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Clifford's chain of theorems in strictly convex Minkowski planes

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Abstract. E. Asplund and B. Grünbaum emphasized in [2] that theorems on circles in the Euclidean plane often remain valid in normed (i.e., Minkowski) planes if they are restricted to Minkowskian circles of equal radii. We present various new results in the spirit of this observation. Namely, for circles of equal radii Clifford's chain of theorems (see [6], p. 262) will be completely extended to strictly convex Minkowski planes, and various new geometric properties of the related configurations are derived, too.

1. Introduction

A well known chain of theorems on systems of straight lines goes back to W. K. Clifford (see [5], pp. 38–54), and suitably using inversive geometry (cf. [19], Chapter VI), H. S. M. Coxeter [6, p. 262] formulated it in equivalent form as follows: *Let C_1, C_2, C_3 be coplanar circles passing through a common point p, and let C_i, C_j meet again in p_{ij}. Then p_{12}, p_{13}, p_{23} lie on a circle C_{123}; circles C_{123}, C_{124}, C_{134}, C_{234} meet in a point p_{1234}; points $p_{1234}, \dots, p_{2345}$ lie on a circle C_{12345}; circles $C_{12345}, \dots, C_{23456}$ meet in a point p_{123456}; and so on ad infinitum.*

In Figure 1 the case "$C_{123}, \dots, C_{234}$ meet in a point p_{1234}" (i.e., four starting circles of equal radii) is shown. For arbitrary radii, this case was found already by J.-V. Poncelet in 1817 and, independently, by J. Steiner in 1827, whereas the next case (five given circles) was added by A. Miquel; see [10], pp. 463–464,

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and, for Steiner's contributions, [23]. It is obvious that this chain of theorems yields the so-called (inverse) Clifford configuration $(2^{n-1}, n)$ of 2^{n-1} points, with n circles through each of those points. This circle configuration has far-reaching relations to well known configurations (e.g., obtained by Cox's chain of theorems in real projective 3-space) and to certain polytopes; for related discussions and results we refer to [27], [7], p. 141, [13], [11], and the review to the latter paper given by H. S. M. Coxeter (MR 49#7904). Special cases with their relations to famous topics of classical geometry (such as Miquel's theorem or the Platonic solids) were presented in [12], [18], [13], and [9]. Furthermore, in [27] and, independently, in [3] it was proved that the subcase of n starting circles with *equal* radii implies that *all* 2^{n-1} circles of the respective Clifford configuration have the same radius, which is directly related to our considerations here, since we want to investigate Clifford's chain of theorems for circles of equal radii in strictly convex normed planes.

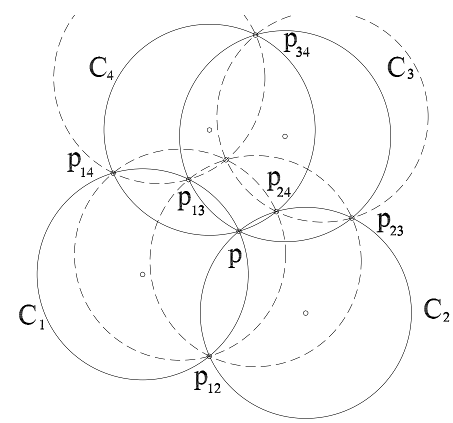

Figure 1.

Let $\mathcal{C}$ denote a bounded, closed, convex curve in the real affine plane $\mathbb{A}_2$ which is centered at the origin O. The curve $\mathcal{C}$ induces a *norm* $\|\cdot\|$, i.e., $(\mathbb{A}_2, \mathcal{C})$ is a *normed or Minkowski plane* with *unit circle* $\mathcal{C}$, see, e.g., [17] and [16], Section 2.2. A normed plane for which $\mathcal{C}$ is strictly convex (i.e., $\mathcal{C}$ does not contain a line segment) is itself called *strictly convex.* For n-dimensional *Minkowski spaces* we refer to Thompson's book [24] and to the surveys [17] and [15].

