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Volatility estimation for different structures of random field interest rate models in discrete time

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Abstract. The general framework of the discrete time forward interest rate model considered in this paper is introduced by GÁLL, PAP and ZUIJLEN in [7]. This paper studies the maximum likelihood estimator of the volatility of forward interest rates driven by geometric spatial AR sheet and considers its asymptotic behaviour, as is done in GÁLL, PAP and ZUIJLEN in [6]. However, we consider the case of a *non*-constant volatility to derive new asymptotic results for far more general structures.

1. The model and the no-arbitrage criterion

In the following we summarise the basics of the model. We will consider discrete time forward interest rate models, that are driven by random fields, see [7]. Note that they are based on an idea of HEATH, JARROW and MORTON [11]. Let $f(k,\ell)$ denote the forward interest rate at time k with time to maturity ℓ, where $k,\ell \in \mathbb{Z}_+$ (where $\mathbb{Z}_+ := \{x \in \mathbb{Z} \mid x \geq 0\}$). This means that we follow the MUSIELA parametrization (see [13]), where ℓ denotes the time *to* maturity. Based on this, the price $P(k,\ell)$ of a zero coupon bond at time k with maturity date ℓ is defined in a recursive way by $P(k,k) := 1$ and

$$P(k,\ell+1) = P(k,\ell)\exp(-f(k,\,\ell-k)), \quad k,\ell \in \mathbb{Z}_+ \text{ with } k \leq \ell. \tag{1.1}$$

The forward rate dynamics in this paper are of the form

$$f(k+1,\ell) = f(k,\ell) + \alpha(k,\ell) + \beta(k,\ell)(S(k+1,\ell) - S(k,\ell)), \tag{1.2}$$

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where $\{S(k,\ell)\}_{k,\ell\in\mathbb{Z}_+}$ is a random field and $\{S(k,\ell)\}_{k\in\mathbb{Z}_+}$, $\{\alpha(k,\ell)\}_{k\in\mathbb{Z}_+}$, $\{\beta(k,\ell)\}_{k\in\mathbb{Z}_+}$ are all adapted to a certain filtration $\{\mathcal{F}_k\}_{k\in\mathbb{Z}_+}$ for each $\ell\in\mathbb{Z}_+$. In another form

$$f(k+1,\ell)=f(0,\ell)+\sum_{i=0}^{k}\alpha(i,\ell)+\sum_{i=0}^{k}\beta(i,\ell)(S(i+1,\ell)-S(i,\ell)),\quad k,\ell\in\mathbb{Z}_+, \tag{1.3}$$

with initial values $f(0,\ell)\in\mathbb{R}$, $\ell\in\mathbb{Z}_+$. In this paper we will study different structures of the volatility β.

The key feature of the model is that the forward rates corresponding to different time to maturity values can be driven by different discrete time processes, that is, the forward rates are driven by a random field. Hence, different market 'shocks' may impact at different forward rate processes. Such a generalisation of the classical HJM type models has been proposed by Kennedy [12] in the continuous case, and studied by e.g. Goldstein [9] and Santa-Clara and Sornette [15]. For a further discussion of the model one can consult Gáll, Pap and Zuijlen [7] and we refer to [5] for results on the limiting connection of such discrete and continuous models.

In this specific paper the forward rates corresponding to different times to maturity are driven by a Gaussian type of random field, which is built up by a system $\{\eta(i,j)\mid i,j\in\mathbb{Z}_+\}$ of i.i.d. Gaussian random variables with mean zero and variance one on a probability space $(\Omega,\mathcal{F},\mathsf{P})$. Suppose that the filtration $\mathcal{F}_k$ is defined by $\mathcal{F}_k:=\sigma\big(\eta(i,j)\mid 0\le i\le k,\ j\in\mathbb{Z}_+\big)$, $k\in\mathbb{Z}_+$. Consider the doubly geometric spatial autoregressive process $\{S(k,\ell)\mid k,\ell\in\mathbb{Z}_+\}$ generated by

$$\begin{cases} S(k,\ell)=S(k-1,\ell)+\varrho S(k,\ell-1)\\ \qquad\qquad -\varrho S(k-1,\ell-1)+\eta(k,\ell), \qquad k,\ell\in\mathbb{Z}_+,\\ S(k,-1)=S(-1,\ell)=0, \end{cases} \tag{1.4}$$

where $\varrho\in\mathbb{R}$. Gáll, Pap and Zuijlen [5] have shown limit cases where Ornstein–Uhlenbeck sheets occurred in the continuous time counterpart of discrete time autoregressive type of forward rate models.

Clearly

$$S(k,\ell)=\sum_{i=0}^{k}\sum_{j=0}^{\ell}\varrho^{\ell-j}\eta(i,j), \tag{1.5}$$

and hence

$$\Delta_1 S(k,\ell)=\sum_{j=0}^{\ell}\varrho^{\ell-j}\eta(k+1,j), \tag{1.6}$$

where

$$\Delta_1 S(i,\ell) := S(i+1,\ell) - S(i,\ell), \quad i,\ell \in \mathbb{Z}_+. \tag{1.7}$$

We suppose the existence of a stochastic discount factor process $\{M(k) \mid k \in \mathbb{Z}_+\}$ in the market that is given by $M(0) := 1$ and

$$M(k+1) = M(k)\frac{\exp\{-f(k,0) + \sum_{j=0}^{\infty} \phi_j \Delta_1 S(k,j)\}}{\mathbb{E}(\exp\{\sum_{j=0}^{\infty} \phi_j \Delta_1 S(k,j)\} \mid \mathcal{F}_k)}, \quad k \in \mathbb{Z}_+, \tag{1.8}$$

where the factors $\phi_j \in \mathbb{R}$, $j \in \mathbb{Z}_+$, will be called *market prices of risk*. These market prices of risk play an important role in the market for the determination of the market prices of assets. Furthermore, we suppose that $\sum_{j=0}^{\infty} \phi_j \Delta_1 S(k,j)$ is stochastically convergent. The discount factor $M(k+1)$ given by equation (1.8) does not only discount by the current interest rate $f(k,0)$, but with the inclusion of $\sum_{j=0}^{\infty} \phi_j \Delta_1 S(k,j)$ it also takes into account the reaction of the market to the shocks corresponding to time k. It can be seen easily that $\mathbb{E}(M(k+1) \mid \mathcal{F}_k) = \exp(-f(k,0))M(k)$ for all $k \in \mathbb{Z}_+$. We refer to [7] for more details on the choice of the special form of the stochastic discount factors and to [7], [15] and [1] for more on the role of the market price of risk.

