Projectively Osserman manifolds

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Abstract. One says that a smooth manifold M of dimension m is a pseudo-Riemannian manifold of signature (p, q) if the tangent bundle TM is equipped with a smooth non-degenerate symmetric inner product g of signature (p, q) where $p + q = m$. Similarly one says that M is an affine manifold if TM is equipped with a torsion free connection ∇. One says g is Osserman if the eigenvalues of the Jacobi operator are constant on the pseudo-sphere bundles of unit timelike and spacelike vectors. We extend this concept from the pseudo-Riemannian to the affine setting to define the notion of a projectively Osserman manifold. This notion is the focus of the paper. We establish some basic results concerning projectively Osserman manifolds and exhibit examples of this structure which arise in several different geometrical contexts.

1. Introduction

Let $\mathcal{M} = (M, g)$ be an m-dimensional pseudo-Riemannian manifold of signature (p, q) where $p + q = m$. Let ∇ be the Levi–Civita connection defined by g, let $\mathcal{R}(x, y) := \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]}$ be the curvature operator, and let $\mathcal{J}(x) : y \to \mathcal{R}(y, x)x$ be the Jacobi operator. Let $\mathrm{Spec}\{\mathcal{J}(x)\} \subset \mathbb{C}$ be the set of eigenvalues of $\mathcal{J}(x)$ and let $S^{\pm}(\mathcal{M})$ be the pseudo-sphere bundles of unit spacelike $(+)$ and unit timelike $(-)$ tangent vectors. One says that $\mathcal{M}$ is *spacelike Osserman at* $P \in M$ if for every $x, y \in S^+(T_P M, g_P)$, $\mathrm{Spec}\{\mathcal{J}(x)\} = \mathrm{Spec}\{\mathcal{J}(y)\}$. One says that $\mathcal{M}$ is pointwise spacelike Osserman if it is spacelike Osserman at every point of M and that $\mathcal{M}$ is globally spacelike Osserman if the eigenvalue structure

does not in fact depend on the point in question. The notion *timelike Osserman* is defined by replacing S^+ by S^- as appropriate. Note that if $p > 0$ and if $q > 0$, then work of García–Río et al. [4] shows these are equivalent notions. The investigation of Osserman manifolds has been an extremely active and fruitful one in recent years; we refer to [5], [6], [7] for further details.

In this paper, we wish to generalize these notions to the affine setting. Let $\mathcal{A} := (M, \nabla)$ be an *affine manifold* where ∇ is a torsion free connection on TM. Again, let $\mathcal{R}(x, y)$ be the curvature operator and let $\mathcal{J}(x) : y \to \mathcal{R}(y, x)x$ be the associated *Jacobi operator*; we will write $\mathcal{J}_\nabla$ when it is necessary to distinguish the role of the connection. Let $\mathrm{Spec}\{\mathcal{J}(x)\} \subset \mathbb{C}$ be the spectrum of the Jacobi operator; since $\mathcal{J}(x)x = 0$, $0 \in \mathrm{Spec}\{\mathcal{J}(x)\}$. Since $\mathcal{J}(cx) = c^2 \mathcal{J}(x)$, $\mathrm{Spec}\{\mathcal{J}(cx)\} = c^2 \mathrm{Spec}\{\mathcal{J}(x)\}$. In the pseudo-Riemannian setting, we eliminated this rescaling effect by assuming that $g(x, x) = \pm 1$. As this normalization is not available in the affine setting, we must proceed slightly differently.

Recall that two non-zero points $u, v \in \mathbb{R}^m$ are said to be *projectively equivalent* if $u = cv$ for some $0 \neq c \in \mathbb{R}$. This motivates the following definition; the role of $\{0\}$ is distinguished and introduces a small amount of technical fuss.

Definition 1.1. We say that an affine manifold $\mathcal{A} = (M, \nabla)$ is *projectively Osserman* at a point $P \in M$ if there exists a subset $\mathcal{S}_P$ of $\mathbb{C}$ so that for any tangent vector $x \in T_P M$, $\mathrm{Spec}\{\mathcal{J}(x)\} = c(x)\mathcal{S}_P$ for some suitably chosen complex number $c(x)$. We say that $\mathcal{A}$ is *pointwise projectively Osserman* if it is projectively Osserman at every point of M. We say that $\mathcal{A}$ is *globally projectively Osserman* if $\mathcal{S}$ can be chosen independently of P.