Any curve C in $(\mathbb{A}_2, \mathcal{C})$ of the type $x + \lambda\,\mathcal{C} =: C(x, \lambda)$, where $x \in (\mathbb{A}_2, \mathcal{C})$ and λ is a positive real number, is called a *(Minkowski) circle* with *center* x and *radius* λ. As observed by E. Asplund and B. Grünbaum (see [2]), various theorems on circles in the Euclidean plane remain valid in Minkowski geometry if all the circles are of the same radius.

In this sense, we will extend Clifford's chain of theorems (see above and Remark 3.1) for circles of the same radius from the Euclidean plane (see once more [6], p. 262) to all strictly convex Minkowski planes. Also we will present various interesting geometric properties of Clifford configurations. Most likely, some of these properties are even new in the Euclidean plane.

2. Some preliminaries

For later use we present some remarks and lemmas.

Remark 2.1. Any three non-collinear points in a strictly convex Minkowski plane determine at most one circle containing them, and if the plane is strictly convex and smooth, then there exists exactly one such circle; see [2] and [17], p. 107 and pp. 127–128. Any two circles in a strictly convex Minkowski plane intersect in at most two points, see [17], Section 3.2.

Also we will use the following result of Asplund and sc Grünbaum which is proved in [2].

Lemma 2.1. *If x_1, x_2 are two different points in a strictly convex normed plane, and $y_1, y_2 \in C(x_1, \lambda) \cap C(x_2, \lambda)$ with $y_1 \neq y_2$, then $x_1 + x_2 = y_1 + y_2$.*

For circles of the same size in a strictly convex Minkowski plane $(\mathbb{A}_2, \mathcal{C})$ the following statement holds: if p_1, p_2, p_3 are distinct points from a Minkowskian circle $C(x, \lambda)$, and $C(x_i, \lambda)$, $i = 1, 2, 3$, are three Minkowskian circles different from $C(x, \lambda)$ each of which contains two of the three points p_i, then $\bigcap_{i=1}^{3} C(x_i, \lambda)$ is not empty and consists of precisely one point p, where

$$p = p_1 + p_2 + p_3 - 2x.$$

Remark 2.2. The point p is called the $\mathcal{C}$*-orthocenter* of the triangle $p_1p_2p_3$; various properties of $\mathcal{C}$-orthocenters of triangles are discussed in [2] and [14]. In particular, in [14] the relation between $\mathcal{C}$-orthocentricity and James orthogonality of normed linear spaces is clarified. (James orthogonality is defined below, after the proof of Theorem 4.2. For further orthogonality concepts we refer to [1], §3, §4, §7, and §8, [4], and [24], §3.5.)

With the additional assumption of $\mathcal{C}$ being smooth, this result is due to Asplund and Grünbaum [2], for all strictly convex normed planes we refer to

[1], p. 32, and [24], pp. 104–106. The following lemma is a consequence of this observation.

Lemma 2.2. *Let $C(x_i, \lambda)$ be three circles in a strictly convex Minkowski plane all passing through a point p. Let p_i, in each case, be the second intersection point of the circles $C(x_j, \lambda)$ and $C(x_k, \lambda)$, where $\{i, j, k\} = \{1, 2, 3\}$. Then*

$$C\left(\frac{1}{2}\,(p_1 + p_2 + p_3 - p), \lambda\right)$$

is the circumcircle of the triangle $p_1p_2p_3$.

Remark 2.3. The Euclidean case of this lemma is known as Ţiţeica's theorem or Johnson's theorem; see [18], [25], and [13].

3. Clifford's chain of theorems

Theorem 3.1 below gives an extension to strictly convex Minkowski planes of what is shown in Figure 1: if four circles with equal radii pass through one point, then the circumcircles of the four triangles formed by the triples of the "second intersection points" of these four circles again have a point in common.

