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Embeddable probability measures and infinitesimal systems of probability measures on a Moore Lie group

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Abstract. We show that, under natural conditions, a sequence of Poisson measures, close to the row products of the accompanying Poisson system of an infinitesimal system of probability measures on a Moore Lie group, converges to an embeddable probability measure.

1. Introduction

The central limit problem on a Lie group G can be formulated as follows. There is given a system $\{\mu_{n,\ell} : n \in \mathbb{N},\ \ell = 1, \ldots, k_n\}$ of probability measures on G satisfying the infinitesimality condition

$$\max_{1 \leq \ell \leq k_n} \mu_{n,\ell}(G \setminus N) \to 0 \quad \text{as } n \to \infty$$

valid for all Borel neighbourhoods N of the identity e of G. There is also given a probability measure μ on G. One searches for necessary and sufficient conditions on the system so that weak convergence

$$\mu_{n,1} * \cdots * \mu_{n,k_n} \to \mu$$

holds.

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Although the functional version of this problem has already been solved (see Feinsilver [1], Pap [6]), the above non-functional version is still open. There are only some partial results due to Parthasarathy [7] and to Heyer [3].

In this paper we consider the above problem on a Moore Lie group, that is, on a Lie group such that all of its irreducible representations are finite dimensional. (For example, each compact Lie group is a Moore Lie group.) First we show that under some conditions convergence of row products of an infinitesimal system is equivalent to convergence of row products of the accompanying Poisson system, and in case of convergence the limits coincide. Then we prove that under the natural conditions a sequence of Poisson measures, close to the row products of the accompanying Poisson system, converges to an embeddable probability measure. The missing link is to show that under the same natural conditions the row products of the accompanying Poisson system converges to the same embeddable limit measure (see Remarks 7.2 and 7.4).

2. Preliminaries

Let G be a Moore Lie group of dimension d, that is, a Lie group of dimension d such that all of its irreducible representations are finite dimensional. By $\mathcal{N}(e)$ we denote the system of all Borel neighbourhoods of the identity e in G. The Lie algebra of G will be denoted by $\mathfrak{L}(G)$. Let $\exp_G : \mathfrak{L}(G) \to G$ be the exponential mapping. By $\mathfrak{C}(G)$ we denote the space of real-valued continuous functions on G furnished with the supremum norm $\|\cdot\|$. By $\mathfrak{D}(G)$ we denote the space of infinitely differentiable real-valued functions with compact support on G.

If $f \in \mathfrak{C}(G)$ is continuously differentiable in some neighbourhood of a $y \in G$ then for every $D \in \mathfrak{L}(G)$ there exists the left derivative of f in y with respect to D defined by

$$Df(y) := \lim_{t\to 0} \frac{f(\exp_G(tD)y) - f(y)}{t}.$$

Let $\{D_1, \ldots, D_d\}$ be a basis of $\mathfrak{L}(G)$. Let $x_1, \ldots, x_d \in \mathfrak{D}(G)$ be a system of skew-symmetric canonical local coordinates of the first kind adapted to the basis $\{D_1, \ldots, D_d\}$ and valid in a compact neighbourhood $N_0 \in \mathcal{N}(e)$; i.e.,

$$y = \exp_G\left(\sum_{i=1}^{d} x_i(y) D_i\right) \quad \text{for all } y \in N_0,$$

the mapping $(x_1, \ldots, x_d) : N_0 \to \mathbb{R}^d$ is injective, and $x_i(y^{-1}) = -x_i(y)$ for $i = 1, \ldots, d$. Let $\varphi : G \to [0, 1]$ be a *Hunt function* for G; i.e., $1 - \varphi \in \mathfrak{D}(G)$,

$\varphi(y) > 0$ for all $y \in G \setminus \{e\}$, and

$$\varphi(y) = \sum_{i=1}^{d} x_i(y)^2 \quad \text{for all } y \in N_0.$$

Let $\mathfrak{M}^1(G)$ denote the semigroup of probability measures on G. For every $x \in G$, ε_x denotes the Dirac measure in x.

Let $\mathfrak{M}_+(G)$ denote the set of positive measures on G. A measure $\eta \in \mathfrak{M}_+(G)$ is said to be a *Lévy measure* on G if $\eta(\{e\}) = 0$ and $\int_G \varphi(y)\,\eta(\mathrm{d}y) < \infty$.

Let $\mathcal{P}(G)$ be the set of triplets (a, B, η), where $a \in \mathbb{R}^d$, $B \in \mathbb{R}^{d\times d}$ is a symmetric positive semidefinite matrix, and η is a Lévy measure on G.

A family $(\mu_t)_{t\geq 0}$ in $\mathfrak{M}^1(G)$ is called a *continuous convolution semigroup* if $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \geq 0$, $\mu_0 = \varepsilon_e$ and $\lim_{t\downarrow 0} \mu_t = \mu_0$. Its *generating functional* $(A, \mathcal{A})$ is defined by

$$\mathcal{A} := \left\{ f \in \mathfrak{C}(G) \;\middle|\; A(f) := \lim_{t\downarrow 0} \frac{1}{t} \left(\int_G f(y)\,\mu_t(\mathrm{d}y) - f(e) \right) \text{ exists} \right\}.$$

We have $\mathfrak{D}(G) \subset \mathcal{A}$, and there is a uniquely determined triplet $(a, B, \eta) \in \mathcal{P}(G)$ such that on $\mathfrak{D}(G)$ the functional A admits the *canonical decomposition* (Lévy–Khinchine formula)

$$\begin{aligned} A(f) = \sum_{i=1}^{d} a_i (D_i f)(e) &+ \frac{1}{2} \sum_{i,j=1}^{d} b_{i,j} (D_i D_j f)(e) \\ &+ \int_G \left(f(y) - f(e) - \sum_{i=1}^{d} x_i(y)(D_i f)(e) \right) \eta(\mathrm{d}y), \end{aligned} \tag{2.1}$$

where $a = (a_1, \ldots, a_d)$ and $B = (b_{i,j})_{1\leq i,j\leq d}$. Moreover, for each triplet $(a, B, \eta) \in \mathcal{P}(G)$ there exists a uniquely determined continuous convolution semigroup $(\mu_t)_{t\geq 0}$ in $\mathfrak{M}^1(G)$ such that (2.1) holds for all $f \in \mathfrak{D}(G)$. (See, e.g., Heyer [5, 4.2.8 Theorem].)

3. Unitary representations

A *unitary representation* of G is a homomorphism U from G into the group $\mathcal{U}(\mathcal{H}(U))$ of unitary operators on a complex Hilbert space $\mathcal{H}(U)$ such that the mapping $x \mapsto U(x)u$ from G into $\mathcal{H}(U)$ is continuous for all $u \in \mathcal{H}(U)$. The set of all unitary representations of G will be denoted by $\mathrm{Rep}(G)$. A representation $U \in \mathrm{Rep}(G)$ is said to be *irreducible* if there exists no nontrivial closed U-invariant subspace of $\mathcal{H}(U)$. Since G is a Moore group, the set $\mathrm{Irr}(G)$ of irreducible representations of G contains only finite dimensional representations. For $U \in \mathrm{Irr}(G)$ let $\dim(U)$ denote the dimension of the representation space $\mathcal{H}(U)$. Then $\mathcal{H}(U)$ and $\mathcal{U}(\mathcal{H}(U))$ can be identified with $\mathbb{C}^{\dim(U)}$ and with the unitary group $\mathcal{U}(\dim(U))$ consisting of the unitary matrices in $\mathbb{C}^{\dim(U)\times\dim(U)}$, respectively.

The *Fourier transform* $\widehat{\mu}$ of a bounded measure μ on G is given by

$$\langle \widehat{\mu}(U)u, v\rangle = \int_G \langle U(x)u, v\rangle \, \mu(\mathrm{d}x)$$

whenever $U \in \mathrm{Rep}(G)$, $u, v \in \mathcal{H}(U)$. Clearly, for given $U \in \mathrm{Rep}(G)$, $\widehat{\mu}(U)$ belongs to the space $\mathcal{L}(\mathcal{H}(U))$ of bounded linear operators on $\mathcal{H}(U)$, and one has $\|\widehat{\mu}(U)\| \leq 1$ whenever μ is a probability measure on G. If $U \in \mathrm{Irr}(G)$ then

$$\widehat{\mu}(U) = \int_G U(x) \, \mu(\mathrm{d}x) \in \mathbb{C}^{\dim(U)\times\dim(U)}.$$

Moreover, the mapping $\mu \mapsto \widehat{\mu}$ from $\mathfrak{M}^1(G)$ into the set of mappings $\mathrm{Rep}(G) \to \bigcup\{\mathcal{L}(\mathcal{H}(U)) : U \in \mathrm{Rep}(G)\}$ is injective (even on $\mathrm{Irr}(G)$), linear, multiplicative in the sense that $(\mu_1 * \mu_2)\widehat{\ }(U) = \widehat{\mu}_1(U)\widehat{\mu}_2(U)$ for all $U \in \mathrm{Rep}(G)$, and sequentially bicontinuous in the sense of the following equivalences expressed for sequences $(\mu_n)_{n\geq 0}$ of measures in $\mathfrak{M}^1(G)$:

(i) $\mu_n \to \mu_0$.