This is related to earlier work by García–Río et al. [4]. One says $\mathcal{A} = (M, \nabla)$ is *affine Osserman* if $\mathrm{Spec}\{\mathcal{J}(x)\} = \{0\}$ for all tangent vectors x; such a manifold admits a natural neutral signature Osserman metric, called the Riemannian extension, on the cotangent bundle T^*M. Clearly any affine Osserman manifold is projectively Osserman. The Riemannian extension is Osserman if and only if $\mathcal{A}$ is affine Osserman.

We have chosen to work with $\mathrm{Spec}\{\mathcal{J}(x)\}$, it is also possible to work with the unordered collection of eigenvalues $\widetilde{\mathrm{Spec}}\{\mathcal{J}(x)\}$ where each eigenvalue is repeated according to multiplicity; working with $\widetilde{\mathrm{Spec}}\{\mathcal{J}(x)\}$ instead of $\mathrm{Spec}\{\mathcal{J}(x)\}$ gives rise to the notion of *strongly projective Osserman*. Fortunately, these are equivalent concepts as we will show in Section 2.

In this paper, we will exhibit several examples of projectively Osserman affine manifolds. In Section 3, we discuss examples which arise from Osserman geometry. In Section 4, we discuss projectively Osserman Walker manifolds. In Section 5,

we discuss examples from affine hypersurface theory. In Section 6, we discuss the Weyl projective tensor.

2. Eigenvalue multiplicities

This section is devoted to the proof of the following technical result.

Lemma 2.1. *Let* $\mathcal{A} = (M, \nabla)$ *and let* $P \in M$. *Then* $\mathcal{A}$ *is projectively Osserman at* P *if and only if* $\mathcal{A}$ *is strongly projectively Osserman at* P.

PROOF. We must show that $\mathcal{A}$ is projectively Osserman at P implies that $\mathcal{A}$ is strongly projectively Osserman at P as the reverse implication is trivial. If $\mathrm{Spec}\{\mathcal{J}(x)\} = \{0\}$ for all $x \in T_P M$, then $\widetilde{\mathrm{Spec}}\{\mathcal{J}(x)\}$ is the unordered set where 0 is repeated with multiplicity m and there is nothing to prove. We therefore suppose there exists $y \in T_P M$ so $\mathcal{J}(y)$ has a non-zero eigenvalue. As the eigenvalues vary continuously, as long as the spectrum does not degenerate to $\{0\}$, the multiplicities are constant. Thus there is a small neighborhood $\mathcal{O}$ of y in $T_P M$ so $\widetilde{\mathrm{Spec}}\{\mathcal{J}(x)\} = c(x)\widetilde{\mathrm{Spec}}\{\mathcal{J}(y)\}$ for $x \in \mathcal{O}$ where $c(x) \neq 0$. Let $p(x; t) := \det\{\mathcal{J}(x) - t\,\mathrm{id}\}$ be the characteristic polynomial. Decompose

$$p(x; t) = \prod_{i=1}^{m}(\lambda_i(x) - t) = \kappa_m(x) + \kappa_{m-1}(x)t + \cdots + \kappa_k t^{m-k} + \cdots + \kappa_0(x)t^m$$

where the coefficients $\kappa_\nu(x)$ are the elementary symmetric functions of the eigenvalues $\{\lambda_1(x), \ldots, \lambda_m(x)\}$. We note that κ_ν is a polynomial of degree $2k$ in the coordinate functions of x relative to some basis for V. Furthermore, if $x \in \mathcal{O}$, then $\lambda_\nu(x) = \kappa_\nu(x)\lambda_\nu(y)$ so

$$\kappa_\nu(x) = \kappa_\nu(y)c(x)^\nu \quad \text{for } x \in \mathcal{O} \quad \text{and for } 0 \leq \nu \leq m. \tag{2.a}$$

As $p(y; t) \neq (-t)^m$, there is $1 \leq \nu \leq m$ so $\kappa_\nu(y) \neq 0$. Thus

$$c(x) := \left\{ \frac{\kappa_\nu(x)}{\kappa_\nu(y)} \right\}^{1/k}$$

is an analytic function of x. We may then complexify V and $\mathcal{J}$ and consider the open dense subset $\mathcal{U} \subset V \otimes \mathbb{C}$ where $\mathrm{Spec}\{\mathcal{J}(x)\} \neq \{0\}$. We use analytic continuation to see that equation (2.a) holds for all $x \in \mathcal{U} \cap V$. Consequently $p(y; t) = c(x)^{-m}p(x; c(x)t)$ if $\mathrm{Spec}\{\mathcal{J}(x)\} \neq \{0\}$ and the desired result follows. $\square$

3. Osserman Manifolds

One has the following observation:

Theorem 3.1. *Let* $\mathcal{M} = (M, g)$ *be a pseudo-Riemannian manifold. If* $\mathcal{M}$ *is Osserman at* $P \in M$, *then* $\mathcal{M}$ *is projectively Osserman at* P.

