On mininjective and min-flat modules

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Abstract. A left R-module M is said to be mininjective if $\operatorname{Ext}^1(R/I, M) = 0$ for any simple left ideal I of R. A ring R is called left min-coherent in case each simple left ideal of R is finitely presented. It is shown that every left R-module over a left min-coherent ring R has a mininjective cover. We also give some new characterizations of left FS rings, left PS rings and left universally mininjective rings.

1. Introduction

Let $\mathcal{C}$ be a class of R-modules and M an R-module. Following [5], we say that a homomorphism $\phi : C \to M$ is a *$\mathcal{C}$-precover of M* if $C \in \mathcal{C}$ and the abelian group homomorphism $\operatorname{Hom}(C', \phi) : \operatorname{Hom}(C', C) \to \operatorname{Hom}(C', M)$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-precover $\phi : C \to M$ of M is said to be a *$\mathcal{C}$-cover* if every endomorphism $g : C \to C$ such that $\phi g = \phi$ is an isomorphism. Dually we have the definitions of a $\mathcal{C}$-preenvelope and a $\mathcal{C}$-envelope. $\mathcal{C}$-covers ($\mathcal{C}$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism.

The problem of the existence of covers and envelopes by different classes of modules has become an active branch of algebra (see, for example, [1], [2], [4], [5], [6], [8], [12], [16], [20]). For example, every left R-module over a left Noetherian ring R has an injective cover (see [5]). Recently, PINZON has proven that every

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left R-module over a left coherent ring R has an FP-injective cover (see [16]), where a left R-module M is called FP-injective [18] if $\mathrm{Ext}^1(N, M) = 0$ for any finitely presented left R-module N; R is called a *left coherent ring* if every finitely generated left ideal of R is finitely presented.

As a generalization of injectivity, HARADA introduced the concept of mininjective modules (see [9]). A left R-module M is said to be *mininjective* if $\mathrm{Ext}^1(R/I, M) = 0$ for any simple left ideal I of R. Mininjective modules have been studied by many authors (see, for example, [9], [11], [12], [14], [15]). Recall that a right R-module M is called *min-flat* [11] in case $\mathrm{Tor}_1(M, R/I) = 0$ for any simple left ideal I of R. By the standard isomorphism $\mathrm{Tor}_1(M, R/I)^+ \cong \mathrm{Ext}^1(R/I, M^+)$ for any simple left ideal I of R, we get that a right R-module M is min-flat if and only if M^+ is mininjective.

In this paper, we first introduce the concept of min-pure exact sequences, which is used to characterize min-flat and mininjective modules. Then we investigate the existence of mininjective covers. Recall that R is called a *left min-coherent ring* [11] if every simple left ideal of R is finitely presented. We show that every left R-module over a left min-coherent ring R has a mininjective cover. It is also proven that every left R-module has a mininjective cover with the unique mapping property if and only if for any left R-module exact sequence $A \to B \to C \to 0$ with A and B mininjective, C is mininjective. As applications, we give some new characterizations of left FS rings, left PS rings and left universally mininjective rings.

Throughout this paper, R is an associative ring with identity and all modules are unitary. M_R ($_RM$) denotes a right (left) R-module. The character module M^+ is defined by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. $E(M)$ stands for the injective envelope of M. Let M and N be R-modules. $\mathrm{Hom}(M, N)$ (resp. $\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ (resp. $\mathrm{Ext}_R^n(M, N)$), and similarly $M \otimes N$ (resp. $\mathrm{Tor}_n(M, N)$) denotes $M \otimes_R N$ (resp. $\mathrm{Tor}_n^R(M, N)$) for an integer $n \geq 1$. General background materials can be found in [6], [8], [17], [20].

2. Min-pure exact sequences and min-pure-injective modules

We begin with the following

Lemma 2.1. *The following are equivalent for a right R-module exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{\alpha} C \to 0$, where i is regarded as an inclusion:*

(1) *For any $a \in R$ such that Ra is simple, the sequence $0 \to A \otimes (R/Ra) \xrightarrow{i \otimes 1} B \otimes (R/Ra)$ is exact.*

(2) *For any $a \in R$ such that Ra is simple, the sequence $\mathrm{Hom}(R/aR, B) \xrightarrow{\alpha_*}$*
$\mathrm{Hom}(R/aR, C) \to 0$ is exact.

