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On the solution set of nonlinear evolution inclusions depending on a parameter[1]

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Abstract. In this paper examine nonlinear evolution inclusions depending on a parameter. The parameter appears in all the data of the problem, including the nonlinear operator. Using the general concept of G-convergence of operators, we prove three continuous dependence results for both the Vietoris and Hausdorff hyperspace topologies. Then we use these results to study the variational stability of a class of nonlinear, parabolic optimal control problems.

1. Introduction

In a recent paper [13], we examined evolution inclusions depending on a parameter and established the continuity properties of the solution set with respect to the parameter. We proved two such results in [13]: in the first (theorem 3.2), the abstract nonlinear operator modelling the partial differential term, does not depend on the parameter and only the orientor field does; in the second (theorem 3.3), the operator depends on the parameter but is also linear (semilinear inclusions). In theorem 3.2, we proved that the solution set, as a set-valued function (multifunction) of the parameter, is Vietoris and Hausdorff continuous, while in theorem 3.3, we established that the solution set is an upper semicontinuous (*u.s.c.*) multifunction of the parameter. The purpose of this paper is to improve theorems 3.2 and 3.3 of [13], by allowing the abstract operator to depend on the parameter and to be nonlinear. For this general situation, we show that the solution set multifunction is Vietoris and Hausdorff continuous (see theorems 3.1 and 3.3 this paper).

Our approach is different from that of [13], since here we use a recent stability result for the set of fixed points of Lipschitzian multifunctions,

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due to Rybinski [14]. By expressing the solution set of our evolution inclusion as the set of fixed points of an appropriately defined multifunction, we are able to exploit the above mentioned result of Rybinski and prove that the set of solutions of the nonlinear multivalued Cauchy problem is both Vietoris and Hausdorff continuous as a set-valued function of the parameter.

Having established these continuity results, in section 4 we use them to study a class of parametric Meyer-type optimal control problems, monitored by a nonlinear parabolic partial differential equation. We show (theorem 4.1) that the value of these problems, depends continuously on the parameter, while the optimal trajectories (states) multifunction is *u.s.c.* Such a sensitivity analysis, is useful both from the theoretical and applied viewpoints. It produces useful continuous dependence results which help us analyze parametric problems, it can produce robust computational schemes and finally it gives us information about the admissible tolerances in the specification of the mathematical models.

Previously, continuous dependence results were obtained by Vasilev [18] and Lim [7] for differential inclusions in $\mathbb{R}^n$ and by Tolstonogov [16] and Papageorgiou [11], who examined differential inclusions in Banach spaces. However, their continuity and boundedness hypotheses precludes the applicability of their work to multivalued partial differential equations and to distributed parameter optimal control problems.

Preliminaries

Let $T = [0, r]$ and Y a separable Banach space. Throughout this paper, we will be using the following notations:

$$P_{f(c)}(Y) = \{A \subseteq Y : \text{nonempty, closed (and convex)}\}$$
$$\text{and } P_{(w)k(c)}(Y) = \{A \subseteq Y : \text{nonempty, } (w-) \text{ compact (and convex)}\}.$$

A multifunction $F : T \to P_f(Y)$ is said to be measurable, if for all $z \in Y$, the $\mathbb{R}_+$-valued function $t \to d(z, F(t)) = \inf\{\|z - y\| : y \in F(t)\}$ is measurable. By $S_F^p (1 \leq p \leq \infty)$, we will denote the set of selectors of $F(\cdot)$, that belong in the Lebesgue–Bochner space $L^p(Y)$; i.e. $S_F^p = \{f \in L^p(Y) : f(t) \in F(t) \ a.e.\}$. This set may be empty. For a measurable $F(\cdot)$, it is nonempty if and only if $t \to \inf\{\|v\| : v \in F(t)\} \in L_+^p$. On $P_f(Y)$, we can define a generalized metric, known in the literature as the Hausdorff metric, by setting

$$h(A, B) = \max\left[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right]$$