(ii) $\langle \widehat{\mu}_n(U)u, v\rangle \to \langle \widehat{\mu}_0(U)u, v\rangle$ for all $U \in \mathrm{Irr}(G)$, $u, v \in \mathcal{H}(U)$.

(iii) $\widehat{\mu}_n(U)u \to \widehat{\mu}_0(U)u$ for all $U \in \mathrm{Irr}(G)$, $u \in \mathcal{H}(U)$.

(iv) $\widehat{\mu}_n(U) \to \widehat{\mu}_0(U)$ for all $U \in \mathrm{Irr}(G)$.

(For the proof of the equivalence of (i)–(iii) see, for example, Siebert [8]. The equivalence of (iii) and (iv) follows from the assumption that G is a Moore group, so each irreducible representation is finite dimensional. See also Heyer [5, Theorem 1.4.5].)

Let $D \in \mathfrak{L}(G)$ and $U \in \mathrm{Irr}(G)$. Then the mapping $t \mapsto U(\exp_G(tD))$ is a continuous homomorphism from the (real) Lie group $\mathbb{R}$ into the (complex) Lie group $\mathcal{U}(\dim(U))$; hence $t \mapsto U(\exp_G(tD))$ is infinitely differentiable (see, e.g., Varadarajan [11, pp. 92–94]). Consequently the limit

$$D(U) := \lim_{t\to 0} \frac{U(\exp_G(tD)) - U(e)}{t} \in \mathbb{C}^{\dim(U)\times\dim(U)}$$

exists. Moreover, $D(U)$ is a skew-Hermitian matrix. Indeed,

$$\overline{D(U)}^{\top} = \lim_{t\to 0} \frac{\overline{U(\exp_G(tD))}^{\top} - \overline{U(e)}^{\top}}{t} = \lim_{t\to 0} \frac{\big(U(\exp_G(tD))\big)^{-1} - U(e)}{t}$$

$$= \lim_{t\to 0} \frac{U\big(\exp_G(tD)^{-1}\big) - U(e)}{t} = \lim_{t\to 0} \frac{U(\exp_G(-tD)) - U(e)}{t} = -D(U)$$

Lemma 3.1. *For $U \in \operatorname{Irr}(G)$ we have*

$$U(y) = \exp\left(\sum_{i=1}^{d} x_i(y) D_i(U)\right) \quad \textit{for } y \in N_0,$$

where $\exp : \mathbb{C}^{\dim(U)\times\dim(U)} \to \mathbb{C}^{\dim(U)\times\dim(U)}$ *denotes the exponential function defined by*

$$\exp(A) := \mathrm{e}^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Proof. Let $D := \sum_{i=1}^{d} x_i(y) D_i$. Defining $f(t) := U(\exp_G(tD))$ for $t \in \mathbb{R}$, we have

$$f'(t) = \lim_{h\to 0} \frac{U(\exp_G((t+h)D)) - U(\exp_G(tD))}{h}$$

$$= \lim_{h\to 0} \frac{U(\exp_G(hD)) - U(e)}{h} U(\exp_G(tD)) = D(U) f(t)$$

and $f(0) = U(e) = I$ (where I always denotes the appropriate identity matrix); hence $f(t) = \exp(tD(U))$. Substituting $t = 1$ we obtain $U(y) = \exp(D(U))$, since $y \in N_0$ implies

$$y = \exp_G\left(\sum_{i=1}^{d} x_i(y) D_i\right) = \exp_G(D);$$

hence $U(y) = U(\exp_G(D)) = f(1) = \exp(D(U))$. Finally,

$$D(U) = \sum_{i=1}^{d} x_i(y) D_i(U). \tag{3.1}$$

Indeed, $f(t) = g(x_1(y)t, \dots, x_d(y)t)$, where $g : \mathbb{R}^d \to \mathbb{C}^{\dim(U)\times\dim(U)}$, defined by $g(t_1, \dots, t_d) := U\left(\exp_G\left(\sum_{i=1}^{d} t_i D_i\right)\right)$ is differentiable. We have $\partial_i g(0, \dots, 0) = D_i(U)$; hence

$$f'(0) = \sum_{i=1}^{d} x_i(y) \partial_i g(0, \dots, 0) = \sum_{i=1}^{d} x_i(y) D_i(U).$$

We already know that $f'(0) = D(U)$, so (3.1) holds. □

Lemma 3.2. *Let* $U \in \operatorname{Irr}(G)$.

(i) *For the mapping* $y \mapsto U(y)$ *from* G *into* $\mathcal{U}(\dim(U))$ *the* Taylor formula

$$U(y) = U(e) + \sum_{i=1}^{d} x_i(y) D_i(U) + \frac{1}{2} \sum_{i,j=1}^{d} x_i(y) x_j(y) D_i(U) D_j(U) + \frac{1}{6} \sum_{i,j,k=1}^{d} x_i(y) x_j(y) x_k(y) T(U)(y) D_i(U) D_j(U) D_k(U)$$

is valid for all $y \in N_0$. *Here each* $T(U)(y)$ *is a matrix in* $\mathbb{C}^{\dim(U)\times\dim(U)}$ *with* $\|T(U)(y)\| \leq 1$.

(ii) *The following estimates hold for all* $y \in N_0$:

$$\left\| U(y) - U(e) - \sum_{i=1}^{d} x_i(y) D_i(U) \right\| \leq \frac{1}{2} \varphi(y) \sum_{i,j=1}^{d} \|D_i(U) D_j(U)\|$$

and

$$\left\| U(y) - U(e) - \sum_{i=1}^{d} x_i(y) D_i(U) - \frac{1}{2} \sum_{i,j=1}^{d} x_i(y) x_j(y) D_i(U) D_j(U) \right\| \leq \frac{1}{6} \varphi(y)^{3/2} \sum_{i,j,k=1}^{d} \|D_i(U) D_j(U) D_k(U)\|.$$

Proof. Similar to the proof of Lemma 5.1 in Siebert [9]. □

4. Convergence of embeddable measures

If $(\mu_t)_{t\geq 0}$ is a continuous convolution semigroup in $\mathfrak{M}^1(G)$ belonging to a triplet $(a, B, \eta) \in \mathcal{P}(G)$, then $(\widehat{\mu}_t(U))_{t\geq 0}$ is a strongly continuous semigroup of contractions on $\mathcal{H}(U)$ for all $U \in \operatorname{Rep}(G)$. If $U \in \operatorname{Irr}(G)$, then the infinitesimal generator $A(U)$ of $(\widehat{\mu}_t(U))_{t\geq 0}$ is given by

$$A(U) = \sum_{i=1}^{d} a_i D_i(U) + \frac{1}{2} \sum_{i,j=1}^{d} b_{i,j} D_i(U) D_j(U) + \int_G \left(U(y) - U(e) - \sum_{i=1}^{d} x_i(y) D_i(U) \right) \eta(\mathrm{d}y).$$

This is a consequence of the Corollary of Proposition 3.2 in Siebert [9] taking into account that the subspace $\mathcal{H}_0(U)$, consisting of the differentiable vectors in $\mathcal{H}(U)$ for U, coincides with $\mathcal{H}(U)$, since $\mathcal{H}_0(U)$ is dense in $\mathcal{H}(U)$ and $\mathcal{H}(U)$ is finite dimensional. (See Lemma 1.1 in Siebert [9].) Clearly $A(U) \in \mathbb{C}^{\dim(U)\times\dim(U)}$; hence we have

$$\widehat{\mu}_t(U) = \exp\Bigg\{ t\Bigg[\sum_{i=1}^d a_i D_i(U) + \frac{1}{2}\sum_{i,j=1}^d b_{i,j} D_i(U) D_j(U) + \int_G \Big(U(y) - U(e) - \sum_{i=1}^d x_i(y) D_i(U) \Big) \eta(\mathrm{d}y) \Bigg] \Bigg\}$$

for all $U \in \mathrm{Irr}(G)$ and for all $t \geq 0$.

Definition 4.1. A probability measure μ on G is said to be *embeddable* if there exists a convolution semigroup $(\mu_t)_{t\geq 0}$ in $\mathfrak{M}^1(G)$ such that $\mu = \mu_1$.