(3) *For every $a \in R$ such that Ra is simple, $Aa = A \cap Ba$.*

PROOF. $(1) \Rightarrow (2)$. Let $a \in R$ with Ra simple and $f \in \mathrm{Hom}(R/aR, C)$. Then there exist g and h such that the following diagram with exact rows commutes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & aR & \xrightarrow{\lambda} & R & \xrightarrow{\pi} & R/aR & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{\alpha} & C & \longrightarrow & 0.
\end{array}
$$

So $g(a) = g\lambda(a) = ih(a) = h(a) \in A$. Note that $g(a) \otimes \overline{1} = g(1)a \otimes \overline{1} = g(1) \otimes \overline{a} = 0$ in $B \otimes (R/Ra)$, thus $g(a) \otimes \overline{1} = 0$ in $A \otimes (R/aR)$ since the sequence $0 \to A \otimes (R/Ra) \to B \otimes (R/Ra)$ is exact. Hence there exists $k \in A$ such that $g(a) = ka$ by [17, Lemma 3.64]. Define $\beta : R \to A$ by $\beta(r) = kr$ for any $r \in R$, then β is well-defined and $\beta\lambda = h$. Therefore there exists $\gamma : R/aR \to B$ such that $\alpha\gamma = f$ by [7, Lemma 8.4].

$(2) \Rightarrow (3)$. Suppose $a \in R$ with Ra simple. Let $b \in B$ such that $ba \in A$. Define $f : R/aR \to C$ by $f(\overline{r}) = \alpha(br)$ for any $r \in R$. Then f is well-defined. By (2), there exists $\beta : R/aR \to B$ such that $f = \alpha_*(\beta) = \alpha\beta$. So $\alpha(b) = f(\overline{1}) = \alpha\beta(\overline{1})$. Thus $b - \beta(\overline{1}) \in \ker(\alpha) = A$ and hence $ba = (b - \beta(\overline{1}))(a) \in Aa$. So $A \cap Ba \subseteq Aa$. In addition, $Aa \subseteq A \cap Ba$ is clear. Therefore $Aa = A \cap Ba$.

$(3) \Rightarrow (1)$. Assume that $a \in R$ with Ra simple. Let $k \otimes \overline{1} \in A \otimes (R/Ra)$ such that $k \otimes \overline{1} = 0$ in $B \otimes (R/Ra)$. Then there exists $p \in B$ such that $k = pa$ by [17, Lemma 3.64]. Define $g : R \to B$ by $g(r) = pr$ and define $f : R/aR \to C$ by $f(\overline{r}) = \alpha(p)r$ for any $r \in R$. Then f and g are well-defined, and $f\pi = \alpha g$. Therefore there exists $h : aR \to A$ such that the following diagram with exact rows commutes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & aR & \xrightarrow{\lambda} & R & \xrightarrow{\pi} & R/aR & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{\alpha} & C & \longrightarrow & 0.
\end{array}
$$

So $g(1)a = g(a) = h(a) \in A$. By (3), there exists $x \in A$ such that $g(a) = xa$. Thus $k = pa = g(a) = xa$. Consequently, $k \otimes \overline{1} = xa \otimes \overline{1} = x \otimes \overline{a} = 0$ in $A \otimes (R/Ra)$. It follows that $0 \to A \otimes (R/Ra) \to B \otimes (R/Ra)$ is exact. $\square$

Definition 2.1. A right R-module exact sequence $0 \to A \to B \to C \to 0$ is said to *min-pure exact* if it satisfies one of the equivalent conditions of the lemma above. A right R-module M is called *min-pure-injective* if for every right R-module min-pure exact sequence $0 \to A \to B \to C \to 0$, the sequence $\mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \to 0$ is exact.

Obviously, the concept of min-pure exact sequences is a generalization of pure exact sequences, and so any min-pure-injective module is pure-injective.

The following results are easy to show by Lemma 2.1 and we omit the proof.