Theorem 3.1. *Let $C_i = C(x_i, \lambda)$, $i = 1, 2, 3, 4$, be four circles passing through a point p in a strictly convex Minkowski plane $(\mathbb{A}_2, \mathcal{C})$. Let p_{ij} be the second intersection of the circles C_i and C_j, and C_{ijk} be the circumcircle of the triangle $p_{ij}p_{jk}p_{ki}$, where $i, j, k \in \{1, 2, 3, 4\}$ with $i \neq j, j \neq k, k \neq i$. Then the four circles C_{123}, C_{234}, C_{341}, C_{241} all pass through the point p_{1234}, see Figure 2.*

Proof. According to Lemma 2.2, the circle C_{ijk} is

$$C\left(\frac{1}{2}\,(p_{ij} + p_{jk} + p_{ki} - p), \lambda\right). \tag{1}$$

We will prove that the point

$$p_{1234} = x_1 + x_2 + x_3 + x_4 - 3p \tag{2}$$

lies on all the circles C_{ijk}. By (1) and Lemma 2.1 we have

$$\begin{aligned} y_{ijk} &= \frac{1}{2}\,(p_{ij} + p_{jk} + p_{ki} - p) \\ &= \frac{1}{2}\,\big[(x_i + x_j - p) + (x_j + x_k - p) + (x_k + x_i - p) - p\big] \end{aligned}$$

$$= x_i + x_j + x_k - 2p \tag{3}$$

for the center y_{ijk} of the circle C_{ijk}. Therefore

$$\|y_{ijk} - p_{1234}\| = \|(x_i + x_j + x_k - 2\,p) - (x_1 + x_2 + x_3 + x_4 - 3p)\| = \|p - x_l\| = \lambda, \tag{4}$$

where $l = 1, 2, 3, 4$ with $l \neq i$, $l \neq j, l \neq k$. Equation (4) means that the point p_{1234} lies on the circle C_{ijk}. □

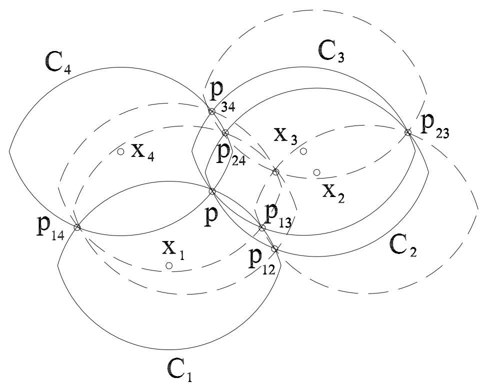

Figure 2.

Remark 3.1. The Euclidean version of Theorem 3.1 (see Figure 1) for circles of arbitrary size is known as *Clifford's first theorem*, see, e.g., [6, p. 262].

The next theorem is the extension of the so-called *second Clifford theorem* (see again [6, p. 262]) for circles of equal radii to all strictly convex Minkowski planes.

Theorem 3.2. *Under the assumption of Theorem 3.1, let $C_5 = C(x_5, \lambda)$ be a fifth circle through p. Then the five points p_{1234}, p_{1235}, p_{1245}, p_{1345}, p_{2345} lie on a circle C_{12345}.*

Proof. The proof of Theorem 3.1 (see, in particular, (2)) implies that

$$p_{ijkl} = x_i + x_j + x_k + x_l - 3p = x_1 + x_2 + x_3 + x_4 + x_5 - x_m - 3p, \tag{5}$$

where $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$. Let

$$y_{12345} = x_1 + x_2 + x_3 + x_4 + x_5 - 4p. \tag{6}$$

We will prove that all the points p_{ijkl} lie on the circle $C(y_{12345}, \lambda)$. Indeed,

$$\|y_{12345} - p_{ijkl}\| = \|(x_1 + x_2 + x_3 + x_4 + x_5 - 4p) \\ - (x_1 + x_2 + x_3 + x_4 + x_5 - x_m - 3p)\| = \|x_m - p\| = \lambda. \quad \square$$

Theorem 3.1 and Theorem 3.2 can be generalized for an arbitrary number of circles.