If μ is an embeddable probability measure then it is clearly infinitely divisible, and there exists a triplet $(a, B, \eta) \in \mathcal{P}(G)$ such that

$$\widehat{\mu}(U) = \exp\Bigg\{ \sum_{i=1}^d a_i D_i(U) + \frac{1}{2}\sum_{i,j=1}^d b_{i,j} D_i(U) D_j(U) + \int_G \Big(U(y) - U(e) - \sum_{i=1}^d x_i(y) D_i(U) \Big) \eta(\mathrm{d}y) \Bigg\}$$

holds for all $U \in \mathrm{Irr}(G)$. In this case we say that μ is an embeddable probability measure with triplet (a, B, η). In general, the triplet $(a, B, \eta) \in \mathcal{P}(G)$ is not uniquely determined by the measure μ.

For a triplet $(a, B, \eta) \in \mathcal{P}(G)$ with $B = (b_{i,j})_{1\leq i,j\leq d}$ we define the matrix $\widetilde{B} = (\widetilde{b}_{i,j})_{1\leq i,j\leq d}$ by

$$\widetilde{b}_{i,j} := b_{i,j} + \int_G x_i(y) x_j(y)\, \eta(\mathrm{d}y). \tag{4.1}$$

Theorem 4.2. *For each $n \in \mathbb{Z}_+$ let $\mu_n \in \mathfrak{M}^1(G)$ be an embeddable probability measure with a triplet $(a^{(n)}, B^{(n)}, \eta^{(n)})$. Suppose that*

(i) $a^{(n)} \to a^{(0)}$ *as* $n \to \infty$,

(ii) $\widetilde{B}^{(n)} \to \widetilde{B}^{(0)}$ *as* $n \to \infty$,

(iii) $\eta^{(n)}(G\setminus N) \to \eta^{(0)}(G\setminus N)$ *as* $n \to \infty$ *for all* $N \in \mathcal{N}(e)$ *with* $\eta^{(0)}(\partial N) = 0$.

Then $\mu_n \to \mu_0$ *as* $n \to \infty$.

PROOF. It suffices to show $\widehat{\mu}_n(U) \to \widehat{\mu}_0(U)$ as $n \to \infty$ for all $U \in \operatorname{Irr}(G)$. Let

$$h(y,U) := U(y) - U(e) - \sum_{i=1}^{d} x_i(y) D_i(U) - \frac{1}{2} \sum_{i,j=1}^{d} x_i(y) x_j(y) D_i(U) D_j(U)$$

for all $y \in G$ and all $U \in \operatorname{Irr}(G)$. Then $\widehat{\mu}_n(U)$ can be written in the form

$$\exp\left\{ \sum_{i=1}^{d} a_i^{(n)} D_i(U) + \frac{1}{2} \sum_{i,j=1}^{d} \widetilde{b}_{i,j}^{(n)} D_i(U) D_j(U) + \int_G h(y,U)\, \eta^{(n)}(\mathrm{d}y) \right\}$$

for all $n \in \mathbb{Z}_+$ and all $U \in \operatorname{Irr}(G)$. Taking into account the assumptions (i) and (ii), it is enough to show that

$$\int_G h(y,U)\, \eta^{(n)}(\mathrm{d}y) \to \int_G h(y,U)\, \eta^{(0)}(\mathrm{d}y) \tag{4.2}$$

as $n \to \infty$ for all $U \in \operatorname{Irr}(G)$.

By Lemma 3.2

$$\|h(y,U)\| \le c_U \varphi(y)^{3/2} \quad \text{for } y \in N_0,$$

where

$$c_U := \frac{1}{6} \sum_{i,j,k=1}^{d} \|D_i(U) D_j(U) D_k(U)\|.$$

Consequently for all $N \in \mathcal{N}(e)$ with $N \subset N_0$

$$\left\| \int_G h(y,U)\, \eta^{(n)}(\mathrm{d}y) - \int_G h(y,U)\, \eta^{(0)}(\mathrm{d}y) \right\| \le I_1^{(n)}(N) + I_2^{(n)}(N),$$

where

$$I_1^{(n)}(N) = c_U \int_N \varphi(y)^{3/2} \, (\eta^{(n)} + \eta^{(0)})(\mathrm{d}y),$$

$$I_2^{(n)}(N) = \left\| \int_{G \setminus N} h(y,U)\, \eta^{(n)}(\mathrm{d}y) - \int_{G \setminus N} h(y,U)\, \eta^{(0)}(\mathrm{d}y) \right\|.$$

We have

$$I_1^{(n)}(N) \le c_U \sup_{y \in N} \varphi(y)^{1/2} \int_N \varphi(y) \, (\eta^{(n)} + \eta^{(0)})(\mathrm{d}y)$$

and

$$\int_N \varphi(y)\,(\eta^{(n)} + \eta^{(0)})(\mathrm{d}y) \le \operatorname{Tr} \widetilde{B}^{(n)} + \operatorname{Tr} \widetilde{B}^{(0)}.$$

By the assumption (ii)

$$\sup_{n\ge 1} \operatorname{Tr} \widetilde{B}^{(n)} < \infty.$$

Let $\varepsilon > 0$. Then there exists $N_1 \in \mathcal{N}(e)$ such that $N_1 \subset N_0$, $\eta^{(0)}(\partial N_1) = 0$ and such that $\sup_{y\in N_1} \varphi(y)^{1/2}$ is small enough to guarantee that

$$c_U \sup_{y\in N_1} \varphi(y)^{1/2} \left(\operatorname{Tr} \widetilde{B}^{(0)} + \sup_{n\ge 1} \operatorname{Tr} \widetilde{B}^{(n)} \right) < \frac{\varepsilon}{2}\ .$$

Then

$$I_1^{(n)}(N_1) < \frac{\varepsilon}{2}\ .$$

By assumption (iii)

$$I_2^{(n)}(N_1) < \frac{\varepsilon}{2}$$

for sufficiently large n. Hence we obtain

$$\left\| \int_G h(y,U)\,\eta^{(n)}(\mathrm{d}y) - \int_G h(y,U)\,\eta^{(0)}(\mathrm{d}y) \right\| < \varepsilon$$

for sufficiently large n, which implies (4.2). $\square$

5. Local mean and local covariance matrix

Definition 5.1. A probability measure μ on G is said to have a *local mean* $m \in N_0$ and a *local covariance matrix* $B = (b_{ij})_{i,j=1,\dots,d}$ if

$$x_i(m) = \int_G x_i(y)\,\mu(\mathrm{d}y) \quad \text{for all } i \in \{1,\dots,d\},$$

and

$$b_{ij} = \int_G (x_i(y) - x_i(m))(x_j(y) - x_j(m))\,\mu(\mathrm{d}y) \quad \text{for all } i,j \in \{1,\dots,d\}.$$

If the numbers $\left|\int_G x_i(y)\,\mu(\mathrm{d}y)\right|$, $i = 1,\dots,d$ are sufficiently small, then μ has a uniquely determined local mean $m \in N_0$. The local covariance matrix always exists and is uniquely determined. Both the local mean and local covariance matrix will depend upon the choice of the coordinate functions on G.

We shall use the local mean for local centering and consider the shifted measure $\mu * \varepsilon_{m^{-1}}$. More specifically, we want to prove convergence theorems for a triangular system $\{\mu_{n,\ell} : n \in \mathbb{N},\ \ell = 1, \dots, k_n\}$ of probability measures on G. We use local centering and consider the sequences of convolutions $(\mu_{n,1} * \varepsilon_{m_{n,1}^{-1}}) * \cdots (\mu_{n,k_n} * \varepsilon_{m_{n,k_n}^{-1}})$, where $m_{n,\ell}$ is the local mean of $\mu_{n,\ell}$. We have to estimate how close a shifted measure $\nu := \mu * \varepsilon_{m^{-1}}$ is to the measure ε_e for a probability measure μ with local mean m. To this end we shall estimate the distance between their Fourier transforms; i.e., the quantity $\|\widehat{\nu}(U) - I\|$. (This will be an analogue of Lemma 1.6 in Siebert [10] providing an estimate for the distance between the convolution operators of μ and ε_e.)

Now we choose an appropriate neighbourhood of the identity e in G. There exists $N_0' \in \mathcal{N}(e)$ such that $N_0'(N_0')^{-1} \subset N_0$. Moreover, there exists $c_0 > 0$ such that $N_0'' := \left\{y \in N_0 : \sum_{i=1}^d x_i(y)^2 \le c_0\right\} \subset N_0'$. Then N_0'' is compact, and it is convex in the sense that $u, v \in N_0''$ implies

$$\lambda u + (1-\lambda)v := \exp_G\left(\sum_{i=1}^d (\lambda x_i(u) + (1-\lambda)x_i(v))D_i\right) \in N_0''$$

for all $\lambda \in [0,1]$.

Lemma 5.2. *For every $U \in \mathrm{Irr}(G)$ there exists a constant $c(U) > 0$ such that*

$$\|(\mu * \varepsilon_{m^{-1}})\widehat{\ }(U) - I\| \le c(U)\big(\mu(G \setminus N_0'') + \mathrm{Tr}(B)\big)$$

whenever μ is a probability measure on G with local mean $m \in N_0''$ and local covariance matrix B.