PROOF. Let $\mathcal{M}$ have signature (p, q). This is immediate if the metric on $\mathcal{M}$ is positive definite since one has $T_P M = \mathbb{R} \cdot S^+(T_P M, g_P)$ and since one also has $\mathcal{J}(cx) = c^2 \mathcal{J}(x)$. The argument is the same if the metric is negative definite and we therefore suppose $p > 0$ and $q > 0$. Let $\mathcal{M}$ be spacelike Osserman at P and let $\mathcal{S}_P^+ := \operatorname{Spec}\{\mathcal{J}(x)\}$ for any $x \in S^+(T_P M, g_P)$. Let $\varrho_\nu(x) := \operatorname{Tr}\{\mathcal{J}(x)^\nu\}$ for any $x \in T_P M$. As $\operatorname{Spec}\{\mathcal{J}(x)\}$ is constant on $S^+(T_P M, g_P)$, the eigenvalues and hence the eigenvalue multiplicities are constant on $S^+(T_P M, g_P)$. This implies $\varrho_\nu(x) = \varrho_\nu$ is constant on $S^+(T_P M, g_P)$. Since $\varrho_\nu(cx) = c^{2i} \varrho_\nu(x)$, we have that $\varrho_\nu(x) = g(x, x)^\nu \varrho_\nu$ if x is spacelike. This polynomial identity holds on an open subset of $T_P M$ and hence holds identically:

$$\operatorname{Tr}\{\mathcal{J}(x)^\nu\} = g(x, x)^\nu \varrho_\nu \quad \text{for all } x \in T_P M.$$

It now follows, of course, that $\operatorname{Spec}\{\mathcal{J}(x)\} = g(x, x)\mathcal{S}_P^+$ for any $x \in T_P M$ and hence $\mathcal{M}$ is projectively Osserman at P. $\qquad \square$

Affine Osserman tensors play a central role:

Theorem 3.2. *Let* $\mathcal{A}_1 := (M_1, \nabla_1)$ *be projectively Osserman at* $P_1 \in M_1$ *and let* $\mathcal{A}_2 := (M_2, \nabla_2)$ *be affine Osserman at* $P_2 \in M_2$. *Then the product* $\mathcal{A} := (M_1 \times M_2, \nabla_1 \oplus \nabla_2)$ *is projectively Osserman at* $P = (P_1, P_2)$.

PROOF. If $x = (x_1, x_2) \in T_{(P_1, P_2)}(M_1 \times M_2)$, then $\mathcal{J}(x) = \mathcal{J}(x_1) \oplus \mathcal{J}(x_2)$ so

$$\begin{aligned}
\operatorname{Spec}\{\mathcal{J}(x)\} &= \operatorname{Spec}\{\mathcal{J}(x_1)\} \cup \operatorname{Spec}\{\mathcal{J}(x_2)\} \\
&= \operatorname{Spec}\{\mathcal{J}(x_1)\} \cup \{0\} = \operatorname{Spec}\{\mathcal{J}(x_1)\} = c(x_1)\mathcal{S}_{P_1}. \qquad \square
\end{aligned}$$

We use this ansatz to construct new examples. Give the sphere S^n and the torus T^k the usual metrics where $n \geq 2$ and $k \geq 1$. Then S^n is Osserman and T^k is flat so $S^n \times T^k$ is projectively Osserman. On the other hand, $S^n \times T^k$ is not Osserman. Thus there are projectively Osserman manifolds which are not Osserman. Furthermore, while any flat manifold is affine Osserman, there are other examples:

Example 3.3. Follow the discussion in [8]. Let $(x_1, \ldots, x_p, \tilde{x}_1, \ldots, \tilde{x}_p)$ be coordinates on $\mathbb{R}^{2p}$. Let ψ be a smooth symmetric 2-tensor field on $\mathbb{R}^p$. Define a pseudo-Riemannian metric of neutral signature (p, p) on $\mathbb{R}^{2p}$ whose non-zero components are, up to the usual $\mathbb{Z}_2$-symmetries, given by:

$$g(\partial_{x_i}, \partial_{x_j}) = \psi_{ij}(x_1, \ldots, x_p) \quad \text{and} \quad g(\partial_{x_i}, \partial_{\bar{x}_j}) = \delta_{ij}.$$

As $\mathcal{J}(x)^2 = 0$ for all x, $\text{Spec}\{\mathcal{J}(x)\} = \{0\}$ for all x as desired.

