sum_{k=n+1}^{2n+1} b_k\right) + \sum_{\nu=n+2}^{\infty} \nu^{-1/p} b_\nu,
\end{aligned}$$

that is, (2.2) is verified.

The proof is complete. □

References

[1] G. H. Hardy and J. E. Littlewood, Elementary theorems concerning power series with positive coefficients and moment constants of positive functions, *J. für Math.* **157** (1927), 141–158.

[2] A. A. Konyushkov, Best approximation by trigonometric polynomials and Fourier coefficients, *Mat. Sb.* **44** (1958), 53–84 (in *Russian*).

[3] L. Leindler, A new class of numerical sequences and its applications to sine and cosine series, *Analysis Math.* **28** (2002), 279–286.

[4] L. Leindler, A theorem on Besov-Nikol'skiĭ class, *Publ. Math. Debrecen* **64**(1–2) (2004), 237–248.

[5] L. Leindler, Best approximation and Fourier coefficients, *Analysis Math.* **31** (2005), 117–129.

[6] L. Leindler, A new extension of monotone sequences and its applications, *J. Inequal. Pure and Appl. Math.* **7**(1) (2006), Art. 39, http://jipam.vu.edu.au/article.php?sid=....

[7] L. Leindler, Embedding results pertaining to strong approximation of Fourier series, *V, Analysis Math.* **33** (2007), 113–121.

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