Proof. Let μ be a probability measure on G with local mean $m \in N_0''$ and local covariance matrix B. Let $U \in \mathrm{Irr}(G)$. Then

$$(\mu * \varepsilon_{m^{-1}})\widehat{\ }(U) - I = \int_G \big(U(ym^{-1}) - U(e)\big)\,\mu(\mathrm{d}y).$$

We are going to find a Taylor formula for $U(ym^{-1})$ valid for $y \in N_0''$. If $g : G \to \mathbb{R}$ is differentiable in $y \in N_0$ then there exist the partial derivatives

$$\partial_i g(y) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g\Big(\exp_G\big(tD_i + \sum_{\ell=1}^d x_\ell(y)D_\ell\big)\Big)$$

for $i = 1, \dots, d$ and $y \in N_0$. Consider the function $f : \mathbb{R}^d \to \mathbb{C}^{\dim(U)\times\dim(U)}$ defined by

$$f(t_1, \dots, t_d) := U\left(\exp_G\left(\sum_{i=1}^d t_i D_i\right) m^{-1}\right).$$

By Lemma 3.1 we have that, if $y \in N_0$ and $t_i = x_i(y)$ for all i, then

$$f(t_1, \ldots, t_d) = \exp\left(\sum_{i=1}^{d} t_i D_i(U)\right) U(m^{-1}).$$

Hence f is infinitely differentiable at each point $(x_1(y), \ldots, x_d(y)) \in \mathbb{R}^d$ such that $y \in N_0$. For $t = (x_1(y), \ldots, x_d(y))$ and $s = (x_1(m), \ldots, x_d(m))$, where $y, m \in N_0''$, the Taylor formula for matrix-valued functions yields

$$\begin{aligned} f(t) = f(s) &+ \sum_{i=1}^{d} (t_i - s_i) \partial_i f(s) \\ &+ \sum_{i,j=1}^{d} (t_i - s_i)(t_j - s_j) \int_0^1 (1-\lambda) \partial_i \partial_j f(\lambda t + (1-\lambda)s) \, \mathrm{d}\lambda. \end{aligned}$$

But, for $y, m \in N_0''$, we have

$$f(t) = U(ym^{-1}) = R_{m^{-1}} U(y),$$

$$f(s) = U(e),$$

$$\partial_i f(s) = \partial_i R_{m^{-1}} U(m),$$

$$\partial_i \partial_j f(\lambda t + (1-\lambda)s) = \partial_i \partial_j R_{m^{-1}} U(\lambda y + (1-\lambda)m),$$

where for a function h on G and $z \in G$ the shifted function $R_z h$ is defined by $R_z h(y) := h(yz)$ for $y \in G$. Hence $U(ym^{-1})$ can be written in the form

$$\begin{aligned} U(e) &+ \sum_{i=1}^{d} (x_i(y) - x_i(m)) \partial_i R_{m^{-1}} U(m) \\ &+ \frac{1}{2} \sum_{i,j=1}^{d} (x_i(y) - x_i(m))(x_j(y) - x_j(m)) \partial_i \partial_j R_{m^{-1}} U(m) + R(U, y, m), \end{aligned} \tag{5.1}$$

where $R(U, y, m)$ denotes the quantity

$$\begin{aligned} &\sum_{i,j=1}^{d} (x_i(y) - x_i(m))(x_j(y) - x_j(m)) \\ &\qquad \times \int_0^1 (1-\lambda) \big(\partial_i \partial_j R_{m^{-1}} U(\lambda y + (1-\lambda)m) - \partial_i \partial_j R_{m^{-1}} U(m) \big) \, \mathrm{d}\lambda. \end{aligned}$$

Consequently

$$
\begin{aligned}
(\mu * \varepsilon_{m^{-1}})\widehat{\ }(U) - I = \int_{G\setminus N_0''} \Bigg(& U(ym^{-1}) - U(e) - \sum_{i=1}^{d} (x_i(y) - x_i(m)) \partial_i R_{m^{-1}} U(m) \\
& - \frac{1}{2} \sum_{i,j=1}^{d} (x_i(y) - x_i(m))(x_j(y) - x_j(m)) \partial_i \partial_j R_{m^{-1}} U(m) \Bigg) \mu(\mathrm{d}y) \\
& + \frac{1}{2} \sum_{i,j=1}^{d} b_{ij} \partial_i \partial_j R_{m^{-1}} U(m) + \int_{N_0''} R(U, y, m)\, \mu(\mathrm{d}y).
\end{aligned}
$$

For $v \in N_0$ we have

$$
\begin{aligned}
\partial_i R_{m^{-1}} U(v) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} U\left(\exp_G\left(tD_i + \sum_{\ell=1}^{d} x_\ell(v) D_\ell\right) m^{-1}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp\left(tD_i(U) + \sum_{\ell=1}^{d} x_\ell(v) D_\ell(U)\right) U(m^{-1}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \sum_{k=0}^{\infty} \frac{1}{k!} \left(tD_i(U) + \sum_{\ell=1}^{d} x_\ell(v) D_\ell(U)\right)^k U(m^{-1}) \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{r=0}^{k-1} \left(\sum_{\ell=1}^{d} x_\ell(v) D_\ell(U)\right)^r D_i(U) \left(\sum_{\ell=1}^{d} x_\ell(v) D_\ell(U)\right)^{k-1-r} U(m^{-1}).
\end{aligned}
$$

Since the coordinate functions $x_1, \dots, x_d$ are continuous, the function

$$(m, v) \mapsto \partial_i R_{m^{-1}} U(v)$$

from $N_0'' \times N_0''$ into $\mathbb{C}^{\dim(U)\times\dim(U)}$ is continuous, hence bounded, because of the compactness of N_0. Similarly, the function $(m, v) \mapsto \partial_i \partial_j R_{m^{-1}} U(v)$ is bounded on $N_0'' \times N_0''$; thus we conclude the assertion. □

6. Infinitesimal systems of probability measures

Definition 6.1. A system $\{\mu_{n,\ell} : n \in \mathbb{N},\ \ell = 1, \dots, k_n\}$ of probability measures on G is said to be *infinitesimal* if

$$\max_{1\le\ell\le k_n} \mu_{n,\ell}(G \setminus N) \to 0 \quad \text{as } n \to \infty$$

for all $N \in \mathcal{N}(e)$.

Lemma 6.2. *A system $\{\mu_{n,\ell} : n \in \mathbb{N},\ \ell = 1, \dots, k_n\}$ of probability measures on G is infinitesimal if and only if for each $U \in \mathrm{Irr}(G)$ we have*

$$\max_{1 \le \ell \le k_n} \|\widehat{\mu}_{n,\ell}(U) - I\| \to 0 \quad \text{as } n \to \infty.$$

Proof. Similar to the proof of Lemma 8.1 in Siebert [9]. □

Lemma 6.3. *If $\{\mu_{n,\ell} : n \in \mathbb{N},\ \ell = 1, \dots, k_n\}$ is an infinitesimal system of probability measures on G then for sufficiently large n, the measures $\mu_{n,1}, \dots, \mu_{n,k_n}$ have local means $m_{n,1}, \dots, m_{n,k_n}$, and the systems $\{\varepsilon_{m_{n,\ell}} : n \in \mathbb{N},\ \ell = 1, \dots, k_n\}$ and $\{\mu_{n,\ell} * \varepsilon_{m^{-1}_{n,\ell}} : n \in \mathbb{N},\ \ell = 1, \dots, k_n\}$ are infinitesimal.*

Proof. For all $N \in \mathcal{N}(e)$, $n \in \mathbb{N}$, $\ell = 1, \dots, k_n$ and $i = 1, \dots, d$ we have

$$\left| \int_G x_i(y)\, \mu_{n,\ell}(\mathrm{d}y) \right| \le \sup_{y \in N} |x_i(y)| + \|x_i\| \mu_{n,\ell}(G \setminus N);$$

hence

$$\limsup_{n \to \infty} \max_{1 \le \ell \le k_n} \left| \int_G x_i(y)\, \mu_{n,\ell}(\mathrm{d}y) \right| \le \sup_{y \in N} |x_i(y)|.$$

Since $N \in \mathcal{N}(e)$ is arbitrary, we conclude

$$\max_{1 \le \ell \le k_n} \left| \int_G x_i(y)\, \mu_{n,\ell}(\mathrm{d}y) \right| \to 0 \quad \text{as } n \to \infty \quad \text{for all } i = 1, \dots, d, \tag{6.1}$$

which implies existence of local mean of the measures $\mu_{n,1}, \dots, \mu_{n,k_n}$ for sufficiently large $n \in \mathbb{N}$. Convergence (6.1) also implies that for each $N \in \mathcal{N}(e)$ we have $m_{n,1}, \dots, m_{n,k_n} \in N$ for sufficiently large $n \in \mathbb{N}$; thus the system $\{\varepsilon_{m_{n,\ell}} : n \in \mathbb{N},\ \ell = 1, \dots, k_n\}$ is infinitesimal.