for all $A, B \in P_f(Y)$. Recall that $(P_f(Y), h)$ is a complete metric space. If Λ is a complete metric space, a multifunction $G : \Lambda \to P_f(Y)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from Λ into $(P_f(Y), h)$. A multifunction G is said to be d-continuous if for all $z \in Y$, $\lambda \to d(z, G(\lambda))$ is a continuous $\mathbb{R}_+$-valued function. Clearly an h-continuous multifunction, is d-continuous, while the converse is not in general true. Also if $h(G(\lambda), G(\lambda')) \leq k d_\Lambda(\lambda, \lambda')$ $k > 0$, then we say that $G(\cdot)$ is h-Lipschitz.

Let $\{A_n, A\}_{n \geq 1} \subseteq 2^Y \setminus \{\emptyset\}$ and denote by s- the strong (norm) topology on Y and by w- the weak topology on Y. We define:

$$s\text{-}\underline{\lim}\, A_n = \{z \in Y : \lim d(z, A_n) = 0\}$$
$$= \{z \in Y : z = s\text{-}\lim z_n,\ z_n \in A_n,\ n \geq 1\}$$
$$s\text{-}\overline{\lim}\, A_n = \{z \in Y : \underline{\lim}\, d(z, A_n) = 0\}$$
$$= \{z \in Y : z = s\text{-}\lim z_{n_k},\ z_{n_k} \in A_{n_k},\ n_1 < n_2 < \dots < n_k < \dots\}$$
$$\text{and } w\text{-}\overline{\lim}\, A_n$$
$$= \{z \in Y : z = w\text{-}\lim z_{n_k},\ z_{n_k} \in A_{n_k},\ n_1 < n_2 < \dots < n_k < \dots\}.$$

From these definitions, it is clear that we always have

$$s\text{-}\underline{\lim}\, A_n \subseteq s\text{-}\overline{\lim}\, A_n \subseteq w\text{-}\overline{\lim}\, A_n.$$

If $s\text{-}\underline{\lim}\, A_n = s\text{-}\overline{\lim}\, A_n = A$, then we say that the A_n's converge to A in the Kuratowski sense, and denote it by $A_n \xrightarrow{K} A$ as $n \to \infty$. If $s\text{-}\underline{\lim}\, A_n = w\text{-}\overline{\lim}\, A_n = A$, then we say that the A_n's converge to A in the Kuratowski–Mosco sense and denote it by $A_n \xrightarrow{K-M} A$ as $n \to \infty$.

A multifunction $G : \Lambda \to P_f(Y)$ is said to be upper semicontinuous ($u.s.c.$) (resp. lower semicontinuous ($l.s.c.$)), if for all $U \subseteq Y$ open, the set $G^+(U) = \{\lambda \in \Lambda : G(\lambda) \subseteq U\}$ (resp. $G^-(U) = \{\lambda \in \Lambda : G(\lambda) \cap U \neq \emptyset\}$) is open in Λ. A multifunction which is both $u.s.c.$ and $l.s.c.$, is said to be continuous or Vietoris continuous, to emphasize that it is continuous into the hyperspace $P_f(Y)$ equipped with the Vietoris topology (see Klein–Thompson [5]). If $\overline{G(\Lambda)} = \overline{\bigcup_{\lambda \in \Lambda} G(\Lambda)}$ is compact in Y, then $G(\cdot)$ is Vietoris continuous if and only if for $\lambda_n \to \lambda$ in Λ, we have $G(\lambda_n) \xrightarrow{K} G(\lambda)$. This follows from remarks 1.6 and 1.8 of DeBlasi–Myjak [2]. Finally since on $P_k(Y)$ the Vietoris and Hausdorff topologies coincide (see Klein–Thompson [5], corollary 4.2.3, p. 41), we deduce that a $P_k(Y)$-valued multifunction is Vietoris continuous if and only if it is h-continuous (see also DeBlasi–Myjak [2], remark 1.9).