Theorem 3.3. *In a strictly convex Minkowski plane* $(\mathbb{A}_2, \mathcal{C})$ *let there be given* $2n$ $(n \geq 2)$ *circles* $C_1 = C(x_1, \lambda)$, $C_2 = C(x_2, \lambda), \ldots,$ $C_{2n} = C(x_{2n}, \lambda)$ *passing through a point* p. *For* $k = 1, \ldots, 2n$ *let*

$$p_{12\ldots\widehat{k}\ldots\widehat{\imath}\ldots 2n} = x_1 + \ldots + x_{2n} - x_k - x_i - (2n-3)\, p,$$

where $i \neq k$ *and* $i = 1, \ldots, 2n$.

(i) *For any* $k = 1, \ldots, 2n$ *all the points* $p_{12\ldots\widehat{k}\ldots\widehat{\imath}\ldots 2n}$ *lie on the circle*

$$C_{12\ldots\widehat{k}\ldots 2n} = C\bigl(x_1 + \ldots + x_{2n} - x_k - (2n-2)\, p, \lambda\bigr).$$

(ii) *For any* $k = 1, \ldots, 2n$ *all the circles* $C_{12\ldots\widehat{k}\ldots 2n}$ *pass through the point*

$$x_1 + \ldots + x_{2n} - (2n-1)\, p.$$

(iii) *If* $C_{2n+1} = C(x_{2n+1}, \lambda)$ *is also a circle through* p *and*

$$p_{12\ldots\widehat{\jmath}\ldots 2n} = x_1 + \ldots + x_{2n} - x_j - (2n-1)\, p,$$

where $j = 1, \ldots, 2n+1$, *then all the points* $p_{12\ldots\widehat{\jmath}\ldots 2n}$ *lie on the circle*

$$C\bigl(x_1 + \ldots + x_{2n+1} - 2n\, p, \lambda\bigr).$$

Proof. (i) Since for any $k = 1, \ldots, 2n$ we have

$$\Bigl\|\bigl[x_1+\ldots+x_{2n}-x_k-(2n-2)\, p\bigr]-\bigl[x_1+\ldots+x_{2n}-x_k-x_i-(2n-3)\, p\bigr]\Bigr\| = \|x_i-p\| = \lambda,$$

it follows that the points $x_1 + \ldots + x_{2n} - x_k - x_i - (2n-3)\, p$ lie on the circles $C_{12\ldots\widehat{k}\ldots 2n}$.

(ii) To prove (ii) it is sufficient to see that for any $k = 1, \ldots, 2n$

$$\Bigl\|\bigl[x_1+\ldots+x_{2n}-(2n-1)\, p\bigr]-\bigl[x_1+\ldots+x_{2n}-x_k-(2n-2)\, p\bigr]\Bigr\| = \|x_k-p\| = \lambda.$$

(iii) This statement follows from the fact that

$$\Bigl\|\bigl(x_1+\ldots+x_{2n+1}-2n\, p\bigr)-\bigl[x_1+\ldots+x_{2n}-x_j-(2n-1)\, p\bigr]\Bigr\|=\|x_j-p\|=\lambda. \qquad \square$$

Remark 3.2. An interesting property of Clifford configurations in strictly convex normed planes is depicted in Figure 3 for the case of three starting circles. Suitably connecting all occurring circle midpoints and all interesting intersection points of the configuration, one gets a cubical dissection (see [22]) of a zonogon, i.e., of a centrally symmetric polygon. (A second cubical dissection of the depicted hexagon can be obtained with the help of the midpoint of the fourth circle spanned by the three pairwise intersection points in the figure.) This allows interesting spatial interpretations of this geometric figure. In particular, for three starting circles one gets the opportunity to use spatial interpretations for generalizing theorems from planar elementary geometry which are related to the Euler line, the Feuerbach circle, the concurrence of the altitudes in the triangles, e.g. formed by the three circle midpoints, etc.; for the Euclidean situation we refer to [20], Ch. 10, and for strictly convex normed planes to [13] and [14].