Example 3.4. Follow the discussion in [9]. For $s \geq 2$, choose coordinates

$$(u_1, \ldots, u_s, t_1, \ldots, t_s, v_1, \ldots, v_s)$$

on $\mathbb{R}^{3s}$. Let $f_i \in C^\infty(\mathbb{R})$ be given. Define a pseudo-Riemannian metric g of signature $(2s, s)$ on $\mathbb{R}^{3s}$ whose non-zero components are given by:

$$g(\partial_{u_i}, \partial_{u_i}) = -2(f_1(u_1) + u_1 t_1 + \cdots + f_s(u_s) + u_s t_s),$$
$$g(\partial_{u_i}, \partial_{v_i}) = g(\partial_{v_i}, \partial_{u_i}) = 1, \quad \text{and} \quad g(\partial_{t_i}, \partial_{t_i}) = -1.$$

Then $\mathcal{J}(x)^3 = 0$ for all x so $\mathcal{A} := (\mathbb{R}^{3s}, \nabla)$ is affine Osserman.

One says that an affine manifold $\mathcal{A} = (M, \nabla)$ is k-affine curvature homogeneous if given any two points $P, Q \in M$, there is an isomorphism $\phi : T_P M \to T_Q M$ so that $\phi^*\{\nabla^i \mathcal{R}_P\} = \nabla^i \mathcal{R}_Q$ for $0 \leq i \leq k$.

Example 3.5. Follow the discussion in [10]. Let $(x, y, z_0, \ldots, z_\ell, \tilde{x}, \tilde{y}, \tilde{z}_0, \ldots, \tilde{z}_\ell)$ be coordinates on $\mathbb{R}^{6+2\ell}$. Let g_f be the pseudo-Riemannian neutral signature metric defined by:

$$g_f(\partial_x, \partial_{\tilde{x}}) = g_f(\partial_y, \partial_{\tilde{y}}) = g_f(\partial_{z_i}, \partial_{\tilde{z}_i}) = 1,$$
$$g_f(\partial_x, \partial_x) = f(y) + y z_0 + \cdots + y^{\ell+1} z_\ell.$$

Assume $f^{(\ell+3)} > 0$ and $f^{(\ell+4)} > 0$. Let $\mathcal{A}_f := (\mathbb{R}^{6+2\ell}, \nabla_f)$ where ∇_f is the Levi–Civita connection defined by g_f. Then $\mathcal{A}_f$ is affine Osserman and $\mathcal{A}_f$ is $(\ell+2)$-affine curvature homogeneous. Furthermore, $\mathcal{A}_f$ is $(\ell+3)$-affine curvature homogeneous if and only if $f^{(\ell+3)}(y) = a e^{by}$ for $a > 0$ and $b > 0$ real constants; this happens if and only if $\mathcal{A}_f$ is affine homogeneous.

The manifolds in Examples 3.3, 3.4, and 3.5 are generalized plane wave manifolds and hence are geodesically complete. We refer to [7] for other examples and to [11] for a further discussion of generalized plane wave manifolds.

4. Walker manifolds

The following family will be crucial for our study.

Definition 4.1. Let (x_1, x_2, x_3, x_4) be coordinates on $\mathbb{R}^4$. Consider the following *Walker manifold* $\mathcal{M} := (\mathbb{R}^4, g)$ of signature $(2, 2)$ where

$$g(\partial_{x_1}, \partial_{x_3}) = g(\partial_{x_2}, \partial_{x_4}) = 1 \quad \text{and} \quad g(\partial_{x_3}, \partial_{x_4}) = g_{34}(x_1, x_2, x_3, x_4).$$

Let $\mathcal{W}$ be the *Weyl conformal curvature operator* of a pseudo-Riemannian manifold and let $\mathcal{J}_W(x) : y \to \mathcal{W}(y, x)x$ be the *conformal Jacobi operator*. One says that $\mathcal{M}$ is *pointwise conformally Osserman* if $\mathcal{J}_W$ has constant eigenvalues on $S^{\pm}(T_P M, g_P)$ for every point $P \in M$. We showed [1] that this is a conformal notion; (M, g) is pointwise conformally Osserman if and only if $(M, e^h g)$ is pointwise conformally Osserman for any $h \in C^{\infty}(M)$.