Proposition 2.2. *The following are equivalent for a right R-module M:*

(1) *M is min-flat.*

(2) *Every exact sequence $0 \to A \to B \to M \to 0$ is min-pure exact.*

(3) *There exists a min-pure exact sequence $0 \to K \to F \to M \to 0$ with F min-flat.*

Proposition 2.3. *Let R be a commutative ring. Then the following are equivalent for an R-module M:*

(1) *M is mininjective.*

(2) *Every exact sequence $0 \to M \to B \to C \to 0$ is min-pure exact.*

(3) *The exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$ is min-pure exact.*

Proposition 2.4. *The following are equivalent for a commutative ring R:*

(1) *$0 \to A \to B \to C \to 0$ is a min-pure exact sequence.*

(2) *$0 \to C^+ \to B^+ \to A^+ \to 0$ is a min-pure exact sequence.*

PROOF. It is easy by Lemma 2.1 and the Adjoint Isomorphism Theorem. $\square$

3. The existence of mininjective covers of modules

For any ring R, it was proven that every R-module has a min-flat cover and has a mininjective preenvelope (see [11, Theorem 3.4]). In this section, we will continue to study the existence of mininjective covers. The following lemmas are needed.

Lemma 3.1 ([11, Proposition 3.5])**.** *For any ring R, the class of mininjective R-modules and the class of min-flat R-modules are closed under pure submodules.*

Lemma 3.2 ([11, Theorems 4.5 and 4.6])**.** *The following are equivalent for a ring R:*

(1) *R is a left min-coherent ring.*

(2) *Any direct product of min-flat right R-modules is min-flat.*

(3) *Any direct limit of mininjective left R-modules is mininjective.*

(4) *A left R-module M is mininjective if and only if M^+ is min-flat.*

(5) *Every right R-module has a min-flat preenvelope.*

Lemma 3.3 ([6, Proposition 5.2.2])**.** *If $\mathcal{F}$ is a class of R-modules closed under direct sums, then an R-module M has an $\mathcal{F}$-precover if and only if there is a cardinal number $\aleph_\alpha$ such that any homomorphism $D \to M$ with $D \in \mathcal{F}$ has a factorization $D \to C \to M$ with $C \in \mathcal{F}$ and $\mathrm{Card}(C) \leq \aleph_\alpha$.*

Lemma 3.4 ([2, Theorem 5])**.** *Let R be an arbitrary ring. Then for each cardinal λ, there is a cardinal κ such that for any R-module M and any $L \leq M$ satisfying $\mathrm{Card}(M) \geq \kappa$ and $\mathrm{Card}(M/L) \leq \lambda$, the submodule L contains a non-zero submodule that is pure in M.*

We are now in a position to prove the main result.

Theorem 3.5. *Let R be a left min-coherent ring. Then every left R-module has a mininjective cover.*

PROOF. Suppose that N is a left R-module with $\mathrm{Card}(N) = \lambda$. Let κ be a cardinal as in Lemma 3.4. By Lemma 3.3, we only need to show that any homomorphism $f : D \to N$ with D mininjective has a factorization $D \to C \to N$ with C mininjective and $\mathrm{Card}(C) \leq \kappa$.

If $\mathrm{Card}(D) \leq \kappa$, then we are done. Next we may assume that $\mathrm{Card}(D) > \kappa$.

Let $K = \ker(f)$. Note that $\mathrm{Card}(D/K) \leq \lambda$ since D/K embeds in N. Thus K contains a non-zero submodule D_0 which is pure in D by Lemma 3.4. The pure exact sequence $0 \to D_0 \to D \to D/D_0 \to 0$ induces the split exact sequence $0 \to (D/D_0)^+ \to D^+ \to D_0^+ \to 0$. Thus $(D/D_0)^+$ is min-flat since D^+ is min-flat by Lemma 3.2. So D/D_0 is mininjective by Lemma 3.2 again.

If $\mathrm{Card}(D/D_0) \leq \kappa$, then we are done by Lemma 3.3 since f factors through D/D_0. Suppose that $\mathrm{Card}(D/D_0) > \kappa$. Put

$$\mathcal{S} = \{X : D_0 \leq X \leq K \text{ and } D/X \text{ is mininjective}\}.$$

Then $\mathcal{S}$ is a nonempty set since $D_0 \in \mathcal{S}$. Let $\{X_i \in \mathcal{S} : i \in I\}$ be an ascending chain. Note that $D_0 \leq \cup X_i \leq K$ and $D/ \cup X_i = D/\varinjlim X_i = \varinjlim(D/X_i)$ is mininjective by Lemma 3.2 since each D/X_i is mininjective. Thus $\cup X_i \in \mathcal{S}$, and so $\mathcal{S}$ has a maximal element C by Zorn's Lemma.