If a probability measure μ on G has local mean m then for all $U \in \mathrm{Irr}(G)$

$$\begin{aligned} \|(\mu * \varepsilon_{m^{-1}})\widehat{\ }(U) - I\| &= \|\widehat{\mu}(U) U(m)^{-1} - I\| = \|(\widehat{\mu}(U) - U(m)) U(m)^{-1}\| \\ &\le \|\widehat{\mu}(U) - U(m)\| = \|\widehat{\mu}(U) - \widehat{\varepsilon_m}(U)\| \le \|\widehat{\mu}(U) - I\| + \|\widehat{\varepsilon_m}(U) - I\|. \end{aligned}$$

Hence Lemma 6.2 implies infinitesimality of the system $\{\mu_{n,\ell} * \varepsilon_{m^{-1}_{n,\ell}} : n \in \mathbb{N},\ \ell = 1, \dots, k_n\}$. □

For a positive bounded measure μ on G, the *Poisson measure* $\nu = \mathrm{e}^{\mu - \mu(G)\varepsilon_e} \in \mathfrak{M}^1(G)$ with exponent μ is defined by

$$\mathrm{e}^{\mu - \mu(G)\varepsilon_e} := \mathrm{e}^{-\mu(G)} \left(\varepsilon_e + \mu + \frac{\mu * \mu}{2!} + \frac{\mu * \mu * \mu}{3!} + \cdots \right).$$

Clearly

$$\widehat{\nu}(U) = \mathrm{e}^{\widehat{\mu}(U) - \mu(G)\cdot I}$$

for all $U \in \mathrm{Rep}(G)$; hence ν is an embeddable probability measure with triplet $(a, 0, \mu)$, where $a = (a_1, \ldots, a_d)$ with $a_i = \int_G x_i(y)\,\mu(\mathrm{d}y)$, $i = 1, \ldots, d$.

Theorem 6.4. *Let $\{\mu_{n,\ell} : n \in \mathbb{N},\ \ell = 1, \ldots, k_n\}$ be an infinitesimal system of probability measures on G. Denote the local mean and the local covariance matrix of $\mu_{n,1}, \ldots, \mu_{n,k_n}$ by $m_{n,1}, \ldots, m_{n,k_n}$ and by $B_{n,1}, \ldots, B_{n,k_n}$ (which exist for sufficiently large $n \in \mathbb{N}$ by Lemma 6.3). Let*

$$\mu'_{n,\ell} := \mu_{n,\ell} * \varepsilon_{m_{n,\ell}^{-1}}, \qquad \nu'_{n,\ell} := \exp(\mu'_{n,\ell} - \varepsilon_e).$$

Suppose that

$$\sup_{n\in\mathbb{N}} \sum_{\ell=1}^{k_n} \big(\mu_{n,\ell}(G \setminus N_0'') + \mathrm{Tr}(B_{n,\ell})\big) < \infty. \tag{6.2}$$

Then

$$\|(\mu_{n,1} * \cdots * \mu_{n,k_n})\widehat{\ }(U) - (\nu'_{n,1} * \varepsilon_{m_{n,1}} * \cdots * \nu'_{n,k_n} * \varepsilon_{m_{n,k_n}})\widehat{\ }(U)\| \to 0$$

as $n \to \infty$ for all $U \in \mathrm{Irr}(G)$. In particular, the sequence $\big(\mu_{n,1} * \cdots * \mu_{n,k_n}\big)_{n\geq 1}$ of row products is convergent if and only if the sequence $\big(\nu'_{n,1} * \varepsilon_{m_{n,1}} * \cdots * \nu'_{n,k_n} * \varepsilon_{m_{n,k_n}}\big)_{n\geq 1}$ of row products is convergent. In the affirmative case the limits of these sequences coincide.

Moreover,

$$\|(\mu'_{n,1} * \cdots * \mu'_{n,k_n})\widehat{\ }(U) - (\nu'_{n,1} * \cdots * \nu'_{n,k_n})\widehat{\ }(U)\| \to 0 \quad \textit{as } n \to \infty$$

for all $U \in \mathrm{Irr}(G)$. In particular, the sequence $\big(\mu'_{n,1} * \cdots * \mu'_{n,k_n}\big)_{n\geq 1}$ of row products is convergent if and only if the sequence $\big(\nu'_{n,1} * \cdots * \nu'_{n,k_n}\big)_{n\geq 1}$ of row products is convergent, and in the affirmative case the limits of these sequences coincide.

Proof. Let

$$\mu_n := \mu_{n,1} * \cdots * \mu_{n,k_n} = \mu'_{n,1} * \varepsilon_{m_{n,1}} * \cdots * \mu'_{n,k_n} * \varepsilon_{m_{n,k_n}},$$

$$\nu'_n := \nu'_{n,1} * \varepsilon_{m_{n,1}} * \cdots * \nu'_{n,k_n} * \varepsilon_{m_{n,k_n}}.$$

If $A_1, \ldots, A_k, B_1, \ldots, B_k \in \mathbb{C}^{\dim(U)\times\dim(U)}$ with $\|A_\ell\| \leq 1$ and $\|B_\ell\| \leq 1$ for all $\ell = 1, \ldots, k_n$, then

$$\|A_1 \cdots A_k - B_1 \cdots B_k\| \leq \sum_{\ell=1}^{k} \|A_\ell - B_\ell\|.$$

Hence

$$
\begin{aligned}
\|\widehat{\mu}_n(U) - (\nu'_n)^\frown(U)\| = \big\| & (\mu'_{n,1})^\frown(U)(\varepsilon_{m_{n,1}})^\frown(U) \cdots (\mu'_{n,k_n})^\frown(U)(\varepsilon_{m_{n,k_n}})^\frown(U) \\
& -(\nu'_{n,1})^\frown(U)(\varepsilon_{m_{n,1}})^\frown(U) \cdots (\nu'_{n,k_n})^\frown(U)(\varepsilon_{m_{n,k_n}})^\frown(U)\big\| \\
\leq \sum_{\ell=1}^{k_n} \big\|(\mu'_{n,\ell})^\frown(U) - (\nu'_{n,\ell})^\frown(U)\big\| &= \sum_{\ell=1}^{k_n} \big\|(\mu'_{n,\ell})^\frown(U) - \exp((\mu'_{n,\ell})^\frown(U) - I)\big\|.
\end{aligned}
$$

For a matrix $A \in \mathbb{C}^{\dim(U)\times\dim(U)}$ we have

$$
\begin{aligned}
\|e^A - I - A\| &= \left\| \sum_{k=2}^{\infty} \frac{A^k}{k!} \right\| \leq \sum_{k=2}^{\infty} \frac{\|A\|^k}{k!} = \|A\|^2 \sum_{k=0}^{\infty} \frac{\|A\|^k}{(k+2)!} \\
&\leq \|A\|^2 \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!} = \|A\|^2 e^{\|A\|}.
\end{aligned}
$$

Applying this for $A = (\mu'_{n,\ell})^\frown(U) - I$ we obtain

$$
\begin{aligned}
\|\widehat{\mu}_n(U) - (\nu'_n)^\frown(U)\| &\leq \sum_{\ell=1}^{k_n} \|(\mu'_{n,\ell})^\frown(U) - I\|^2 e^{\|(\mu'_{n,\ell})^\frown(U) - I\|} \\
&\leq e^2 \left(\max_{1\leq\ell\leq k_n} \|(\mu'_{n,\ell})^\frown(U) - I\| \right) \sum_{\ell=1}^{k_n} \|(\mu'_{n,\ell})^\frown(U) - I\|.
\end{aligned}
$$

By Lemma 6.3, the system $\{\mu'_{n,\ell} : n \in \mathbb{N},\ \ell = 1, \ldots, k_n\}$ is infinitesimal; thus by Lemma 6.2

$$
\max_{1\leq\ell\leq k_n} \|(\mu'_{n,\ell})^\frown(U) - I\| \to 0 \quad \text{as } n \to \infty.
$$

Hence it suffices to show that

$$
\sup_{n\in\mathbb{N}} \sum_{\ell=1}^{k_n} \|(\mu'_{n,\ell})^\frown(U) - I\| < \infty.
$$

By Lemma 5.2, this is a consequence of assumption (6.2).