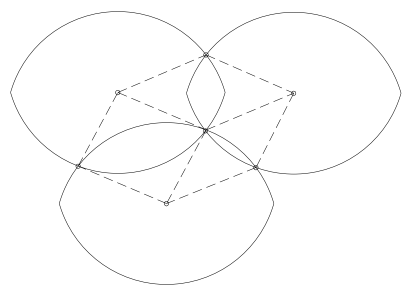

Figure 3.

4. Properties of Clifford configurations in strictly convex normed planes

4.1. An even Clifford configuration. Let $C_i = C(x_i, \lambda)$, p, p_{ij}, C_{ijk}, and p_{1234} be as in Theorem 3.1. Due to certain symmetry properties it makes sense to use the following notations throughout this section: $C_l^* := C_{ijk}$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$, and $p^* := p_{1234}$. So by (2) we get

$$p^* = x_1 + x_2 + x_3 + x_4 - 3p. \tag{7}$$

The center of C_l^* is denoted by x_l^*. It is clear (see Lemma 2.2) that the radius of C_l^* is also λ, and that

$$x_l^* = \frac{1}{2}\,(p_{ij} + p_{jk} + p_{ki} - p). \tag{8}$$

In view of (3), equation (8) can be rewritten as

$$x_l^* = x_1 + x_2 + x_3 + x_4 - x_l - 2p. \tag{9}$$

We call the configuration $\mathbf{C} = \{C_1,\ C_2,\ C_3,\ C_4,\ C_1^*,\ C_2^*,\ C_3^*,\ C_4^*,\ p_{ij},\ p,\ p^*\}$ of circles and points an *even*[1] *Clifford configuration*, the set $\{C_1, C_2, C_3, C_4\}$ the *first bunch of* $\mathbf{C}$, and the centers x_1, x_2, x_3, x_4 of C_1, C_2, C_3, C_4 the *first skeleton of* $\mathbf{C}$. Analogously, $\{C_1^*, C_2^*, C_3^*, C_4^*\}$ and x_1^*, x_2^*, x_3^*, x_4^* will be called the *second bunch of* $\mathbf{C}$ and the *second skeleton of* $\mathbf{C}$, respectively.

It is clear that the first skeleton of any even Clifford configuration forms a cyclic quadrangle. The next proposition guarantees that the second skeleton also forms a cyclic quadrangle.

Proposition 4.1. *Let* $\mathbf{C} = \{C_i,\ C_i^*,\ p_{ij},\ p,\ p^*\}$ *be an even Clifford configuration, where* $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, *and* $C_i = C(x_i, \lambda)$, $C_i^* = C(x_i^*, \lambda)$. *Then all the points* x_i^* *lie on the circle* $C(p^*, \lambda)$.

Proof. Using (7) and (9) we obtain

$$\begin{aligned} \|p^* - x_i^*\| &= \|(x_1 + x_2 + x_3 + x_4 - 3p) - (x_1 + x_2 + x_3 + x_4 - x_i - 2p)\| \\ &= \|x_i - p\| = \lambda. \end{aligned}$$

□

For strictly convex normed planes, in [14] the *Feuerbach circle* (also called *Euler circle*) of an n-gon $x_1 x_2 \dots x_n$ inscribed to the circle $C(p, \lambda)$ is defined by

$$C\left(\frac{1}{2}\left[\sum_{i=1}^{n} x_i - (n-2)\, p\right], \frac{1}{2}\,\lambda\right). \tag{10}$$

Thus for a triangle in the Euclidean plane the defined Feuerbach circle coincides with the well known *nine-point circle* (or *classical Feuerbach circle*) of this triangle, see, e.g., [8], § 1.8, p. 20, and [25], p. 159. For a cyclic quadrangle in the Euclidean plane the circle (10) also coincides with the known *Feuerbach circle of a quadrangle*, see pp. 22–23 and pp. 108–109 in [26], as well as [21]. Various properties of Feuerbach circles in strictly convex Minkowski planes are collected in [14].