We claim that $\mathrm{Card}(D/C) \leq \kappa$. Otherwise, let $\mathrm{Card}(D/C) > \kappa$. Since $C \subseteq K$, there exists $g : D/C \to N$ such that $\ker(g) = K/C$. Note that $\mathrm{Card}((D/C)/(K/C)) = \mathrm{Card}(D/K) \leq \lambda$, and so K/C contains a non-zero submodule C_1/C which is pure in D/C by Lemma 3.4. Thus $D/C_1 \cong (D/C)/(C_1/C)$ is mininjective by the proof above, and hence $C_1 \in \mathcal{S}$, which contradicts the maximality of C. It is clear that D/C is mininjective and f factors through D/C. So N has a mininjective precover by Lemma 3.3, and hence has a mininjective cover by [6, Corollary 5.2.7]. □

As applications of the result above, we list some corollaries as follows.

Corollary 3.6. *If R is a left coherent ring or a domain, then every left R-module has a mininjective cover.*

Recall that R is called a *left mininjective ring* [14] if $_RR$ is mininjective.

Corollary 3.7. *The following are equivalent for a left min-coherent ring R:*
(1) *Every left R-module has an epic mininjective cover.*
(2) *R is a left mininjective ring.*
(3) *Every injective right R-module is min-flat.*

PROOF. (1) $\Rightarrow$ (2). Let $f : N \to {_RR}$ be an epic mininjective cover. Then $_RR$ is isomorphic to a direct summand of N, and so R is left mininjective.

(2) $\Rightarrow$ (3). Let M be an injective right R-module. Then M embeds in $\Pi(_RR)^+$. So M is isomorphic to a direct summand of $\Pi(_RR)^+ \cong (\oplus_R R)^+$. Since $\oplus_R R$ is mininjective by (2), we have $(\oplus_R R)^+$ is min-flat by Lemma 3.2. Thus M is also min-flat.

(3) $\Rightarrow$ (1). Since the injective right R-module $(_RR)^+$ is min-flat by (3), $_RR$ is mininjective by Lemma 3.2. Let M be a left R-module, then there is an exact sequence $F \to M \to 0$, where F is free and so is mininjective. Since M has a mininjective cover g by Theorem 3.5, we have g is an epimorphism. □

Corollary 3.8. *Let R be a commutative min-coherent ring. Then every min-pure-injective R-module M has a mininjective cover $f : F \to M$ such that F is injective.*

PROOF. By Theorem 3.5, M has a mininjective cover $f : F \to M$. There is an exact sequence $0 \to F \xrightarrow{i} E \to L \to 0$ with E injective. By Proposition 2.3, the exact sequence is min-pure exact. Since M is min-pure-injective, there exists $g : E \to M$ such that $gi = f$. So there exists $\varphi : E \to F$ such that $f\varphi = g$ since f is a cover. Therefore $f\varphi i = f$ and hence φi is an isomorphism. It follows that F is isomorphic to a direct summand of E, and so F is injective. □

Recall that a $\mathcal{C}$-cover $\phi : C \to M$ is said to *have the unique mapping property* [4] if for any homomorphism $f : C' \to M$ with $C' \in \mathcal{C}$, there is a unique homomorphism $g : C' \to C$ such that $\phi g = f$.

Theorem 3.9. *The following are equivalent for a ring R:*

(1) *Every left R-module has a mininjective cover with the unique mapping property.*

(2) *For any left R-module exact sequence $A \to B \to C \to 0$ with A and B mininjective, C is mininjective.*

PROOF. (1) $\Rightarrow$ (2). Suppose that $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence of left R-modules with A and B mininjective. Let $\theta : H \to C$ be a mininjective cover with the unique mapping property. Then there exists $\delta : B \to H$ such that $g = \theta\delta$. Thus $\theta\delta f = gf = 0 = \theta 0$, and hence $\delta f = 0$, which implies that $\ker(g) = \operatorname{im}(f) \subseteq \ker(\delta)$. Therefore there exists $\gamma : C \to H$ such that $\gamma g = \delta$, and so $\theta\gamma g = \theta\delta = g$. Thus $\theta\gamma = 1_C$ since g is epic. It follows that C is isomorphic to a direct summand of H, and hence C is mininjective.