As is natural in financial mathematics, we are interested only in models where arbitrage opportunities are excluded in the market. We assume that the $M(k)$-discounted bond price processes $\{M(k)P(k,\ell)\}_{0\leqslant k\leqslant \ell}$ are P-martingales for all $\ell \in \mathbb{Z}_+$. Gáll, Pap and Zuijlen [7] showed that arbitrage is excluded for this case. They also found that under the assumption that the common distribution of $\eta(i,j)$, $i,j \in \mathbb{Z}_+$, is the standard normal distribution, the no-ar criterion implies

$$\begin{aligned} f(k,\ell+1) = f(k,\ell) + \alpha(k,\ell) - \frac{1}{2}\beta(k,\ell)^2 c(\ell,\ell) - \beta(k,\ell)\sum_{j=0}^{\ell-1}\beta(k,j)c(\ell,j) \\ + \beta(k,\ell)\sum_{j=0}^{\infty}\phi_j c(\ell,j), \quad k,\ell \in \mathbb{Z}_+, \end{aligned} \tag{1.9}$$

where the covariances are

$$c(\ell_1,\ell_2) := \operatorname{cov}(\Delta_1 S(k,\ell_1), \Delta_1 S(k,\ell_2)) = \sum_{j=0}^{\ell_1\wedge\ell_2} \varrho^{\ell_1+\ell_2-2j}, \quad \ell_1,\ell_2 \in \mathbb{Z}_+. \tag{1.10}$$

Notice that they do not depend on k. Together with (1.2), we can obtain for all $k,\ell \in \mathbb{Z}_+$

$$\begin{aligned} f(k+1,\ell) - f(k,\ell+1) = \frac{1}{2}\beta(k,\ell)^2 c(\ell,\ell) + \beta(k,\ell)\sum_{j=0}^{\ell-1}\beta(k,j)c(\ell,j) \\ + \beta(k,\ell)\Delta_1 S(k,\ell) - \beta(k,\ell)\sum_{j=0}^{\infty}\phi_j c(\ell,j). \end{aligned} \tag{1.11}$$

Note that $\alpha(k, \ell)$ does not appear in equation (1.11) like in the well known drift conditions in interest rate models.

First, we will study volatility estimation in the 'martingale' case. For this 'martingale' case, we assume that the market prices of risk ϕ_j, $j \in \mathbb{Z}_+$, are equal to zero, which means that the real measure of the market is a martingale measure. Here the terminology, for e.g. 'martingale' case, is the same as in Föllmer and Sondermann [3], Föllmer and Schweizer [2]. Under the no-arbitrage criterion, we will find the maximum likelihood estimator of the volatility parameter defined below (Section 2) and consider its asymptotic behaviour (Section 3), both for the 'martingale' case. In Section 4 we will see that the results obtained for the 'martingale' case can be easily generalised to the general model as described above.

In [6] the authors considered a constant volatility, which we do not find very realistic. The choice of the structure of the non-constant volatility as introduced in Section 2, makes it possible to consider more complex and more dynamic models for forward interest rates. In Section 3 we will first create a general framework for the consideration of the asymptotic behaviour and then derive results for special structures on the volatility, given different sampling and different autoregressive features. In [14] more asymptotic results for the volatility are stated for different structures on the different parameters and the sample size. Not all cases may be realistic but our objective was to develop the tools for the asymptotic behaviour and to study the differences e.g. in convergence. Recently we have been doing emperical tests and we studied goodness of fit and model selection problems together with the testing of selection criterion (Akaike and modifications). For the latter part we study certain cases of the general model as proposed by Gáll, Pap and Zuijlen in [7]. However, we do not study the classical HJM model (see e.g. [10]) because in Remark 2.3 it is stated as well as in Gáll, Pap and Zuijlen [6], that the classical HJM model is not the most realistic model. For a similar model to the one discussed in Section 4, Gáll, Pap and Zuijlen [8] considered the joint estimation of all parameters for a constant volatility. In later work we plan to consider the estimation of all parameters for a *non*-constant volatility.

2. ML estimation in the 'martingale' case

In Sections 2 and 3 we assume that the market prices of risk ϕ_j, $j \in \mathbb{Z}_+$, are equal to zero, the stochastic discount factor process is given by $M(0) := 1$ and

$$M(k+1) = \mathsf{e}^{-f(k,0)} M(k), \quad k \in \mathbb{Z}_+. \tag{2.1}$$

Then, assuming that the common distribution of $\eta(i,j)$, $i,j \in \mathbb{Z}_+$, is standard normal, (1.9) simplifies to

$$f(k,\ell+1) = f(k,\ell) + \alpha(k,\ell) - \frac{1}{2}\beta(k,\ell)^2 c(\ell,\ell) - \beta(k,\ell)\sum_{j=0}^{\ell-1}\beta(k,j)c(\ell,j). \quad (2.2)$$

Recalling (1.11), we obtain for all $k,\ell \in \mathbb{Z}_+$

$$\begin{aligned} f(k+1,\ell) - f(k,\ell+1) = \frac{1}{2}\beta(k,\ell)^2 c(\ell,\ell) + \beta(k,\ell)\sum_{j=0}^{\ell-1}\beta(k,j)c(\ell,j) \\ + \beta(k,\ell)\Delta_1 S(k,\ell). \end{aligned} \quad (2.3)$$

Assume that there exists a sequence $\{a_i\}_{i=1}^{\infty} \in \mathbb{R}$ where $a_i \neq 0$ $\forall i \in \mathbb{N}^*$ (the set of positive integers) and a volatility parameter $\beta \in \mathbb{R}\backslash\{0\}$ such that

$$\begin{aligned} &\beta(k,0) = \beta && k \in \mathbb{Z}_+; \\ &\beta(k,\ell+1) = a_{\ell+1}\beta(k,\ell) && k,\ell \in \mathbb{Z}_+. \end{aligned} \quad (2.4)$$

This structure on the volatility is chosen in a way that volatilities corresponding to different times to maturity might have different values and thus different impacts on the forward rates. In the lemma below, we will obtain an explicit expression for the maximum likelihood estimator of β^2, that is based on a sample of forward rates.

Lemma 2.1. *Consider a forward interest rate curve model* $\{f(k,\ell) \mid k,\ell \in \mathbb{Z}_+\}$ *as given in* (1.3). *Let K and L be positive integers. Assume that the parameters ϱ and $\{a_i\}_{i=1}^{\infty}$ are known, $a_i \neq 0$ and $a_L = a_{L+j}$ for all $j \in \mathbb{N}^*$. Then, under the assumption that equation* (2.2) *holds, the maximum likelihood estimator $\widehat{\beta^2}_{K,L}$ of β^2 based on the sample of forward rates*

$$\{f(k,\ell) \mid 1 \le k \le K,\ 0 \le \ell \le L\} \quad (2.5)$$

is given by

$$\widehat{\beta^2}_{K,L} := \frac{-B_{K,L} + \sqrt{B_{K,L}^2 + 4A_{K,L}C_{K,L}}}{2A_{K,L}}, \quad (2.6)$$

where

$$\begin{aligned} A_{K,L} &:= \frac{K}{4} + K\sum_{\ell=1}^{L-1}\frac{\theta^2(\ell)}{\tau(\ell)} + \sum_{k=1}^{K}\frac{\bar{\theta}^2(k,L)}{\bar{\tau}(k,L)}, \\ B_{K,L} &:= K(L+1), \\ C_{K,L} &:= \sum_{k=1}^{K}\left[y_{k,0}^2 + \sum_{\ell=1}^{L-1}\frac{y_{k,\ell}^2}{\tau(\ell)} + \frac{\tilde{y}_{k,L}^2}{\bar{\tau}(k,L)}\right], \end{aligned} \quad (2.7)$$