Recall that in dimension 4, $\mathcal{M}$ is conformally Osserman if and only if $\mathcal{M}$ is either self-dual or anti-self-dual. We take the orientation $dx_1 dx_2 dx_3 dx_4$ for $\mathbb{R}^4$. If $f = f(x_1, x_2, x_3, x_4)$, let $f_{/i} := \partial_{x_i} f$ and $f_{/ij} : \partial_{x_i} \partial_{x_j} f$. One has the following result concerning these manifolds [2], [3]:

Theorem 4.2. *Let $\mathcal{M}$ be as in Definition 4.1. Then*

(1) $\mathcal{M}$ *is self-dual if and only if* $g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4)$.

(2) $\mathcal{M}$ *is anti-self-dual if and only if* $g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4) + \xi(x_1, x_4) + \eta(x_2, x_3)$ *for* $p_{/3} = q_{/4}$ *and* $g_{34} p_{/3} - x_1 p_{/34} - x_2 p_{/33} - s_{/34} = 0$.

(3) *The following assertions are equivalent:*

   (a) $\mathcal{M}$ *is Osserman.*

   (b) $\mathcal{M}$ *is Einstein.*

   (c) *The Ricci tensor is zero.*

   (d) $g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4)$ *where* $p^2 = 2p_{/4}$, $q^2 = 2q_{/3}$, *and* $pq = p_{/3} + q_{/4}$.

Remark 4.3. Let $\mathcal{M}$ be as in Definition 4.1. Results of [3] show that if $\mathcal{M}$ is Einstein, then $g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4)$ where p and q have one of the following forms:

(1) $p = q = 0$.

(2) $p = 0$ and $q = -2(x_3 + b(x_4))^{-1}$.

(3) $p = -2(x_4 + a(x_3))^{-1}$ and $q = 0$.

(4) $p = -2(x_4 + a)^{-1}$ and $q = -2(x_3 + b)^{-1}$.

(5) $p = -2(x_3 + b_0 + b_1 x_4)^{-1}$ and $q = -2(x_4 + a_0 + a_1 x_3)^{-1}$ where $a_1 b_1 = 1$ and $a_0 = b_0 a_1$.

We now come to the main result of this section:

Theorem 4.4. *Let $\mathcal{M}$ be as in Definition 4.1. Then $\mathcal{M}$ is globally projectively Osserman if and only if at least one of the following conditions holds:*

(1) $g_{34} = p(x_1, x_4) + s(x_3)$.

(2) $g_{34} = q(x_2, x_3) + s(x_4)$.

(3) $g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4)$.

PROOF. This is a computer assisted computation. We begin by verifying the defining functions of Theorem 4.4 (1)–(3) define projectively Osserman manifolds. Let $\xi := \sum_i v_i \partial_{x_i}$. Suppose that

$$g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4).$$

Then $\mathrm{Spec}\{\mathcal{J}(\xi)\} = \{0, a(\xi)\}$ where 0 and $a(\xi)$ appear with multiplicity 2 with

$$a(\xi) = \frac{1}{4}\left\{-v_4^2 p(x_3, x_4)^2 + 2v_3 v_4 p(x_3, x_4) q(x_3, x_4) - v_3^2 q(x_3, x_4)^2 \right.$$
$$\left. +2v_4^2 p_{/4}(x_3, x_4) - 2v_3 v_4 q_{/4}(x_3, x_4) - 2v_3 v_4 p_{/3}(x_3, x_4) + 2v_3^2 q_{/3}(x_3, x_4)\right\}.$$

Next suppose that $g_{34} = p(x_1, x_4) + s(x_3)$; the case $g_{34} = q(x_2, x_3) + s(x_4)$ being analogous. One has $\mathrm{Spec}\{\mathcal{J}(\xi)\} = \{0, a(\xi)\}$ where 0 and $a(\xi)$ appear with multiplicity 2 with

$$a(\xi) = -\frac{1}{4}v_4\{v_4 p_{/1}(x_1, x_4)^2 - 2v_4 p_{/14}(x_1, x_4) - 2v_1 p_{/11}(x_1, x_4)\}.$$