(2) $\Rightarrow$ (1). Note that R is a left min-coherent ring by [11, Theorem 4.6]. Let M be a left R-module. Then M has a mininjective cover $f : F \to M$ by Theorem 3.5. It is enough to show that, for any mininjective left R-module G and any homomorphism $g : G \to F$ such that $fg = 0$, we have $g = 0$. In fact, there exists $\beta : F/\operatorname{im}(g) \to M$ such that $\beta\pi = f$ since $\operatorname{im}(g) \subseteq \ker(f)$, where $\pi : F \to F/\operatorname{im}(g)$ is the natural map. Note that $F/\operatorname{im}(g)$ is mininjective by (2). Thus there exists $\alpha : F/\operatorname{im}(g) \to F$ such that $\beta = f\alpha$, and so $f\alpha\pi = \beta\pi = f$. Hence $\alpha\pi$ is an isomorphism since f is a cover. Therefore π is monic, and so $g = 0$. $\square$

To study the kernels of mininjective precovers and the cokernels of min-flat preenvelopes, we introduce the following definitions.

Definition 3.1. A left R-module M is called *MI-injective* if $\operatorname{Ext}^1(G, M) = 0$ for any mininjective left R-module G.

A right R-module N is said to be *MI-flat* if $\operatorname{Tor}_1(N, G) = 0$ for any mininjective left R-module G.

Remark 3.1. (1) By Wakamutsu's Lemma (see [20, Lemma 2.1.1]), any kernel of a mininjective cover is *MI*-injective.

(2) A right R-module N is *MI*-flat if and only if N^+ is *MI*-injective by the standard isomorphism $\operatorname{Ext}^1(M, N^+) \cong \operatorname{Tor}_1(N, M)^+$ for any mininjective left R-module M.

Proposition 3.10. *The following are equivalent for a left R-module M:*

(1) *M is MI-injective.*

(2) *For every exact sequence $0 \to M \to E \to L \to 0$ with E mininjective, $E \to L$ is a mininjective precover of L.*

(3) *M is a kernel of a mininjective precover $f : A \to B$ with A injective.*

(4) *M is injective with respect to every exact sequence $0 \to A \to B \to C \to 0$, where C is mininjective.*

PROOF. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) are clear by definitions.

(2) $\Rightarrow$ (3) is obvious since there exists a short exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$.

(3) $\Rightarrow$ (1). Let M be a kernel of a mininjective precover $f : A \to B$ with A injective. Then we have an exact sequence $0 \to M \to A \to A/M \to 0$. So, for any mininjective left R-module N, the sequence $\operatorname{Hom}(N, A) \to \operatorname{Hom}(N, A/M) \to \operatorname{Ext}^1(N, M) \to 0$ is exact. It is easy to verify that the sequence $\operatorname{Hom}(N, A) \to \operatorname{Hom}(N, A/M) \to 0$ is exact since f is a mininjective precover. So $\operatorname{Ext}^1(N, M) = 0$, and (1) follows.

(4) $\Rightarrow$ (1). For each mininjective left R-module N, there exists a short exact sequence $0 \to K \to P \to N \to 0$ with P projective, which induces an exact sequence $\operatorname{Hom}(P, M) \to \operatorname{Hom}(K, M) \to \operatorname{Ext}^1(N, M) \to 0$. Note that $\operatorname{Hom}(P, M) \to \operatorname{Hom}(K, M) \to 0$ is exact by (4). Hence $\operatorname{Ext}^1(N, M) = 0$, as desired. □

The following result may be regarded as a dual of the proposition above.