The proof for the other system is similar. $\square$

7. Further investigations: convergence of a system to an embeddable measure

Theorem 7.1. *Let $\{\mu_{n,\ell} : n \in \mathbb{N},\ \ell = 1, \dots, k_n\}$ be an infinitesimal system of probability measures on G. Denote the local mean and the local covariance matrix of $\mu_{n,1}, \dots, \mu_{n,k_n}$ by $m_{n,1}, \dots, m_{n,k_n}$ and by $B_{n,1}, \dots, B_{n,k_n}$ (which exist for sufficiently large $n \in \mathbb{N}$ by Lemma 6.3). Let*

$$\mu'_{n,\ell} := \mu_{n,\ell} * \varepsilon_{m_{n,\ell}^{-1}}.$$

Suppose that there exists $(0, B, \eta) \in \mathcal{P}(G)$ such that

(i) $\sum_{\ell=1}^{k_n} B_{n,\ell} \to \widetilde{B}$ *as* $n \to \infty$,

(ii) $\sum_{\ell=1}^{k_n} \mu_{n,\ell}(G \setminus N) \to |, \eta(G \setminus N)$ *as* $n \to \infty$ *for all* $N \in \mathcal{N}(e)$ *with* $\eta(\partial N) = 0$,

where $\widetilde{B}$ is defined in (4.1). *Then*

$$\exp\left(\sum_{\ell=1}^{k_n} (\mu'_{n,\ell} - \varepsilon_e)\right) \to \nu,$$

where ν is an embeddable probability measure on G with triplet $(0, B, \eta)$.

Proof. The measure

$$\nu'_n := \exp\left(\sum_{\ell=1}^{k_n} (\mu'_{n,\ell} - \varepsilon_e)\right)$$

is a Poisson measure with

$$(\nu'_n)\widehat{\ }(U) = \exp\left(\sum_{\ell=1}^{k_n} \big((\mu'_{n,\ell})\widehat{\ }(U) - I\big)\right)$$

for all $U \in \mathrm{Irr}(G)$. Moreover $\widehat{\nu}(U)$ has the form

$$\exp\left\{\frac{1}{2}\sum_{i,j=1}^{d} b_{i,j} D_i(U) D_j(U) + \int_G \left(U(y) - U(e) - \sum_{i=1}^{d} x_i(y) D_i(U)\right) \eta(\mathrm{d}y)\right\}$$

for all $U \in \mathrm{Irr}(G)$. Hence it is enough to show that

$$\sum_{\ell=1}^{k_n} \big((\mu'_{n,\ell})\widehat{\ }(U) - I\big) \to \frac{1}{2}\sum_{i,j=1}^{d} b_{i,j} D_i(U) D_j(U) + \int_G \left(U(y) - U(e) - \sum_{i=1}^{d} x_i(y) D_i(U)\right) \eta(\mathrm{d}y) \tag{7.1}$$

as $n \to \infty$ for all $U \in \mathrm{Irr}(G)$. By the Taylor formula (5.1) the quantity $(\mu'_{n,\ell})\widehat{\ }(U) - I$ can be written in the form

$$\frac{1}{2}\sum_{i,j=1}^{d}\int_N (x_i(y)-x_i(m_{n,\ell}))(x_j(y)-x_j(m_{n,\ell}))\partial_i\partial_j R_{m_{n,\ell}^{-1}}U(m_{n,\ell})\,\mu_{n,\ell}(\mathrm{d}y)$$
$$+\int_{G\setminus N}\left(U(ym_{n,\ell}^{-1})-U(e)-\sum_{i=1}^{d}(x_i(y)-x_i(m_{n,\ell}))\partial_i R_{m_{n,\ell}^{-1}}U(m_{n,\ell})\right)\mu_{n,\ell}(\mathrm{d}y)$$
$$+\int_N R(U,y,m_{n,\ell})\,\mu_{n,\ell}(\mathrm{d}y) \tag{7.2}$$

for each $N \in \mathcal{N}(e)$ with $N \subset N_0''$. Hence (7.1) follows from the following five limiting relationships:

$$\sum_{\ell=1}^{k_n}\int_{G\setminus N}\left(U(ym_{n,\ell}^{-1})-U(e)-\sum_{i=1}^{d}(x_i(y)-x_i(m_{n,\ell}))\partial_i R_{m_{n,\ell}^{-1}}U(m_{n,\ell})\right)\mu_{n,\ell}(\mathrm{d}y)$$
$$\to \int_{G\setminus N}\left(U(y)-U(e)-\sum_{i=1}^{d}x_i(y)D_i(U)\right)\eta(\mathrm{d}y) \quad \text{as } n\to\infty, \tag{7.3}$$

$$\int_N\left(U(y)-U(e)-\sum_{i=1}^{d}x_i(y)D_i(U)\right)\eta(\mathrm{d}y)\to 0 \quad \text{as} N\to\{e\}, \tag{7.4}$$

$$\sum_{i,j=1}^{d}\sum_{\ell=1}^{k_n}\int_N (x_i(y)-x_i(m_{n,\ell}))(x_j(y)-x_j(m_{n,\ell}))\partial_i\partial_j R_{m_{n,\ell}^{-1}}U(m_{n,\ell})\,\mu_{n,\ell}(\mathrm{d}y)$$
$$\to \sum_{i,j=1}^{d}\left(b_{i,j}+\int_N x_i(y)x_j(y)\,\eta(\mathrm{d}y)\right)D_i(U)D_j(U) \qquad \text{as } \ n\to\infty, \tag{7.5}$$

$$\int_N x_i(y)x_j(y)\,\eta(\mathrm{d}y)\to 0 \qquad \text{as } \ N\to\{e\}, \tag{7.6}$$

$$\limsup_{n\to\infty}\sum_{\ell=1}^{k_n}\int_N \|R(U,y,m_{n,\ell})\|\;\mu_{n,\ell}(\mathrm{d}y)\to 0 \quad \text{as } N\to\{e\}, \tag{7.7}$$

where (7.3) and (7.5) are valid for $N \in \mathcal{N}(e)$ with $N \subset N_0''$ and $\eta(\partial N)=0$.

In order to show (7.3) it is sufficient to prove that

$$\sum_{\ell=1}^{k_n}\int_{G\setminus N}\big(U(ym_{n,\ell}^{-1})-U(y)\big)\,\mu_{n,\ell}(\mathrm{d}y)\to 0, \tag{7.8}$$

$$\sum_{\ell=1}^{k_n}\int_{G\setminus N}(x_i(y)-x_i(m_{n,\ell}))\big(\partial_i R_{m_{n,\ell}^{-1}}U(m_{n,\ell})-\partial_i U(e)\big)\mu_{n,\ell}(\mathrm{d}y)\to 0, \tag{7.9}$$

$$\sum_{\ell=1}^{k_n}\int_{G\setminus N} x_i(m_{n,\ell})\partial_i U(e)\mu_{n,\ell}(\mathrm{d}y)\to 0, \tag{7.10}$$

$$\sum_{\ell=1}^{k_n}\int_{G\setminus N}\left(U(y)-U(e)-\sum_{i=1}^{d}x_i(y)\partial_i U(e)\right)\mu_{n,\ell}(\mathrm{d}y)$$
$$\to\int_{G\setminus N}\left(U(y)-U(e)-\sum_{i=1}^{d}x_i(y)D_i(U)\right)\eta(\mathrm{d}y) \tag{7.11}$$

as $n\to\infty$. Clearly (7.8) follows from

$$\max_{1\le\ell\le k_n}\sup_{y\in G}\|U(ym_{n,\ell}^{-1})-U(y)\|\to 0 \qquad \text{as} \quad n\to\infty$$

(which is a consequence of the uniform continuity of U and the uniform convergence $m_{n,\ell}\to e$ as $n\to\infty$, see (6.1)), and from assumption (ii).

The mapping $y\mapsto\partial_i R_{y^{-1}}U(y)$ from N_0 into $\mathbb{C}^{\dim(U)\times\dim(U)}$ is continuous (see the exact formula for $\partial_i R_{y^{-1}}U(y)$ in the proof of Lemma 5.2); hence it is uniformly continuous because of the compactness of N_0. Thus by the uniform convergence $m_{n,\ell}\to e$ as $n\to\infty$ we conclude

$$\max_{1\le\ell\le k_n}\|\partial_i R_{m_{n,\ell}^{-1}}U(m_{n,\ell})-\partial_i U(e)\|\to 0 \qquad \text{as } n\to\infty,$$

which together with assumption (ii) imply (7.9). Obviously (7.10) and (7.11) follows from assumption (ii), since $\partial_i U(e)=D_i(U)$ and $y\mapsto U(y)-U(e)-\sum_{i=1}^{d}x_i(y)D_i(U)$ is a bounded continuous function.

By the estimate (ii) in Lemma 3.2 we obtain

$$\left\|\int_N\left(U(y)-U(e)-\sum_{i=1}^{d}x_i(y)D_i(U)\right)\eta(\mathrm{d}y)\right\|$$
$$\le\frac{1}{2}\int_N\varphi(y)\,\eta(\mathrm{d}y)\sum_{i,j=1}^{d}\|D_i(U)D_j(U)\|\to 0$$

as $N\to\{e\}$; hence (7.4) holds.