with

$$y_{k,\ell} := \begin{cases} f(k,\ell) - f(k-1,\ell+1) - \varrho a_\ell(f(k,\ell-1) - f(k-1,\ell)) & k,\ell \geq 1 \\ f(k,0) - f(k-1,1) & k \geq 1,\ \ell = 0, \end{cases}$$

$$\begin{aligned} \tilde{y}_{k,L} := f(k,L) - f(0,L+k) - \varrho a_L(f(k,L-1) \\ - f(0,L+k-1)) \qquad k \geq 1, \end{aligned} \tag{2.8}$$

and for all k, ℓ

$$\theta(\ell) := \frac{1}{2}\left(\prod_{i=1}^{\ell-1} a_i\right)^2 a_l(a_l+\varrho)\sum_{j=0}^{\ell-1}\varrho^{2j} + \frac{1}{2}\left(\prod_{i=1}^{\ell} a_i\right)^2 \varrho^{2\ell}, \qquad \tau(\ell) := \prod_{i=1}^{\ell} a_i^2,$$

$$\bar{\theta}(k,L) := \sum_{j=0}^{k-1}\theta(L+j) \qquad \textit{and} \qquad \bar{\tau}(k,L) := \sum_{j=0}^{k-1}\tau(L+j). \tag{2.9}$$

For $L = 1$ we define the empty sum to be zero and hence $A_{K,L} = \frac{K}{4} + \sum_{k=1}^{K} \frac{\bar{\theta}^2(k,L)}{\bar{\tau}(k,L)}$. A similar covention takes place for $C_{K,L}$ and $\theta(1)$.

Proof. The aim of the following discussion is to find the joint density of $\{f(k,\ell) \mid 1 \leq k \leq K,\ 0 \leq \ell \leq L\}$. We will find the conditional expectations of the forward rates in three steps for the case where $L > 1$ (if $L = 1$ then the proof is about the same). With these conditional expectations we can find the joint density. The discussion is similar to the one for Lemma 3.1 in [6]. With the help of (2.3), we obtain

(i) $f(k+1,0) - f(k,1) = \frac{1}{2}\beta^2 + \beta\eta(k+1,0)$ and hence $(\forall k \geq 0)$ the conditional distribution of $f(k+1,0)$, given $f(k,1)$, is a normal distribution with mean $f(k,1) + \frac{1}{2}\beta^2$ and variance β^2.

(ii) through simple calculus

$$\begin{aligned} &\frac{f(k+1,\ell) - f(k,\ell+1)}{\beta(k,\ell)} - \varrho\left(\frac{f(k+1,\ell-1) - f(k,\ell)}{\beta(k,\ell-1)}\right) \\ &= \frac{1}{2}\beta\left(\prod_{i=1}^{\ell-1} a_i\right)(a_l+\varrho)\sum_{j=0}^{\ell-1}\varrho^{2j} + \frac{1}{2}\beta\left(\prod_{i=1}^{\ell} a_i\right)\varrho^{2\ell} + \eta(k+1,\ell). \end{aligned}$$

From this it can be seen easily that the conditional distribution of $f(k+1,\ell)$, given $f(k,\ell+1)$, $f(k+1,\ell-1)$ and $f(k,\ell)$, is a normal distribution with mean

$$f(k,\ell+1) + \varrho a_l(f(k+1,\ell-1) - f(k,\ell)) + \beta^2\theta(\ell) \tag{2.10}$$

and variance $\beta^2\tau(\ell)$ for all $k \geq 0$ and $\ell \geq 1$, with θ, τ as defined in (2.9).

(iii) Because (using (ii) for k replaced by $k-1$ and taking $\ell = L$) $f(k, L) = f(k-1, L+1) + \varrho a_L(f(k, L-1) - f(k-1, L)) + \beta^2\theta(L) + \beta\sqrt{\tau(L)}\eta(k, L)$ and using the fact that $a_L = a_{L+j}$ for all $j \in \mathbb{N}^*$ k times, we see that

$$\begin{aligned} f(k, L) &= f(0, L+k) + \varrho a_L(f(k, L-1) - f(0, L+k-1)) \\ &\quad + \beta^2 \sum_{j=0}^{k-1} \theta(L+j) + \beta \sum_{j=0}^{k-1} \sqrt{\tau(L+j)}\eta(k-j, L+j). \end{aligned} \tag{2.11}$$

From this one can easily see that the conditional distribution of $f(k+1, L)$, given $f(0, L+k+1)$, $f(k+1, L-1)$ and $f(0, L+k)$, is a normal distribution with mean

$$f(0, L+k+1) + \varrho a_L(f(k+1, L-1) - f(0, L+k)) + \beta^2\bar{\theta}(k+1, L) \tag{2.12}$$

and variance $\beta^2\bar{\tau}(k+1, L)$ for all $k \geq 0$ and $L \geq 1$, with $\bar{\theta}, \bar{\tau}$ as defined in (2.9).

Now we can find the joint density of the sample by using the distribution derived at the first step for $\{f(k, 0) \mid 1 \leq k \leq K\}$, the second step for $\{f(k, \ell) \mid 1 \leq k \leq K, 1 \leq \ell \leq L-1\}$ and the third step for $\{f(k, L) \mid 1 \leq k \leq K\}$. By the independence of $\{\eta(i, j) \mid i, j \in \mathbb{Z}_+\}$ and the chain rule for conditional distributions, we obtain that the joint density of $\{f(k, \ell) \mid 1 \leq k \leq K, 0 \leq \ell \leq L\}$ has the form

$$\begin{aligned} g(\beta^2) &:= \left(\prod_{\ell=1}^{L-1} (2\pi\beta^2\tau(\ell))^{-\frac{K}{2}}\right)(2\pi\beta^2)^{-\frac{K}{2}}\left(\prod_{k=1}^{K}(2\pi\beta^2\bar{\tau}(k, L))^{-\frac{1}{2}}\right) \\ \exp&\left[-\sum_{k=1}^{K}\sum_{\ell=1}^{L-1}\frac{(y_{k,\ell} - \beta^2\theta(\ell))^2}{2\beta^2\tau(\ell)} - \sum_{k=1}^{K}\frac{(y_{k,0} - \frac{1}{2}\beta^2)^2}{2\beta^2} - \sum_{k=1}^{K}\frac{(\tilde{y}_{k,L} - \beta^2\bar{\theta}(k, L))^2}{2\beta^2\bar{\tau}(k, L)}\right], \end{aligned}$$

where $y_{k,\ell}$ and $\tilde{y}_{k,L}$ are defined in (2.8). The above implies that the maximum likelihood estimator $\widehat{\beta^2}_{K,L}$ of β^2 can be obtained by minimizing

$$\begin{aligned} h(\beta^2) &:= K(L+1)\log(\beta^2) + \sum_{k=1}^{K}\sum_{\ell=1}^{L-1}\frac{(y_{k,\ell} - \beta^2\theta(\ell))^2}{\beta^2\tau(\ell)} \\ &\quad + \sum_{k=1}^{K}\frac{(y_{k,0} - \frac{1}{2}\beta^2)^2}{\beta^2} + \sum_{k=1}^{K}\frac{(\tilde{y}_{k,L} - \beta^2\bar{\theta}(k, L))^2}{\beta^2\bar{\tau}(k, L)}. \end{aligned} \tag{2.13}$$