Theorem 4.1. *For any even Clifford configuration the Feuerbach circles of the first and of the second skeleton coincide.*

[1] Clifford's chain of theorems can be "separated" into two subfamilies of theorems, depending on whether the number of the "starting circles" is even or odd. We will follow this separation into two types of theorems, but for the sake of convenient representation only when the number of the given circles is four and five, cf. again [6], p. 262.

Proof. For $i, j \in \{1,2,3,4\}, i \neq j$, and $C_i = C(x_i, \lambda)$, $C_i^* = C(x_i^*, \lambda)$ let $\mathbf{C} = \{C_i,\ C_i^*,\ p_{ij},\ p,\ p^*\}$ be an arbitrary even Clifford configuration. Hence p is the circumcenter of $x_1x_2x_3x_4$, and p^* is the circumcenter of $x_1^*x_2^*x_3^*x_4^*$ (see Proposition 4.1). For the Feuerbach circle of $x_1x_2x_3x_4$ and $x_1^*x_2^*x_3^*x_4^*$ we get by (10)

$$C\left(\frac{1}{2}\,(x_1+x_2+x_3+x_4)-p, \frac{1}{2}\,\lambda\right) \tag{11}$$

and

$$C\left(\frac{1}{2}\,(x_1^*+x_2^*+x_3^*+x_4^*)-p^*, \frac{1}{2}\,\lambda\right), \tag{12}$$

respectively. Applying (7) and (9) to the center f^* of the second circle (12), we obtain

$$\begin{aligned} f^* &= \frac{1}{2}\,\big[4(x_1+x_2+x_3+x_4)-(x_1+x_2+x_3+x_4)-8p\big] \\ &\quad -(x_1+x_2+x_3+x_4-3p) = \frac{1}{2}\,(x_1+x_2+x_3+x_4)-p. \end{aligned}$$

This means that the circles (11) and (12) coincide. □

Remark 4.1. The point

$$\frac{1}{2}\,(x_1+x_2+x_3+x_4)-p \tag{13}$$

is called the *Feuerbach point* of the configuration $\mathbf{C}$.

Theorem 4.2. *If $\mathbf{C} = \{C_i,\ C_i^*,\ p_{ij},\ p,\ p^*\}$ is an even Clifford configuration, where $i, j \in \{1,2,3,4\}$, $i \neq j$, and $C_i = C(x_i, \lambda)$, $C_i^* = C(x_i^*, \lambda)$, then*

(i) *the first and the second skeleton of* $\mathbf{C}$,

(ii) *the first and the second bunch of* $\mathbf{C}$,

(iii) *the points p and p^*,*

(iv) *the points p_{ij} and p_{kl}, where $\{i,j,k,l\} = \{1,2,3,4\}$,*

are symmetric with respect to the Feuerbach point of $\mathbf{C}$.

Proof. Let $f = \frac{1}{2}\,(x_1+x_2+x_3+x_4)-p$ be the Feuerbach point of $\mathbf{C}$.

(i) We have to prove that

$$\frac{1}{2}\,(x_i+x_i^*) = f.$$

Indeed, by (9) and (13) we get

$$\frac{1}{2}\,\big[x_i+(x_1+x_2+x_3+x_4-x_i-2p)\big] = \frac{1}{2}\,(x_1+x_2+x_3+x_4)-p.$$

The proofs of (ii)–(iv) are analogous to that of (i). □

For $i,j \in \{1,2,3,4\}, i \neq j$, and $C_i = C(x_i,\lambda)$, $C_i^* = C(x_i^*,\lambda)$ let us consider the first and the second skeleton of any even configuration $\mathbf{C} = \{C_i,\ C_i^*,\ p_{ij},\ p,\ p^*\}$ as complete quadrilaterals. We will prove that the lines $\langle pp_{ij}\rangle$ are James orthogonal to the sides of the quadrangle $x_1x_2x_3x_4$. The same holds for the point p^* and the sides of $x_1^*x_2^*x_3^*x_4^*$. (Note that the vector $x \in (\mathbb{A}_2,\mathcal{C})$ is *James* (*or isosceles*) *orthogonal* to $y \in (\mathbb{A}_2,\mathcal{C})$ if

$$\|x+y\| = \|x-y\|,$$

in which case we will write $x\#y$. Also we note that in the Euclidean plane the notion of James orthogonality coincides with that of usual orthogonality.) In [14] the following statement is proved.