In order to show (7.5) it is sufficient to prove that

$$\sum_{\ell=1}^{k_n}\int_N(x_i(y)-x_i(m_{n,\ell}))(x_j(y)-x_j(m_{n,\ell}))\,\mu_{n,\ell}(\mathrm{d}y)$$

$$\to b_{i,j} + \int_N x_i(y)x_j(y)\,\eta(\mathrm{d}y), \tag{7.12}$$

$$\sum_{\ell=1}^{k_n} \int_N (x_i(y) - x_i(m_{n,\ell}))(x_j(y) - x_j(m_{n,\ell})) \\ \times \big(\partial_i\partial_j R_{m_{n,\ell}^{-1}} U(m_{n,\ell}) - \partial_i\partial_j U(e)\big)\mu_{n,\ell}(\mathrm{d}y) \to 0 \tag{7.13}$$

as $n \to \infty$, since

$$\partial_i\partial_j U(e) = \frac{1}{2}\big(D_i(U)D_j(U) + D_j(U)D_i(U)\big).$$

Indeed, by Lemma 3.1 we have

$$\begin{aligned}\partial_j U(\exp_G(tD_i)) &= \frac{\mathrm{d}}{\mathrm{d}h}\bigg|_{h=0} U(\exp_G(hD_j + tD_i)) \\ &= \frac{\mathrm{d}}{\mathrm{d}h}\bigg|_{h=0} \exp\big(hD_j(U) + tD_i(U)\big) = \frac{\mathrm{d}}{\mathrm{d}h}\bigg|_{h=0} \sum_{k=0}^{\infty} \frac{\big(hD_j(U) + tD_i(U)\big)^k}{k!} \\ &= \sum_{k=1}^{\infty} \frac{t^{k-1}}{k!} \sum_{r=0}^{k-1} D_i(U)^r D_j(U) D_i(U)^{k-1-r},\end{aligned}$$

implying

$$\partial_i\partial_j U(e) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \partial_j U(\exp_G(tD_i)) = \frac{1}{2}\big(D_i(U)D_j(U) + D_j(U)D_i(U)\big).$$

Convergence (7.12) will follow from

$$\sum_{\ell=1}^{k_n} \int_{G\setminus N} (x_i(y) - x_i(m_{n,\ell}))(x_j(y) - x_j(m_{n,\ell}))\,\mu_{n,\ell}(\mathrm{d}y) \\ \to \int_{G\setminus N} x_i(y)x_j(y)\,\eta(\mathrm{d}y) \tag{7.14}$$

and assumption (i). Assumption (ii) implies

$$\sum_{\ell=1}^{k_n} \int_{G\setminus N} x_i(y)x_j(y)\,\mu_{n,\ell}(\mathrm{d}y) \to \int_{G\setminus N} x_i(y)x_j(y)\,\eta(\mathrm{d}y) \quad \text{as } n \to \infty.$$

By (6.1) and assumption (ii),

$$\sum_{\ell=1}^{k_n} \int_{G\setminus N} \big(x_i(m_{n,\ell})x_j(m_{n,\ell}) - x_i(y)x_j(m_{n,\ell}) - x_j(y)x_i(m_{n,\ell})\big)\, \mu_{n,\ell}(\mathrm{d}y) \to 0,$$

and we conclude (7.14). Convergence (7.13) follows from (7.12) and

$$\max_{1\leq \ell\leq k_n} \|\partial_i\partial_j R_{m_{n,\ell}^{-1}} U(m_{n,\ell}) - \partial_i\partial_j U(e)\| \to 0 \quad \text{as } n\to\infty$$

(which is again a consequence of the uniform continuity of the mapping $y \mapsto \partial_i\partial_j R_{y^{-1}}U(y)$ on N_0 and the uniform convergence $m_{n,\ell} \to e$ as $n \to \infty$); hence (7.5) holds.

We have

$$\left|\int_N x_i(y)x_j(y)\,\eta(\mathrm{d}y)\right| \leq \int_N \varphi(y)\,\eta(\mathrm{d}y);$$

thus we get (7.6).

In order to show (7.7) it suffices to prove that

$$\begin{aligned} \lim_{n\to\infty} \sum_{\ell=1}^{k_n} \int_N &\big|(x_i(y) - x_i(m_{n,\ell}))(x_j(y) - x_j(m_{n,\ell}))\big| \int_0^1 \big|\partial_i\partial_j R_{m_{n,\ell}^{-1}} \\ &\times U(\lambda y + (1-\lambda)m_{n,\ell}) - \partial_i\partial_j U(\lambda y + (1-\lambda)e)\big|\,\mathrm{d}\lambda\,\mu_{n,\ell}(\mathrm{d}y) = 0, \end{aligned} \tag{7.15}$$

$$\begin{aligned} \lim_{n\to\infty} \sum_{\ell=1}^{k_n} \int_N &\big|(x_i(y) - x_i(m_{n,\ell}))(x_j(y) - x_j(m_{n,\ell}))\big| \\ &\times \big|\partial_i\partial_j R_{m_{n,\ell}^{-1}} U(m_{n,\ell}) - \partial_i\partial_j U(e)\big|\,\mu_{n,\ell}(\mathrm{d}y) = 0, \end{aligned} \tag{7.16}$$

$$\begin{aligned} \lim_{N\to\{e\}} \limsup_{n\to\infty} \sum_{\ell=1}^{k_n} \int_N &\big|(x_i(y) - x_i(m_{n,\ell}))(x_j(y) - x_j(m_{n,\ell}))\big| \\ &\times \int_0^1 \big|\partial_i\partial_j U(\lambda y + (1-\lambda)e) - \partial_i\partial_j U(e)\big|\,\mathrm{d}\lambda\,\mu_{n,\ell}(\mathrm{d}y) = 0. \end{aligned} \tag{7.17}$$

The convergences (7.15) and (7.16) follow from (7.12) and

$$\max_{1\leq\ell\leq k_n} \sup_{y\in N} \|\partial_i\partial_j R_{m_{n,\ell}^{-1}} U(\lambda y + (1-\lambda)m_{n,\ell}) - \partial_i\partial_j U(\lambda y + (1-\lambda)e)\| \to 0$$

as $n \to \infty$. Finally, (7.17) is a consequence of (7.12) and

$$\sup_{y\in N} \sup_{\lambda\in[0,1]} \|\partial_i\partial_j U(\lambda y + (1-\lambda)e) - \partial_i\partial_j U(e)\| \to 0 \quad \text{as } N \to \{e\},$$

which follows from the uniform continuity of the function $y \mapsto \partial_i\partial_j U(\lambda y+(1-\lambda)e)$ on N_0''. $\square$

Remark 7.2. If under the assumptions of Theorem 7.1

$$\|(\nu'_{n,1} * \cdots * \nu'_{n,k_n})^\frown(U) - (\nu'_n)^\frown(U)\| \to 0 \qquad \text{as} \quad n \to \infty \tag{7.18}$$

for all $U \in \mathrm{Irr}(G)$, where

$$\nu'_{n,\ell} := \exp(\mu'_{n,\ell} - \varepsilon_e), \qquad \nu'_n := \exp\left(\sum_{\ell=1}^{k_n} (\mu'_{n,\ell} - \varepsilon_e)\right),$$

then Theorems 7.1 and 6.4 would imply

$$\mu'_{n,1} * \cdots * \mu'_{n,k_n} \to \nu,$$

but it is not clear whether (7.18) holds. In fact, (7.18) is equivalent to

$$\|\mathrm{e}^{A_{n,1}} \cdots \mathrm{e}^{A_{n,k_n}} - \mathrm{e}^{A_{n,1}+\cdots+A_{n,k_n}}\| \to 0 \quad \text{as } n \to \infty,$$

where the matrices $A_{n,\ell}$, $n \in \mathbb{N}$, $\ell = 1, \dots, k_n$ are defined by

$$A_{n,\ell} := (\mu'_{n,\ell})^\frown(U) - I,$$

and we have the Taylor formula (7.2).

Theorem 7.3. *Let $\{\mu_{n,\ell} : n \in \mathbb{N},\ \ell = 1, \dots, k_n\}$ be an infinitesimal system of probability measures on G. Denote the local mean and the local covariance matrix of $\mu_{n,1}, \dots, \mu_{n,k_n}$ by $m_{n,1}, \dots, m_{n,k_n}$ and by $B_{n,1}, \dots, B_{n,k_n}$ (which exist for sufficiently large $n \in \mathbb{N}$ by Lemma 6.3). Suppose that there exists $(a, B, \eta) \in \mathcal{P}(G)$ such that*

(i) $\sum_{\ell=1}^{k_n} x_i(m_{n,\ell}) \to a_i$ *as* $n \to \infty$ *for all* $i = 1, \dots, d$,
(ii) $\sum_{\ell=1}^{k_n} B_{n,\ell} \to \widetilde{B}$ *as* $n \to \infty$,
(iii) $\sum_{\ell=1}^{k_n} \mu_{n,\ell}(G \setminus N) \to \eta(G \setminus N)$ *as* $n \to \infty$ *for all* $N \in \mathcal{N}(e)$ *with* $\eta(\partial N) = 0$,

where $\widetilde{B}$ is defined in (4.1). *Then*

$$\exp\left(\sum_{\ell=1}^{k_n} (\mu_{n,\ell} - \varepsilon_e)\right) \to \mu,$$

where μ is an embeddable probability measure on G with triplet (a, B, η).