Because $\beta^4\frac{\delta h(\beta^2)}{\delta\beta^2} = A_{K,L}\beta^4 + B_{K,L}\beta^2 - C_{K,L}$, where $A_{K,L}$, $B_{K,L}$ and $C_{K,L}$ defined as in (2.7), we see that (the positive root) $\widehat{\beta^2}_{K,L}$ is defined as in equation (2.6). $\square$

Remark 2.2. If $a_i = a$ for all $i \in \mathbb{N}^*$, then $\beta(k,\ell) = a^\ell \beta$ (for all k,ℓ) and (2.9) can be simplified to

$$\theta(\ell) := \frac{1}{2} a^{2\ell-1}(a+\varrho)\sum_{j=0}^{\ell-1} \varrho^{2j} + \frac{1}{2} a^{2\ell}\varrho^{2\ell} \qquad \text{and} \qquad \tau(\ell) := a^{2\ell}. \tag{2.14}$$

If we also have $a_i = 1$ for all $i \in \mathbb{N}^*$, then $\beta(k,\ell) = \beta(k,0) = \beta$ is a constant (for all k,ℓ) and (2.9) can be simplified even more to $\theta(\ell) := \frac{1}{2}\sum_{i=0}^{2\ell} \varrho^i$ and $\tau(\ell) := 1$.

Remark 2.3. A discrete time version of a classical HJM model can be obtained replacing in (1.2) the autoregressive process $\{S(k,\ell) \mid k,\ell \in \mathbb{Z}_+\}$ defined by equation (1.4) by

$$\begin{cases} S(k) = S(k-1) + \eta(k) = \sum_{i=0}^{k} \eta(i), \\ S(-1) = 0, \end{cases} \qquad k \in \mathbb{Z}_+, \tag{2.15}$$

where $\{\eta(i) \mid i \in \mathbb{Z}_+\}$ is a system of independent standard normally distributed random variables on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. For our martingale case Gáll, Pap and Zuijlen [7] have proved that the no-arbitrage criterion implies

$$f(k,\ell+1) = f(k,0) + \sum_{j=0}^{\ell} \alpha(k,j) - \frac{1}{2}\left(\sum_{j=0}^{\ell} \beta(k,j)\right)^2, \qquad k,\ell \in \mathbb{Z}_+. \tag{2.16}$$

Defining

$$\theta(\ell) := \left(\prod_{i=1}^{\ell} a_i\right) \sum_{j=0}^{\ell-1}\left(\prod_{i=1}^{j} a_i\right) + \frac{1}{2}\left(\prod_{i=1}^{\ell} a_i\right)^2, \qquad \ell \geq 0, \tag{2.17}$$

($\bar{\theta}(k,\ell)$, $\tau(\ell)$ and $\bar{\tau}(k,\ell)$ are the same as in (2.9) for all $k,\ell \in \mathbb{Z}_+$), it can be seen that for all $k,\ell \in \mathbb{Z}_+$

(i) $f(k,0) = f(k-1,1) + \frac{\beta^2}{2} + \beta\eta(k)$ for all k, and hence

$$\eta(k) = \frac{f(k,0) - f(k-1,1) - \frac{\beta^2}{2}}{\beta}.$$

(ii)
$$\begin{aligned} f(k,\ell) &= f(k-1,\ell+1) + \beta^2\theta(\ell) + \beta\sqrt{\tau(\ell)}\eta(k) \\ &= f(k-1,\ell+1) + \beta^2\theta(\ell) + \sqrt{\tau(\ell)}\Big(f(k,0) - f(k-1,1) - \tfrac{\beta^2}{2}\Big). \end{aligned}$$

Hence, if the sample $\{f(k,\ell) \mid 1 \leq k \leq K,\ 0 \leq \ell \leq 1\}$ is known, then we can determine $f(k,\ell)$ for all $k,\ell \in \mathbb{Z}_+$. This means that if we obtain more data, we do not necessarily obtain more information. Looking back we see that all this is possible because for fixed k the same 'shocks' have effect to all forward rates $f(k,\ell)$, where $\ell \in \mathbb{Z}_+$. As is discussed in the introduction of [6], applying the same 'shocks' to all forward rates seems not to be very realistic.

3. Asymptotic behaviour of the volatility estimator

Consider a sequence of discrete-time forward interest rate curve models $\{f_n(k,\ell) \mid k,\ell \in \mathbb{Z}_+\}$, $n \in \mathbb{Z}_+$, with initial values $\{f_n(0,\ell) \mid \ell \in \mathbb{Z}_+\}$, with coefficients $\{\alpha_n(k,\ell), \beta_n(k,\ell) \mid k,\ell \in \mathbb{Z}_+\}$, and with driving process $\{S_n(k,\ell) \mid k,\ell \in \mathbb{Z}_+\}$ with autoregression parameter ϱ_n. Suppose that the common distribution of $\{\eta_n(i,j) \mid i,j \in \mathbb{Z}_+\}$ is the standard normal distribution for each model $\{f_n(k,\ell) \mid k,\ell \in \mathbb{Z}_+\}$, $n \in \mathbb{N}^*$, and the no-arbitrage criterion (2.2) is satisfied in the models.

Assume that, for every $n \in \mathbb{N}^*$, there exists $a_n, \beta_n \in \mathbb{R}$, $a_n, \beta_n \neq 0$, such that

$$\beta_n(k,\ell) = a_n^\ell \beta_n \quad k,\ell \in \mathbb{Z}_+. \tag{3.1}$$

Note that for fixed n, (3.1) is a special case of (2.4). In this special case we will consider the asymptotic behaviour of the maximum likelihood estimator of β_n^2 and after this we will briefly comment on the case where $\beta_n(k,\ell)$ is dependent on the time to maturity as in (2.4).

We will often consider variables that do not only depend on K and L, but also depend on n. In this case we will simply use the subscript n, for which it will be clear which value of K and L we have taken. E.g. the sample size B_n for a sample $\{f_n(k,\ell) \mid 1 \le k \le K_n,\ 0 \le \ell \le L_n\}$ is defined as $B_n := B_{K_n,L_n} = K_n(L_n+1)$ (see (2.7)). Furthermore, due to (3.1), $\theta_n(\ell)$ and $\tau_n(\ell)$ are defined similar as in (2.14).