Lemma 4.1. *If $x, y \in (\mathbb{A}_2,\mathcal{C})$, then*

$$\|x\| = \|y\| \iff x+y \# x-y.$$

Applying Lemma 4.1 to the parallelogram $x_ip_{ij}x_jp$ (see Lemma 2.1) in any even Clifford configuration $\mathbf{C} = \{C_i,\ C_i^*,\ p_{ij},\ p,\ p^*\}$, where $i,j \in \{1,2,3,4\}, i \neq j$, and $C_i = C(x_i,\lambda), C_i^* = C(x_i^*,\lambda)$, we get

$$x_i - x_j \# p - p_{ij}.$$

Thus we can state

Theorem 4.3. *For any even Clifford configuration $\mathbf{C} = \{C_i,\ C_i^*,\ p_{ij},\ p,\ p^*\}$, where $i,j \in \{1,2,3,4\}$, $i \neq j$, and $C_i = C(x_i,\lambda)$, $C_i^* = C(x_i^*,\lambda)$, the relation*

$$x_i - x_j \# p - p_{ij}$$

holds.

4.2. An odd Clifford configuration. Let $C_i = C(x_i,\lambda)$, $i = 1,\dots,5$, and p, $p_{1234}, p_{1235}, p_{1245}, p_{1345}, p_{2345}$ be as in Theorem 3.2. We write

$$p_{ijkl} =: p_m,$$

where $\{i,j,k,l,m\} = \{1,\dots,5\}$. Thus by (5) we have

$$p_i = x_1 + \ldots + x_5 - x_i - 3p. \tag{14}$$

We call the configuration $\{C_i,\ p,\ p_i\}$, $i = 1,\dots,5$, of circles and points an *odd Clifford configuration*, the pentagon $x_1 \dots x_5$ the corresponding *skeleton pentagon*, and $p_1 \dots p_5$ the corresponding *Clifford pentagon*. Since p is the common point of the circles $C_i = C(x_i, \lambda)$, the skeleton pentagon is inscribed to the circle $C(p, \lambda)$. Theorem 3.2 (Second Clifford Theorem) implies that the Clifford pentagon is also concyclic. By the proof of Theorem 3.2 we get that the circumcircle of the Clifford pentagon $p_1 \dots p_5$ is

$$C(x_1 + \ldots + x_5 - 4p,\ \lambda). \tag{15}$$

Theorem 4.4. *The Feuerbach circles of the skeleton pentagon and of the Clifford pentagon of any odd Clifford configuration coincide.*

Proof. Let $\mathbf{C} = \{C_i,\ p,\ p_i\}$, where $i = 1, \ldots, 5$, and $C_i = C(x_i, \lambda)$ be an odd Clifford configuration. The Feuerbach circle of the skeleton pentagon $x_1 \dots x_5$ is

$$C\left(\frac{1}{2}\,(x_1 + \ldots + x_5 - 3p), \frac{1}{2}\,\lambda\right), \tag{16}$$

see (10). Again by (10) and in view of (15) we obtain

$$C\left(\frac{1}{2}\,\big[p_1 + \ldots + p_5 - 3(x_1 + \ldots + x_5 - 4p)\big], \frac{1}{2}\,\lambda\right) \tag{17}$$

for the Feuerbach circle of the Clifford pentagon $p_1 \dots p_5$. Using (14) it is easy to check that the centers of the circles (16) and (17) coincide. □

Remark 4.2. We call the point

$$\frac{1}{2}\,(x_1 + \ldots + x_5 - 3p) \tag{18}$$

the *Feuerbach point* of the odd Clifford configuration $\mathbf{C}$.

In view of (14) and (18) we get immediately

Theorem 4.5. *The skeleton pentagon and the Clifford pentagon of any odd Clifford configuration are symmetric with respect to its Feuerbach point.*

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