Conversely, suppose $\mathcal{M}$ is as given in Definition 4.1 and that $\mathcal{M}$ is projectively Osserman. We suppose first that 0 is at least a double eigenvalue. The coefficients of $\lambda v_1^3 v_3^2 v_4$ and of $\lambda v_1 v_2^2 v_3^2 v_4$ in the characteristic polynomial are seen to be

$$\frac{1}{2}g_{34/11}(g_{34/12}^2 - g_{34/11}g_{34/22}) \quad \text{and} \quad \frac{1}{2}g_{34/22}(g_{34/12}^2 - g_{34/11}g_{34/22}).$$

To ensure 0 is at least a double eigenvalue, we set these two terms to zero and obtain two cases:

Case 1: $(g_{34/12}^2 - g_{34/11}g_{34/22}) \neq 0$. We then have $g_{34/11} = g_{34/22} = 0$ so

$$g_{34} = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + x_1 x_2 r(x_3, x_4) + s(x_3, x_4).$$

One computes the coefficient of $\lambda v_1 v_2 v_3^3 v_4 x_1^2$ in the characteristic polynomial to be $-\frac{1}{2} r(x_3, x_4)^4$. Setting this to zero implies $r = 0$ so g_{34} is affine in $\{x_1, x_2\}$ which is one of the possibilities enumerated in the Theorem.

Case 2: We have the identity $g_{34/12}^2 = g_{34/11} g_{34/22}$. Only the first and second derivatives appear in the calculation of the curvature tensor. Thus we may approximate by the second order Taylor polynomial. For simplicity, we suppose the point in question to be $x_1 = x_2 = 0$. Set

$$g_{34} = a_0(x_3, x_4) + x_1 a_1(x_3, x_4) + x_2 a_2(x_3, x_4)$$
$$+ x_1^2 a_{11}(x_3, x_4)^2 + x_2^2 a_{22}(x_3, x_4)^2 + 4 x_1 x_2 a_{11}(x_3, x_4) a_{22}(x_3, x_4).$$

When $x_1 = x_2 = 0$, the coefficient of $\lambda v_1 v_2^2 v_3^2 v_4$ in the characteristic polynomial is seen to be $12 a_{11}(x_3, x_4)^2 a_{22}(x_3, x_4)^4$. Thus we have $g_{34/12} = 0$ and, without loss of generality $g_{34/11} = 0$. This means that

$$g_{34} = x_1 p(x_3, x_4) + q(x_2, x_3, x_4).$$

Setting the coefficient of λ in the characteristic polynomial to zero then leads to:

$$0 = -\frac{1}{8} \{-2 p_{/3}(x_3, x_4) + p(x_3, x_4) q_{/2}(x_2, x_3, x_4)\}^2 q_{/22}(x_2, x_3, x_4),$$

$$0 = \frac{1}{4} \{p(x_3, x_4)^2 - 2 p_{/4}(x_3, x_4)\} \{-2 p_{/3}(x_3, x_4) + p(x_3, x_4) q_{/2}(x_2, x_3, x_4)\}$$
$$\times q_{/22}(x_2, x_3, x_4),$$

$$0 = -\frac{1}{8} \{p(x_3, x_4)^2 - 2 p_{/4}(x_3, x_4)\}^2 q_{/22}(x_2, x_3, x_4).$$

One possibility is $q_{/22} = 0$. This implies g_{34} is affine in $\{x_1, x_2\}$ as desired. The other possibility is $q_{/22} \neq 0$ so $2 p_{/3}(x_3, x_4) = p(x_3, x_4) q_{/2}(x_2, x_3, x_4)$. Differentiating this relation with respect to ∂_{x_2} yields $0 = p q_{/22}$ so $p = 0$ and

$$g_{34} = q(x_2, x_3, x_4).$$

Zero is at least a double eigenvalue and the other eigenvalues are

$$\lambda_{\pm}(x_2, x_3, x_4) = \frac{1}{4}(-v_3^2 q_{/2}^2 - 2 v_3 v_4 q_{/24} + 2 v_3^2 q_{/23} + 2 v_2 v_3 q_{/22})$$

$$\pm \sqrt{-v_3^3 v_4 q_{/34} q_{/22} - v_2 v_3^2 v_4 q_{/24} q_{/22}}.$$

Since $q_{/22} \neq 0$, $q_{/2} \neq 0$. Setting $v_4 = 0$ and $v_3 = 2$ yields $\lambda_{\pm} = -q_{/2}^2 \neq 0$. By Lemma 2.1, the remaining eigenvalue must be always be double eigenvalue so the square root must vanish identically. Consequently $q_{/34} = q_{/24} = 0$ so

$g_{34} = q(x_2, x_3) + s(x_4)$ which is the possibility given in (2) of the Theorem.