In Theorem 3.1 we will provide the general framework for the consideration of the asymptotic behaviour of the maximum likelihood estimator of β_n^2. The theorem tells us that if K_n goes to infinity (as $n \to \infty$) and at least one out of two technical assumptions is satisfied, then we have that the maximum likelihood estimator $\widehat{\beta_n^2}$ of β_n^2 converges to β_n^2. For the second technical assumption we define for all $n \in \mathbb{N}^*$ (compare this definition with the definition of $A_{K,L}$ as stated in (2.7))

$$\tilde{A}_n := \frac{K_n}{4} + K_n \sum_{\ell=1}^{L_n-1} \frac{\theta_n^4(\ell)}{\tau_n^2(\ell)} + \sum_{k=1}^{K_n} \frac{\bar{\theta}_n^4(k,L_n)}{\bar{\tau}_n^2(k,L_n)}. \tag{3.2}$$

Theorem 3.1. *If*

$$K_n \to \infty \tag{3.3}$$

as $n \to \infty$ and

$$\sup_{n\in\mathbb{N}} \frac{B_n}{|\beta_n|\sqrt{A_n}} < \infty \tag{3.4}$$

or

$$\lim_{n\to\infty} \frac{\beta_n^4 \tilde{A}_n}{(B_n + 2A_n\beta_n^2)^2} = 0, \tag{3.5}$$

then

$$\sqrt{\frac{B_n}{2\beta_n^4} + \frac{A_n}{\beta_n^2}}(\widehat{\beta_n^2} - \beta_n^2) \xrightarrow{D} N(0,1) \tag{3.6}$$

as $n \to \infty$.

Proof. Theorem 3.1 is a consequence of the following two statements:
Statement 1: If (3.3) and (3.4) are satisfied, then statement (3.6) is valid as $n \to \infty$.
Proof of statement 1: We have (see (2.6))

$$\widehat{\beta^2}_{K,L} - \beta^2 = \frac{2(C_{K,L} - \beta^4 A_{K,L} - \beta^2 B_{K,L})}{B_{K,L} + 2\beta^2 A_{K,L} + \sqrt{B_{K,L}^2 + 4A_{K,L}C_{K,L}}}. \tag{3.7}$$

Defining $\nu(k,\ell) := a^\ell \eta(k,\ell)$ and $\bar{\nu}(k,L) := \sum_{j=0}^{k-1} \nu(k-j, L+j)$ for all $1 \le k \le K$ and $1 \le \ell \le L-1$, we see that (see the proof of Lemma 2.1)

$$\begin{aligned}
C_{K,L} - \beta^4 A_{K,L} - \beta^2 B_{K,L} &= \Bigg[\sum_{k=0}^{K-1} \Big(\big(\beta^2\bar{\theta}(k+1,L) + \beta\bar{\nu}(k+1,L)\big)^2 \bar{\tau}^{-1}(k+1,L) \\
&\quad + \sum_{\ell=1}^{L-1} (\beta^2\theta(\ell) + \beta\nu(k+1,\ell))^2 \tau^{-1}(\ell)\Big) + \left(\frac{1}{2}\beta^2 + \beta\eta(k+1,0)\right)^2\Bigg] \\
&\quad - \beta^4 \left[\frac{K}{4} + K\sum_{\ell=1}^{L-1} \frac{\theta^2(\ell)}{\tau(\ell)} + \sum_{k=0}^{K-1} \frac{\bar{\theta}^2(k+1,L)}{\bar{\tau}(k+1,L)}\right] - \beta^2 K(L+1) \\
&= \beta^2 \sum_{k=0}^{K-1} \left(\left[(\eta^2(k+1,0) - 1) + \sum_{\ell=1}^{L-1} \left(\frac{\nu^2(k+1,\ell)}{\tau(\ell)} - 1\right)\right] + \left(\frac{\bar{\nu}^2(k+1,L)}{\bar{\tau}(k+1,L)} - 1\right)\right) \\
&\quad + \beta^3 \sum_{k=0}^{K-1} \left(\left[\eta(k+1,0) + \sum_{\ell=1}^{L-1} \frac{2\theta(\ell)\nu(k+1,\ell)}{\tau(\ell)}\right] + \frac{2\bar{\theta}(k+1,L)\bar{\nu}(k+1,L)}{\bar{\tau}(k+1,L)}\right).
\end{aligned}$$

(Notice that $\mathbb{E}(\nu^2(k,\ell)) = \tau(\ell)$ and that $\mathbb{E}(\bar{\nu}^2(k,L)) = \bar{\tau}(k,L)$ for all $1 \le \ell \le L-1$, $1 \le k \le K$.) For $K = K_n$ and $L = L_n$, we will consider the asymptotic behaviour of $C_{K,L} - \beta^4 A_{K,L} - \beta^2 B_{K,L}$. Applying the standard normality of $\eta_n(i,j)$, $i,j \in \mathbb{Z}_+$, the fact that they are independent and the Law of Large Numbers, we see that if $n \to \infty$, then

$$\frac{1}{K_n L_n} \sum_{k=0}^{K_n-1} \left[(\eta_n^2(k+1,0) - 1) + \sum_{\ell=1}^{L_n-1} \left(\frac{\nu_n^2(k+1,\ell)}{\tau_n(\ell)} - 1\right)\right] \to 0, \qquad \mathbb{P}\text{-a.s.}$$

$$\frac{1}{K_n}\sum_{k=0}^{K_n-1}\left(\frac{\bar{\nu}_n^2(k+1,L_n)}{\bar{\tau}_n(k+1,L_n)}-1\right)\to 0, \qquad \mathbb{P}\text{-a.s.}$$

$$\sum_{k=0}^{K_n-1}\left[\eta_n(k+1,0)+\sum_{\ell=1}^{L_n-1}\frac{2\theta_n(\ell)\nu_n(\ell)}{\tau_n(\ell)}\right]\overset{D}{=}N\left(0,K_n+4K_n\sum_{\ell=1}^{L_n-1}\frac{\theta_n^2(\ell)}{\tau_n(\ell)}\right),$$

and

$$\sum_{k=0}^{K_n-1}\frac{2\bar{\theta}_n(k+1,L_n)\bar{\nu}_n(k+1,L_n)}{\bar{\tau}_n(k+1,L_n)}\overset{D}{=}N\left(0,4\sum_{k=0}^{K_n-1}\frac{\bar{\theta}_n^2(k+1,L_n)}{\bar{\tau}_n(k+1,L_n)}\right).$$

Consequently (as $n\to\infty$)

$$\frac{C_n-\beta_n^4A_n-\beta_n^2B_n}{2\beta_n^3\sqrt{A_n}}\xrightarrow{D}N(0,1).$$

Moreover, due to $B_n^2+4A_nC_n=(B_n+2\beta_n^2A_n)^2+4A_n(C_n-\beta_n^4A_n-\beta_n^2B_n)$ it can be proved that under assumption (3.3) we have that $\lim_{n\to\infty}\frac{\sqrt{B_n^2+4A_nC_n}}{B_n+2\beta_n^2A_n}=1$ $\mathbb{P}$-a.s.