Proposition 3.11. *Let R be a ring.*

(1) *If M is a finitely presented MI-flat right R-module, then M is a cokernel of a min-flat preenvelope.*

(2) *If R is a left min-coherent ring and L is a cokernel of a min-flat preenvelope $f : K \to F$ with F flat, then L is MI-flat.*

PROOF. (1) Let M be a finitely presented MI-flat right R-module. There is an exact sequence $0 \to K \to P \to M \to 0$ with P projective and both P and K finitely generated. We claim that $K \to P$ is a min-flat preenvelope. In fact, for any min-flat right R-module F, we have $\operatorname{Tor}_1(M, F^+) = 0$ since F^+ is

mininjective. So we get the commutative diagram with the first row exact:

$$
\begin{array}{ccc}
0 \longrightarrow K \otimes F^+ & \xrightarrow{\ \alpha\ } & P \otimes F^+ \\
\tau_1 \downarrow & & \tau_2 \downarrow \\
\mathrm{Hom}(K,F)^+ & \xrightarrow{\ \theta\ } & \mathrm{Hom}(P,F)^+.
\end{array}
$$

By [3, Lemma 2], τ_1 is an epimorphism and τ_2 is an isomorphism. Thus θ is a monomorphism, and hence $\mathrm{Hom}(P,F) \to \mathrm{Hom}(K,F)$ is epic, as required.

(2) It is clear that the inclusion $i : \mathrm{im}(f) \to F$ is a min-flat preenvelope. For any mininjective left R-module N, N^+ is min-flat by Lemma 3.2. Thus we obtain an exact sequence $\mathrm{Hom}(F,N^+) \to \mathrm{Hom}(\mathrm{im}(f),N^+) \to 0$, which yields the exactness of $(F \otimes N)^+ \to (\mathrm{im}(f) \otimes N)^+ \to 0$. So the sequence $0 \to \mathrm{im}(f) \otimes N \to F \otimes N$ is exact. Thus the exactness of $0 \to \mathrm{im}(f) \xrightarrow{i} F \to L \to 0$ induces the exact sequence $0 \to \mathrm{Tor}_1(L,N) \to \mathrm{im}(f) \otimes N \to F \otimes N$. So $\mathrm{Tor}_1(L,N) = 0$. $\square$

4. Applications

Recall that R is called a *left FS ring* [10], [19] if every simple left ideal of R is flat, equivalently if the left socle of R is flat. We will call a right R-module C *min-cotorsion* provided that $\mathrm{Ext}^1(F,C) = 0$ for any min-flat right R-module F. Clearly, any min-pure-injective module is min-cotorsion by Proposition 2.2.

Theorem 4.1. *The following are equivalent for a ring R:*

(1) *R is a left FS ring.*

(2) *Every submodule of any flat right R-module is min-flat.*

(3) *Every right ideal of R is min-flat.*

In this case, any min-cotorsion right R-module has injective dimension ≤ 1.

PROOF. $(1) \Rightarrow (2)$. Let A be a submodule of a flat right R-module B and I be a simple left ideal of R. Then the exactness of $0 \to A \to B \to B/A \to 0$ implies the exact sequence $0 = \mathrm{Tor}_2(B,R/I) \to \mathrm{Tor}_2(B/A,R/I) \to \mathrm{Tor}_1(A,R/I) \to \mathrm{Tor}_1(B,R/I) = 0$, and so $\mathrm{Tor}_2(B/A,R/I) \cong \mathrm{Tor}_1(A,R/I)$. On the other hand, the exact sequence $0 \to I \to R \to R/I \to 0$ gives rise to the exactness of $0 \to \mathrm{Tor}_2(B/A,R/I) \to \mathrm{Tor}_1(B/A,I) = 0$ since I is flat. Thus $\mathrm{Tor}_2(B/A,R/I) = 0$. So $\mathrm{Tor}_1(A,R/I) = 0$ and (2) follows.

$(2) \Rightarrow (3)$ is trivial.

$(3) \Rightarrow (1)$. Let I be a simple left ideal and K a right ideal of R. Then

the exactness of $0 \to K \to R \to R/K \to 0$ implies the exact sequence $0 \to \text{Tor}_2(R/K, R/I) \to \text{Tor}_1(K, R/I) = 0$ since K is min-flat. So $\text{Tor}_2(R/K, R/I) = 0$. On the other hand, the exact sequence $0 \to I \to R \to R/I \to 0$ induces the exactness of $0 \to \text{Tor}_2(R/K, R/I) \to \text{Tor}_1(R/K, I) \to 0$. Hence $\text{Tor}_1(R/K, I) \cong \text{Tor}_2(R/K, R/I) = 0$, and so I is flat.