We complete the proof by analyzing what happens if there is a tangent vector ξ_0 so that 0 is a simple eigenvalue of $\mathcal{J}(\xi_0)$. Lemma 2.1 then shows that for any tangent vector ξ that either $\mathrm{Spec}\{\mathcal{J}(\xi)\} = \{0\}$ or 0 is a simple eigenvalue of $\mathcal{J}(\xi)$.

We specialize and set $v_4 = 0$. Zero is then a double eigenvalue and the other eigenvalue $a(\xi)$ is double eigenvalue as well where:

$$a(\xi) = \frac{1}{4} v_3 \{ -v_3 g_{34/2}(x_1, x_2, x_3, x_4)^2 + 2v_3 g_{34/23}(x_1, x_2, x_3, x_4)$$
$$+ 2v_2 g_{34/22}(x_1, x_2, x_3, x_4) + 2v_1 g_{34/12}(x_1, x_2, x_3, x_4) \}.$$

Consequently, $a(\xi) = 0$. This implies $g_{34/12} = g_{34/22} = 0$; specializing to set $v_3 = 0$ instead of $v_4 = 0$ yields $g_{34/11} = 0$ as well. Thus g_{34} is affine in (x_1, x_2). $\square$

5. Relative hypersurfaces

We refer to [12], [13] for further material concerning the theory of relative hypersurfaces. Let $\mathcal{A} = (M, \nabla)$. We begin with a technical observation. Suppose there is a quadratic form τ so that

$$\mathcal{J}_\nabla(x)y = \tau(x, x)y - \tau(x, y)x. \tag{5.a}$$

Because $\mathcal{J}_\nabla(x)x = 0$, $0 \in \mathrm{Spec}\{\mathcal{J}_\nabla(x)\}$. Suppose that $\tau(x, x) = 0$. One then has that $\mathcal{J}_\nabla(x)y = -\tau(x, y)x$ and thus $\mathcal{J}_\nabla(x)^2 y = 0$. Since $\mathcal{J}_\nabla(x)$ is nilpotent, $\mathrm{Spec}\{\mathcal{J}_\nabla(x)\} = \{0\}$. On the other hand, suppose $\tau(x, x) \neq 0$. Let $y \in x^\perp$, i.e. $\tau(x, y) = 0$. Then $\mathcal{J}(x)y = \tau(x, x)y$ and we conclude $\mathrm{Spec}\{\mathcal{J}_\nabla(x)\} = \{0, \tau(x, x)\}$. Thus ∇ is projectively Osserman by equation (5.a) as $\mathrm{Spec}\{\mathcal{J}_\nabla\} = \{0, \tau(x, x)\}$.

Fix a relative normalization for a hypersurface M in affine space $\mathbb{A}^{m+1}$. Let (∇, g, ∇^*) be the induced connection, the relative metric, and the conormal connection; this forms a *conjugate triple* and one has that:

$$xg(y, z) = g(\nabla_x y, z) + g(y, \nabla_x^* z).$$

Let $\mathcal{R}$ and $\mathcal{R}^*$ be the associated curvature operators, let Ric^* be the Ricci tensor of ∇^*, and let ρ^* be the Ricci operator of ∇^*. One then has

$$\mathcal{R}^*(v_1, v_2)w = \frac{1}{m-1} \{ \mathrm{Ric}^*(v_2, w)v_1 - \mathrm{Ric}^*(v_1, w)v_2 \},$$

$$\mathcal{R}(v_1, v_2)w = \frac{1}{m-1} \{ g(v_2, w)\rho^* v_1 - g(v_1, w)\rho^* v_2 \},$$

$$\mathcal{J}_{\nabla^*} y = \mathcal{R}^*(y, x)x = \frac{1}{m-1} \left\{ \mathrm{Ric}^*(x, x)y - \mathrm{Ric}^*(y, x)x \right\},$$

$$\mathcal{J}_{\nabla} y = \mathcal{R}(y, x)x = \frac{1}{m-1} \left\{ g(x, x)\rho^* y - g(y, x)\rho^* x \right\}.$$

Let H_P be the relative mean curvature; M is said to be a *relative umbilic* at a point P of M if $\mathrm{Ric}^* = (m-1)H_P g$ at P. In this setting, $\mathrm{Ric}^* = \mathrm{Ric}$ so

$$\mathcal{J}_{\nabla^*}(x)y = \mathcal{J}_{\nabla}(x)y = H_P \left\{ g(x, x)y - g(x, y)x \right\}$$

and consequently both (M, ∇) and (M, ∇^*) are projectively Osserman at P. If all points on M are relative umbilics, then the hypersurface with its relative normalization is said to be a *relative sphere*; (M, ∇) and (M, ∇^*) are globally projectively Osserman in such a situation. In particular: the spectra of the two connections, which are in general different, coincide on M.