By (3.3), (3.4), (3.7) and the fact that $\lim_{n\to\infty}\frac{B_n+2\beta_n^2A_n}{2\beta_n^3\sqrt{A_n}}\cdot\frac{\beta_n}{\sqrt{A_n}}=1$, we obtain

$$\frac{\sqrt{A_n}}{\beta_n}(\widehat{\beta_n^2}-\beta_n^2)\xrightarrow{D}N(0,1) \tag{3.8}$$

as $n\to\infty$. Under assumption (3.4), (3.8) is the same as (3.6), so statement 1 has been shown.

Statement 2: If (3.3) and (3.5) are satisfied, then statement (3.6) is valid as $n\to\infty$.

Proof of statement 2: The proof is similar to the proof of the first statement. For $1\le k\le K_n$, $0\le\ell\le L_n$, $n\in\mathbb{N}^*$, we define

$$\xi_n(k,\ell):=\begin{cases}\beta_n^2(\eta_n^2(k,0)-1)+\beta_n^3\eta_n(k,0) & \text{if } \ell=0;\\[2ex] \beta_n^2\left(\dfrac{\nu_n^2(k,\ell)}{\tau_n(\ell)}-1\right)+\beta_n^3\left(\dfrac{2\nu_n(k,\ell)\theta_n(\ell)}{\tau_n(\ell)}\right) & \text{if } 0<\ell<L_n;\\[2ex] \beta_n^2\left(\dfrac{\bar{\nu}_n^2(k,\ell)}{\bar{\tau}_n(k,\ell)}-1\right)+\beta_n^3\left(\dfrac{2\bar{\nu}_n(k,\ell)\bar{\theta}_n(k,\ell)}{\bar{\tau}_n(k,\ell)}\right) & \text{for } \ell=L_n.\end{cases} \tag{3.9}$$

This definition allows us to write (see the proof of statement 1)

$$C_n-\beta_n^4A_n-\beta_n^2B_n=\sum_{k=1}^{K_n}\sum_{\ell=0}^{L_n}\xi_n(k,\ell). \tag{3.10}$$

It is easy to see that (where $\mathbb{V}(X) := \mathbb{E}[(X - \mathbb{E}[X])^2]$)

(i) $\mathbb{E}(C_n - \beta_n^4 A_n - \beta_n^2 B_n) = 0$ for all n,

(ii) $\mathbb{V}(C_n - \beta_n^4 A_n - \beta_n^2 B_n) = \sum_{k=1}^{K_n} \sum_{\ell=0}^{L_n} \mathbb{E}(\xi_n^2(k,\ell)) = 2\beta_n^4 K_n(L_n+1) + 4\beta_n^6 A_n$
$= 2\beta_n^4(B_n + 2\beta_n^2 A_n)$.

(iii) $0 \le \frac{\sum_{k=1}^{K_n}\sum_{\ell=0}^{L_n} \mathbb{E}(\xi_n^4(k,\ell))}{\beta_n^8(B_n+2\beta_n^2A_n)^2} = \frac{60\beta_n^8 B_n + 240\beta_n^{10}A_n + 48\beta_n^{12}\tilde{A}_n}{\beta_n^8(B_n+2\beta_n^2A_n)^2}$
$\le \frac{120}{B_n+2\beta_n^2A_n} + \frac{48\beta_n^4\tilde{A}_n}{(B_n+2\beta_n^2A_n)^2} \to 0$
as $n \to \infty$, because of equations (3.3) and (3.5).

Hence, by Lyapounov's Limit Theorem we obtain:

$$\frac{C_n - \beta_n^4 A_n - \beta_n^2 B_n}{\beta_n^2\sqrt{B_n + 2\beta_n^2 A_n}} \xrightarrow{D} N(0,2). \tag{3.11}$$

Thus:

$$\frac{\sqrt{B_n^2 + 4A_nC_n}}{B_n + 2\beta_n^2 A_n} = \frac{\sqrt{[(B_n + 2\beta_n^2A_n)^2 + 4A_n(C_n - \beta_n^4A_n - \beta_n^2B_n)]}}{B_n + 2\beta_n^2A_n} \to 1$$

as $n \to \infty$. So (see equation (3.7)):

$$\frac{\sqrt{B_n + 2\beta_n^2A_n}(\widehat{\beta_n^2} - \beta_n^2)}{\sqrt{2}\beta_n^2} = \frac{2(C_n - \beta_n^4A_n - \beta_n^2B_n)\sqrt{B_n + 2\beta_n^2A_n}}{\sqrt{2}\beta_n^2\big(B_n + 2\beta_n^2A_n + \sqrt{B_n^2 + 4A_nC_n}\big)} \xrightarrow{D} N(0,1)$$

as $n \to \infty$. □

Remark 3.2. Consider the case where $\lim_{n\to\infty} \frac{B_n}{A_n\beta_n^2} = 0, \lim_{n\to\infty} \frac{A_n\beta_n^2}{B_n^2} = 0$ and $\tilde{A}_n = O(A_n^2)$. It can be seen easily that (3.4) and (3.5) are not satisfied and that (under the assumption that $K_n \to \infty$) this is the only case where statements (3.4) and (3.5) are both not valid. Hence, concerning this case we will not be able to use Theorem 3.1. This can only happen if β_n^2 tends to zero with a certain speed, which is dependent on the values of the different parameters.

With the help of the Lindeberg Theorem, condition (3.5) can be replaced by the following weaker condition: for all $k \in \{1, 2, \ldots, K_n\}$

$$\lim_{n\to\infty} \frac{\beta_n^2\bar{\theta}_n^2(k, L_n)}{\bar{\tau}_n(k, L_n)(B_n + 2A_n\beta_n^2)} = 0. \tag{3.12}$$

Unfortunately, this condition does rarely help us for the cases where equations (3.4) and (3.5) are both not satisfied. For more on this subject we refer to [14].

Remark 3.3. As said before, in later work we plan to do some computer tests. In [4] some empirical results are shown. They show that when the volatility is a constant, the ML estimator of β^2 is already very close to the true value of β^2 for small values of K and L. If we include known parameters a_ℓ, $\ell \in \mathbb{N}^*$, the estimations have again fairly nice performances. In our discussion we will always assume that the parameters a_ℓ are known. If we do not assume this, then finding the ML estimator for all unknown variables becomes very complicated, but we can still do this numerically for some cases. For the sake of the reader we show the results for the estimation of a, which is defined below, and β based on generated data (according to no-arbitrage conditions, of course), where the following setting has been used: $f(0,\ell) = 0.03$, $0 \leq \ell \leq L$, $\beta = 0.002$, $\varrho = -0.1$, $a := a_1 = 0.9$ and $a_\ell = a_{\ell+1}$ for all $\ell \in \mathbb{N}^*$. In later work, the choice of the values of the parameters and the working of the program used for the calculations will be discussed in detail. Table 1 shows the fairly good results for the speed of convergence. For every case 400 runs of the program have been done. If we do not assume that $a_\ell = a_{\ell+1}$ for all $\ell \in \mathbb{N}^*$, but for instance that $a_L = a_{L+i}$ for all $i \in \mathbb{Z}_+$ and a certain $L \in \mathbb{N}^*$, then the estimation of the unknown variables becomes more difficult, but can be achieved by well known numerical optimisation procedures. In later work we plan to consider the estimation of all parameters for a non-constant volatility.