Next we prove the last statement. Let M be a min-cotorsion right R-module and K a right ideal of R. Since $\text{Ext}^1(K, M) = 0$, we have $\text{Ext}^2(R/K, M) = 0$. It follows that M has injective dimension ≤ 1. $\qquad\square$

Recall that a ring R is called *left PS* [13] if every simple left ideal of R is projective, equivalently if the left socle of R is projective. Obviously, R is a left PS ring if and only if R is left min-coherent and left FS.

Theorem 4.2. *The following are equivalent for a ring R:*

(1) *R is a left PS ring.*

(2) *Every quotient of any mininjective left R-module is mininjective.*

(3) *R is left min-coherent and every MI-injective left R-module is injective.*

(4) *The class of all mininjective left R-modules is closed under cokernels of monomorphisms, and every MI-injective left R-module is mininjective.*

PROOF. (1) $\Leftrightarrow$ (2) follows from [12, Theorem 2.5].

(2) $\Rightarrow$ (3). R is left min-coherent since (2) is equivalent to (1). Let M be an MI-injective left R-module. Then there is an exact sequence $0 \to M \to E \to L \to 0$ with E injective. Note that L is mininjective by (2) and so $\text{Ext}^1(L, M) = 0$. Thus the exact sequence is split, and hence M is injective.

(3) $\Rightarrow$ (2). Let M be a quotient of a mininjective left R-module. Note that M has a mininjective cover $f : F \to M$ by Theorem 3.5. Thus f is an epimorphism. By Remark 3.1 (1), $\ker(f)$ is MI-injective, and hence it is injective by (3). So M is mininjective.

(3) $\Rightarrow$ (4) is clear since (3) is equivalent to (2).

(4) $\Rightarrow$ (2). Suppose that M is a quotient of a mininjective left R-module. Let $f : F \to M$ be a mininjective cover of M. Then f is an epimorphism and $\ker(f)$ is MI-injective. By (4), we have $\ker(f)$ is mininjective and so M is mininjective. $\quad\square$

Following [14], a ring R is called *left universally mininjective* if every left R-module is mininjective. It is clear that R is a left universally mininjective ring if and only if R is a left mininjective and left PS ring.

Theorem 4.3. *Let R be a ring. Then the following are equivalent:*

(1) *R is a left universally mininjective ring.*

(2) *Every right R-module is min-flat.*

(3) *Every min-cotorsion right R-module is injective.*

(4) *R is a left mininjective ring and every min-cotorsion right R-module is min-injective.*

(5) *Every right R-module exact sequence $0 \to A \to B \to C \to 0$ is min-pure exact.*

PROOF. (1) $\Leftrightarrow$ (2) holds by [11, Theorem 5.10].

(2) $\Leftrightarrow$ (3) follows from the fact that a right R-module M is min-flat if and only if $\text{Ext}^1(M, C) = 0$ for any min-cotorsion right R-module C (see [11, Theorem 3.4]).

(1) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1). Let M be any min-cotorsion right R-module. For a simple left ideal Ra, aR is simple by [14, Theorem 2.21(c)] since R is left mininjective. The exact sequence $0 \to aR \to R \to R/aR \to 0$ induces an exact sequence $\text{Hom}(R, M) \to \text{Hom}(aR, M) \to \text{Ext}^1(R/aR, M) \to 0$. Note that the homomorphism $\text{Hom}(R, M) \to \text{Hom}(aR, M)$ is epic by (4), and so $\text{Ext}^1(R/aR, M) = 0$. Thus R/aR is min-flat by [11, Theorem 3.4]. So R/aR is projective by [11, Corollary 3.3] since Ra is simple. It follows that aR is a direct summand of R_R, and so Ra is a direct summand of $_RR$, which implies that R is a left universally mininjective ring by [12, Theorem 2.6].

(2) $\Leftrightarrow$ (5) comes from Proposition 2.2. $\square$

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