5.1. Centroaffine geometry. The centroaffine normalization of an affine hypersurface is given by a transversal position vector; this is a relative normalization, and in terms of this relative normalization, any centroaffine hypersurface is a relative sphere. Thus, for all centroaffine hypersurfaces, the discussion of the last section applies.

5.2. Examples. Affine spheres in the unimodular hypersurface theory, relative spheres and centroaffine hypersurfaces are very big classes of hypersurfaces that play an important role in the theory. Thus the foregoing sections give further examples having the projectively Osserman property either pointwise or globally.

6. The Weyl projective tensor

There is no analogue of the Weyl conformal tensor in the geometry of a manifold M equipped with an affine connection ∇, and thus we can not speak of conformally Osserman in this context. Instead, a similar role is played by the Weyl projective curvature tensor which is an invariant of the projective class generated by ∇.

Recall that two connections ∇ and $\nabla^\sharp$ are said to be *projectively equivalent* if there exists a 1-form θ so that

$$\nabla_u v - \nabla^\sharp_u v = \theta(u)v + \theta(v)u. \tag{6.a}$$

We remark that the unparametrized geodesics of ∇ and $\nabla^\sharp$ coincide if and only if both connections are projectively equivalent. Thus the algebraic definition given in equation (6.a) has great geometric significance.

The *projective curvature operator* $\mathcal{P} = \mathcal{P}_\nabla$ is defined by:

$$\mathcal{P}(v_1, v_2)w := \mathcal{R}(v_1, v_2)w + \frac{1}{m^2 - 1}\left\{m\operatorname{Ric}(v_1, w)v_2 + \operatorname{Ric}(w, v_1)v_2\right\}$$
$$- \frac{1}{m^2 - 1}\left\{m\operatorname{Ric}(v_2, w)v_1 + \operatorname{Ric}(w, v_2)v_1\right\}$$
$$+ \frac{1}{m + 1}\left\{\operatorname{Ric}(v_1, v_2)w - \operatorname{Ric}(v_2, v_1)w\right\}.$$

If ∇ is Ricci symmetric, we have a somewhat simpler form:

$$\mathcal{P}(v_1, v_2)w := \mathcal{R}(v_1, v_2)w - \frac{1}{m - 1}\left\{\operatorname{Ric}(v_2, w)v_1 - \operatorname{Ric}(v_1, w)v_2\right\}.$$

This operator satisfies the identities:

$$\mathcal{P}(v_1, v_2) = -\mathcal{P}(v_2, v_1), \quad \text{and}$$
$$\mathcal{P}(v_1, v_2)v_3 + \mathcal{P}(v_2, v_3)v_1 + \mathcal{P}(v_3, v_1)v_2 = 0.$$

If ∇ and $\nabla^\sharp$ are projectively equivalent and Ricci symmetric, then $\mathcal{P}_\nabla = \mathcal{P}_{\nabla^\sharp}$. We use $\mathcal{P}$ to define the *projective Jacobi operator* $\mathcal{J}_P(x) : y \rightarrow \mathcal{P}(y, x)x$; the fundamental observation is then that if ∇ is projectively equivalent to $\nabla^\sharp$, then

$$\mathcal{J}_{P,\nabla}(x) = \mathcal{J}_{P,\nabla^\sharp}(x) \quad \text{for all} \quad x.$$

We note that we are using the word *projective* in two different settings. We shall say that (M, ∇) is *pointwise projectively Weyl Osserman* if for every point $P \in M$, there is a subset $\mathcal{S}_P \subset \mathbb{C}$ so that for any tangent vector $x \in T_P M$, $\operatorname{Spec}\{\mathcal{J}_{P,\nabla}(x)\} = c(x)\mathcal{S}_P$ for a suitably chosen constant $c(x)$. This notion plays the role in the affine setting that the notion 'conformal Osserman' plays in the geometric setting. In particular, any projectively flat manifold is necessarily projectively Weyl Osserman.

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