	$\widehat{\beta^2}$		$\widehat{a}$	
(K,L)	AMD	SE	AMD	SE
		$*10^{-5}$	$*10^{-1}$	$*10^{-3}$
(10,3)	13.253	33.838	52.016	58.080
(20,6)	7.9721	20.280	17.607	19.901
(30,9)	5.8817	14.712	10.041	11.190
(40,12)	4.4713	11.277	5.7182	6.4187
(50,15)	3.6894	9.2628	3.9774	4.4646
(60,18)	3.0235	7.4781	2.8071	3.1650

Table 1. Average mean difference (=AMD), defined by $100 * |\widehat{\beta^2}_{K,L} - \beta^2|/\beta^2$, and its standard error (=SE) for β^2 and a as calculated by R. Note that for the sake of simplicity of the table the values of the AMD and the SE are multiplied by some factors which are shown in line 3.

Remark 3.4. If statement (3.6) is valid for $n \to \infty$ and $\lim_{n\to\infty} \frac{B_n}{A_n \beta_n^2} = 0$, then (as $n \to \infty$)

$$\sqrt{\frac{A_n}{\beta_n^2}} (\widehat{\beta_n^2} - \beta_n^2) \xrightarrow{D} N(0,1). \tag{3.13}$$

Similarly, if $\lim_{n\to\infty} \frac{A_n \beta_n^2}{B_n} = 0$, then (as $n \to \infty$)

$$\sqrt{\frac{B_n}{2\beta_n^4}} (\widehat{\beta_n^2} - \beta_n^2) \xrightarrow{D} N(0,1). \tag{3.14}$$

With the help of Theorem 3.1, the asymptotic behaviour of the maximum likelihood estimator of β^2 is considered in [14] for all cases where a_n converges to $a \in \mathbb{R}$, ϱ_n converges to $\varrho \in \mathbb{R}$ and L_n is of order n or L_n is a constant for all n. In practice the shortage of available (traded) assets in the market does not always make it possible for us to work with large values of L_n, whereas, the value of K_n can well increase as time goes on.

Here we will state only one case, namely the *unstable* case, where a_n and ϱ_n converge to 1 and L_n is a constant for all n. For the other cases we refer to [14].

Theorem 3.5. *Consider the maximum likelihood estimator $\widehat{\beta_n^2}$ of β_n^2 based on a sample $\{f_n(k,\ell) \mid 1 \le k \le K_n,\ 0 \le \ell \le L_n\}$, where $K_n = nK + o(n)$ as $n \to \infty$ and $L_n = L$ for all $n \in \mathbb{N}^*$ with some $K, L > 0$. Assume that $\varrho_n = 1 + \frac{\gamma}{n} + o(n^{-1})$ and $a_n = 1 + \frac{\delta}{n} + o(n^{-1})$ as $n \to \infty$, where $\delta, \gamma \in \mathbb{R}$. Then (3.6) is valid and (as $n \to \infty$)*

1. *if $n^3\beta_n^2 \to \infty$ then*

$$n^2 \beta_n^{-1} (\widehat{\beta_n^2} - \beta_n^2) \xrightarrow{D} N(0, \sigma^2), \tag{3.15}$$

2. *if $n^3\beta_n^2 \to 0$ then*

$$\sqrt{n} \beta_n^{-2} (\widehat{\beta_n^2} - \beta_n^2) \xrightarrow{D} N\left(0, \frac{2}{K(L+1)}\right), \tag{3.16}$$

3. *if $\beta_n^2 = O(n^{-3})$ then*

$$\sqrt{n} \sqrt{\frac{K(L+1)\sigma^2 + 2n^3\beta_n^2}{2\sigma^2\beta_n^4}} (\widehat{\beta_n^2} - \beta_n^2) \xrightarrow{D} N(0,1), \tag{3.17}$$

where

$$\frac{1}{\sigma^2} := \int_0^K \left(\int_0^t \exp(2\delta y) dy \right)^{-1} \left[\int_0^t \exp(2\delta u) \left(\int_0^u \exp(2\gamma z) dz \right) du \right]^2 dt. \tag{3.18}$$

The proof consists of the calculation of (the limits for) A_n and B_n, and showing that (3.3) and (3.4) or (3.5) holds. The definition of σ^2 can be simplified for the cases where $\gamma = 0$ or $\delta = 0$.

If (3.1) is replaced by (for fixed n, (3.19) is similar to (2.4))

$$\begin{aligned} &\beta_n(k,0) = \beta_n && k \in \mathbb{Z}_+; \\ &\beta_n(k,\ell+1) = a_{n,\ell+1}\beta_n(k,\ell) && k,\ell \in \mathbb{Z}_+, \end{aligned} \tag{3.19}$$

then explicit theorems for the asymptotic behaviour of the maximum likelihood estimator of β^2 are given in [14] for all cases where a_n converges to $a \in \mathbb{R}$, ϱ_n converges to ϱ and L_n is a constant for all n. Here we will only state the theorem that is analogous to Theorem 3.5.

Theorem 3.6. *Consider the maximum likelihood estimator $\widehat{\beta_n^2}$ of β_n^2 based on a sample $\{f_n(k,\ell) \mid 1 \le k \le K_n,\ 0 \le \ell \le L_n\}$, where $K_n = nK + o(n)$ as $n \to \infty$ and $L_n = L$ for all $n \in \mathbb{N}^*$ with some $K, L > 0$. Assume that $a_{n,L} = 1 + \frac{\delta}{n} + o(n^{-1})$ and $\varrho_n = 1 + \frac{\gamma}{n} + o(n^{-1})$ as $n \to \infty$, where $\delta, \gamma \in \mathbb{R}$. If $a_{n,i} \to \tilde{a}_i \in \mathbb{R}\backslash\{0\}$ as $n \to \infty$ for all $1 \le i \le L-1$ and $a_{n,L} = a_{n,L+j}$ for all $n, j \in \mathbb{Z}_+$, then (as $n \to \infty$) the statements of Theorem 3.5 are valid, with*

$$\begin{aligned} \frac{1}{\sigma^2} := \left(\prod_{i=1}^{L-1} \tilde{a}_i^2\right) \int_0^K \left(\int_0^t \exp(2\delta y)dy\right)^{-1} \\ \times \left[\int_0^t \exp(2\delta u)\left(\int_0^u \exp(2\gamma z)dz\right)du\right]^2 dt. \end{aligned} \tag{3.20}$$

The proof is about the same as the one of Theorem 3.5. Notice that the definition of σ^2 is only slightly different from the one given in Theorem 3.5.

For the calculation of A_n, we have for all the cases where L_n is a constant, that $K_n \sum_{\ell=1}^{L_n-1} \frac{\theta_n^2(\ell)}{\tau_n(\ell)}$ (where $\theta_n(\ell)$ and $\tau_n(\ell)$ are defined similar to (2.9)) is relatively small compared to $\sum_{k=1}^{K_n} \frac{\bar{\theta}_n^2(k,L_n)}{\bar{\tau}_n(k,L_n)}$. However, if L_n is of order n we do not always know the order of A_n. Of course, there are some special cases for which we can state explicit formula's for A_n, e.g. when $\prod_{p=1}^{\ell} a_{n,p}$ is finite for all $n \in \mathbb{N}^*$ and $1 \le \ell \le L_n$, but general theorems have not been found yet.