Proof. The measure

$$\nu_n := \exp\left(\sum_{\ell=1}^{k_n}(\mu_{n,\ell} - \varepsilon_e)\right) belowdisplayskip = 0pt$$

is a Poisson measure with

$$\widehat{\nu}_n(U) = \exp\left(\sum_{\ell=1}^{k_n}\big(\widehat{\mu}_{n,\ell}(U) - I\big)\right)$$

for all $U \in \mathrm{Irr}(G)$. Moreover

$$\begin{aligned}\widehat{\mu}(U) = \exp\Bigg\{ &\sum_{i=1}^{d} a_i D_i(U) + \frac{1}{2}\sum_{i,j=1}^{d} b_{i,j} D_i(U) D_j(U) \\ &+ \int_G \left(U(y) - U(e) - \sum_{i=1}^{d} x_i(y) D_i(U)\right) \eta(\mathrm{d}y)\Bigg\}\end{aligned}$$

holds for all $U \in \mathrm{Irr}(G)$. Hence it is enough to show that

$$\begin{aligned}\sum_{\ell=1}^{k_n}\big(\widehat{\mu}_{n,\ell}(U) - I\big) \to &\sum_{i=1}^{d} a_i D_i(U) + \frac{1}{2}\sum_{i,j=1}^{d} b_{i,j} D_i(U) D_j(U) \\ &+ \int_G \left(U(y) - U(e) - \sum_{i=1}^{d} x_i(y) D_i(U)\right) \eta(\mathrm{d}y)\end{aligned} \tag{7.19}$$

as $n \to \infty$ for all $U \in \mathrm{Irr}(G)$. By the Taylor formula (5.1) with $m = e$ we obtain

$$\begin{aligned}\widehat{\mu}_{n,\ell}(U) - I = &\sum_{i=1}^{d} D_i(U) \int_G x_i(y)\, \mu_{n,\ell}(\mathrm{d}y) \\ &+ \frac{1}{2}\sum_{i,j=1}^{d} D_i(U) D_j(U) \int_N x_i(y) x_j(y)\, \mu_{n,\ell}(\mathrm{d}y) \\ &+ \int_{G\setminus N} \Big(U(y) - U(e) - \sum_{i=1}^{d} x_i(y) D_i(U)\Big) \mu_{n,\ell}(\mathrm{d}y) \\ &+ \int_N R(U, y, e)\, \mu_{n,\ell}(\mathrm{d}y)\end{aligned} \tag{7.20}$$

for each $N \in \mathcal{N}(e)$ with $N \subset N_0''$. Hence (7.19) follows from the following six limiting relationships:

$$\sum_{\ell=1}^{k_n} \int_{G\setminus N} \left(U(y) - U(e) - \sum_{i=1}^{d} x_i(y) D_i(U)\right) \mu_{n,\ell}(\mathrm{d}y)$$

$$\to \int_{G\setminus N} \left(U(y) - U(e) - \sum_{i=1}^{d} x_i(y) D_i(U) \right) \eta(\mathrm{d}y) \quad \text{as } n \to \infty, \tag{7.21}$$

$$\int_{N} \left(U(y) - U(e) - \sum_{i=1}^{d} x_i(y) D_i(U) \right) \eta(\mathrm{d}y) to 0 \quad \text{as } N \to \{e\}, \tag{7.22}$$

$$\sum_{\ell=1}^{k_n} \int_N x_i(y) x_j(y)\, \mu_{n,\ell}(\mathrm{d}y) \to b_{i,j} + \int_N x_i(y) x_j(y)\, \eta(\mathrm{d}y) \quad \text{as } n \to \infty, \tag{7.23}$$

$$\int_N x_i(y) x_j(y)\, \eta(\mathrm{d}y) \to 0 \quad \text{as } N \to \{e\}, \tag{7.24}$$

$$\limsup_{n\to\infty} \sum_{\ell=1}^{k_n} \int_N \|R(U,y,e)\|\, \mu_{n,\ell}(\mathrm{d}y) \to 0 \quad \text{as } N \to \{e\}, \tag{7.25}$$

$$\sum_{\ell=1}^{k_n} \int_G x_i(y)\, \mu_{n,\ell}(\mathrm{d}y) \to a_i \quad \text{as } n \to \infty, \tag{7.26}$$

where (7.21) and (7.23) are valid for $N \in \mathcal{N}(e)$ with $N \subset N_0''$ and $\eta(\partial N) = 0$.

Clearly (7.21), (7.22), (7.23) and (7.24) are the same as (7.11), (7.4), (7.12) and (7.6), respectively. Moreover, (7.25) can be proved similarly to (7.7). Finally, (7.26) follows from assumption (i). $\Box$

Remark 7.4. If under the assumptions of Theorem 7.3

$$\|(\nu'_{n,1} * \varepsilon_{m_{n,1}} * \cdots * \nu'_{n,k_n} * \varepsilon_{m_{n,k_n}})\widehat{\ }(U) - \widehat{\nu}_n(U)\| \to 0 \quad \text{as } n \to \infty \tag{7.27}$$

for all $U \in \operatorname{Irr}(G)$, where

$$\nu_{n,\ell} := \exp(\mu_{n,\ell} - \varepsilon_e), \qquad \nu_n := \exp\left(\sum_{\ell=1}^{k_n} (\mu_{n,\ell} - \varepsilon_e) \right),$$

then Theorems 7.3 and 6.4 would imply

$$\mu_{n,1} * \cdots * \mu_{n,k_n} \to \mu,$$

but it is not clear whether (7.27) holds. In fact, (7.27) can be written in the form

$$\left\| \mathrm{e}^{A_{n,1} C_{n,1}^{-1} - I} C_{n,1} \cdots \mathrm{e}^{A_{n,k_n} C_{n,k_n}^{-1} - I} C_{n,k_n} - \mathrm{e}^{\sum_{\ell=1}^{k_n} (A_{n,\ell} - I)} \right\| \to 0 \quad \text{as } n \to \infty,$$

where the matrices $A_{n,\ell}$, $C_{n,\ell}$, $n \in \mathbb{N}$, $\ell = 1, \dots, k_n$ are defined by

$$A_{n,\ell} := \widehat{\mu}_{n,\ell}(U), \qquad C_{n,\ell} := U(m_{n,\ell}),$$

and we have the Taylor formula (7.20).

References

[1] Ph. Feinsilver, Processes with independent increments on a Lie group, *Trans. Amer. Math. Soc.* **242** (1978), 73–121.

[2] E. Hewitt and K. A. Ross, Abstract Harmonic Analysis, Vol. 1, *Springer-Verlag*, *Berlin Göttingen, Heidelberg, New York*, 1963.

[3] H. Heyer, A central limit theorem for compact Lie groups, Papers from the 'Open House for Probabilists' (Mat. Inst., Aarhus Univ., Aarhus, 1971), Various Publ. Ser., No. 21 (Mat. Inst., Aarhus Univ., Aarhus, 1972), 101–117.

[4] H. Heyer, Infinitely divisible probability measures on compact groups, Lectures on Operator Algebras, Vol. 247, (dedicated to the memory of David M. Topping; Tulane Univ. Ring and Operator Theory Year, 1970–1971, Vol. II), Lecture Notes in Mathematics, *Springer-Verlag*, *Berlin*, 1972, 55–249.

[5] H. Heyer, Probability Measures on Locally Compact Groups, *Springer-Verlag*, *Berlin, Heidelberg, New York*, 1977.

[6] G. Pap, General solution of the functional central limit problems on a Lie group, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **7(1)** (2004), 43–87.

[7] K. R. Parthasarathy, The central limit theorem for the rotation group, *Theory Probab. Appl.* **9** (1964), 248–257.

[8] E. Siebert, A new proof of the generalized continuity theorem of Paul Lévy, *Math. Ann.* **233** (1978), 257–259.

[9] E. Siebert, Fourier analysis and limit theorems for convolution semigroups on a locally compact group, *Adv. Math.* **39** (1981), 111–154.

[10] E. Siebert, Continuous hemigroups of probability measures on a Lie group, Probability Measures on Groups, Vol. 928, (Proceedings, Oberwolfach 1981), Lecture Notes in Mathematics, (H. Heyer, ed.), *Springer-Verlag*, *Berlin, New York*, 1982, 362–402.

[11] V. S. Varadarajan, Lie Groups, Lie Algebras and Their Representations, *Prentice-Hall*, *New York*, 1974.

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