4. A general case

In this section we turn back to the model introduced in Section 1. Since the driving fields follow an autoregressive structure, which implies a 'geometric' feature (see e.g. (1.6)), we shall suppose that the market price of risk parameters behave in a similar way. Therefore, in what follows we assume that

$$\phi_j = \beta b q^j, \quad j \in \mathbb{Z}_+, \tag{4.1}$$

where $b \in \mathbb{R}$ and $|q| < 1$ such that $|q\varrho| < 1$. Note that the latter condition is sufficient for the convergence of $\sum_{j=0}^{\infty} \phi_j \Delta_1 S(k,j)$ with probability one. The parameter b is included for the sake of generality, although the assumption $b = 1$ would already lead to a quite general model. The reason why ϕ_j is defined relative to β will be discussed later on.

Now we turn to the maximum likelihood estimator of the volatility. The next lemma is a generalisation of Lemma 2.1 and provides us with a method to find this maximum likelihood estimator.

Lemma 4.1. *Consider a forward interest rate curve model* $\{f(k,\ell) \mid k, \ell \in \mathbb{Z}\}$ *as given in* (1.3) *and suppose that equations* (1.9) *and* (4.1) *are valid, with* $b \in \mathbb{R}$ *and* $|q| < 1$ *such that* $|q\varrho| < 1$. *Let* K *and* L *be positive integers. Assume that the parameters* ϱ, b, q *and* $\{a_i\}_{i=1}^{\infty}$ *are known,* $a_i \neq 0$ *and that* $a_L = a_{L+j}$ *for all* $j \in \mathbb{N}^*$. *Then the maximum likelihood estimator* $\widehat{\beta^2}_{K,L}$ *of* β^2 *based on the sample* (2.5) *is given by statement* (2.6) *where* $A_{K,L}, B_{K,L}, C_{K,L}, \tau, \bar{\tau}, \bar{\theta}$ *are the same as in Lemma 2.1 and* θ *is given by*

$$\theta(\ell) := \frac{1}{2}\left(\prod_{i=1}^{\ell-1} a_i\right)^2 a_l(a_l + \varrho)\sum_{j=0}^{\ell-1} \varrho^{2j} + \frac{1}{2}\left(\prod_{i=1}^{\ell} a_i\right)^2 \varrho^{2\ell} - b\left(\prod_{i=1}^{\ell} a_i\right)\frac{q^\ell}{1 - q\varrho}. \tag{4.2}$$

Proof. One can derive the statement of this lemma by following the steps of the proof of Lemma 2.1, where one should use the no-arbitrage criterion stated above (see [14]. □

If $a := a_1$ and $a_i = a_{i+1}$ for all $i \in \mathbb{N}^*$, then $\beta(k,\ell) = a^\ell \beta$ (for all k, ℓ) and (4.2) becomes

$$\theta(\ell) = \frac{1}{2} a^{2\ell-1}(a + \varrho)\sum_{j=0}^{\ell-1} \varrho^{2j} + \frac{1}{2} a^{2\ell} \varrho^{2\ell} + \frac{b a^\ell q^\ell}{q\varrho - 1}. \tag{4.3}$$

If we also have that $a_i = 1$ for all $i \in \mathbb{N}^*$, then $\beta(k,\ell) = \beta(k,0) = \beta$ is a constant (for all k, ℓ) and (4.2) can be simplified even more: $\theta(\ell) = \frac{bq^\ell}{q\varrho - 1} + \frac{1}{2}\sum_{i=0}^{2l} \varrho^i$.

The market price of risk is defined relative to the volatility in our setup. One could of course parametrise the market price of risk without the inclusion of the volatility. However, we remark that without the inclusion of the volatility, the maximum likelihood estimator will not be a solution of a second-order equation and can not be expressed explicitly. On the other hand, it is important to emphasise that this approach does not cause any loss of generality. It is, in fact, just a matter of parametrization.

Consider a sequence of discrete-time forward interest rate curve models like the one defined in Section 3 and suppose that the no-arbitrage criterion (1.9) is satisfied in the models. Assume that, for every $n \in \mathbb{N}^*$, there exists $a_n, \beta_n \in \mathbb{R}$, $a_n, \beta_n \neq 0$, such that $\beta_n(k,\ell)$ is defined as in equation (3.1) and that $\phi_{n,j}$ is defined by

$$\phi_{n,j} = \beta_n b_n q_n^j, \quad n, j \in \mathbb{Z}_+,$$

where $b_n \in \mathbb{R}$ and $|q_n| < 1$ such that $|q_n \varrho_n| < 1$.

The next theorem generalises the results of Theorem 3.5, concerning two different cases:

(i) $q := \lim_{n\to\infty} q_n$ with $|q_n|, |q| < 1$, and

(ii) $q := \lim_{n\to\infty} q_n$ with $|q_n| < 1$ and $q = 1$.

Theorem 4.2. *Suppose that $b := \lim_{n\to\infty} b_n$ with $b \in \mathbb{R}$, and that the driving process S_n and the parameters K_n, L_n, a_n, ϱ_n are as in Theorem 3.5. Furthermore, suppose that $q := \lim_{n\to\infty} q_n$ and $|q_n \varrho_n| < 1$ for all $n \in \mathbb{N}^*$.*

(i) *If $|q_n| < 1$ and $|q| < 1$, then statement* (3.6) *and the statements of Theorem 3.5 are valid with σ^2 given by* (3.18).

(ii) *If $q_n := 1 - \frac{\kappa}{n} + o(n^{-1})$, with $\kappa \in \mathbb{R}_{>0}$ and $\kappa > \gamma$, then statement* (3.6) *and the statements of Theorem 3.5 are valid with*

$$\frac{1}{\sigma^2} := \int_0^K \left(\int_0^t \exp(2\delta y) dy \right)^{-1} \times \left[\int_0^t \exp(\delta u) \left(\exp(\delta u) \int_0^u \exp(2\gamma z) dz - b \frac{\exp(-\kappa u)}{\kappa - \gamma} \right) du \right]^2 dt. \tag{4.4}$$

The proof is about the same as the proof of Theorem 3.5. Note that the condition $\kappa > \gamma$ in the second part of the theorem is included to ensure that $|q_n \varrho_n| < 1$ for all $n \in \mathbb{N}^*$. If $\kappa \leq \gamma$, then we can not use Lemma 4.1 to prove a theorem similar to Theorem 3.5. Theorem 3.6 and others can be generalised in a similar way. For the results and the proofs we refer to [14].

We note that one could, of course, take some other forms for the market price of risk. E.g., finitely many factors $\phi_0, \ldots, \phi_N \neq 0$ could be considered (with $\phi_j = 0$ for $j > N$). This case is studied e.g. in [8]. For a general discussion on the role of the market price of risk one can consult e.g. [1].

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