The following theorem was first proved by the author (see [11], theorem 3.1) and was recently improved by Rybinski (see [14] theorem 1 and the remark on page 33). Actually the result of Rybinski [14], is more general, but for the purpose of this paper, we only need the following special case:

Theorem 2.1. *If Z is a Banach space, $K \in P_{wk}(Z)$, $F_n, F : K \to P_{fc}(K)$ are h-Lipschitz multifunctions with the same Lipschitz constant $k \in (0,1)$ s.t. if $z_n \xrightarrow{s} z$ in Z, then $F_n(z_n) \xrightarrow{K-M} F(z)$,*

then if $L_n = \{z \in Z : z \in F_n(z)\}$ and $L = \{z \in Z : z \in F(z)\}$, we have $L_n \xrightarrow{K} L$.

Remark. The fixed point sets L_n, L are nonempty by Nadler's fixed point theorem [9].

Now let H be a Hilbert space and let X be a dense subspace of H carrying the structure of a separable, reflexive Banach space, which embeds into H continuously. Identifying H with its dual (pivot space), we have $X \to H \to X^\star$, with all embeddings being continuous and dense. Such a triple of spaces, is known in the literature as "evolution triple" or "Gelfand triple" (see Zeidler [20]). We will also assume that the embedding of X into H is also compact (in fact, this implies that $H \to X^\star$ is compact too). To have a concrete example in mind, let m be a positive integer and $2 \le p < \infty$. Let $Z \subseteq \mathbb{R}^N$ be a bounded domain and set $X = W_0^{m,p}(Z)$, $H = L^2(Z)$ and $X^\star = W^{-m,q}(Z)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then from the Sobolev embedding theorem, we know that $(X, H, X^\star)$ is an evolution triple and all embeddings are compact. By $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_\star$) we will denote the norm of X (resp. of H, $X^\star$). Also by $(\cdot\,,\cdot)$ we will denote the inner product of H and by $\langle\cdot\,,\cdot\rangle$ the duality brackets of the pair $(X, X^\star)$. The two are compatible in the sense that $\langle\cdot\,,\cdot\rangle|_{X\times H} = (\cdot\,,\cdot)$. Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. We define:

$$W_{pq}(T) = \{x(\cdot) \in L^p(X) : \dot{x} \in L^q(X^\star)\}.$$

The derivative involved in this definition is understood in the sense of vector valued distributions. Equipped with the norm $\|x\|_{W_{pq}(T)} = \left[\|x\|^2_{L^p(X)} + \|\dot{x}\|^2_{L^q(X^\star)}\right]^{1/2}$, the space $W_{pq}(T)$ becomes a separable, reflexive Banach space. It is well known that $W_{pq}(T)$ embeds continuously in $C(T, H)$; i.e. every element in $W_{pq}(T)$ has a unique representative in $C(T, H)$. Since we have assumed that $X \to H$ compactly, we have that $W_{pq}(T) \to L^p(H)$ compactly (see Zeidler [20], p. 450). Finally if $p = 2$, we simply write $W(T)$ for $W_{2,2}(T)$, and this is a Hilbert space if X is. Furthermore from Nagy [10], we know that in this case $W(T) \to C(T, H)$ compactly.

Let $(X, H, X^\star)$ be an evolution triple with X embedding into H compactly. Let A_n, $A : X \to X^\star$ be a sequence of operators. Following Kolpakov [6], we say that the sequence $\{A_n\}_{n\ge1}$ G-converges to A if and only if for every $n \geq 1$, A_n^{-1}, $A^{-1} : X^\star \to X$ are defined and for any $x^\star \in X^\star$, $A_n^{-1}x^\star \xrightarrow{w} A^{-1}x$ in X (and hence strongly in H). This is the abstract nonlinear formulation of a convergence notion which was first introduced by Spagnolo [15], in order to study the convergence of the solutions of a sequence of elliptic problems. The abstract, linear case was studied in detail by Zhikov–Kozlov–Oleinik–Kha Ten Ngoan [21] and by Zhikov–Kozlov–Oleinik [22].

Let $T = [0, r]$ and $(X, H, X^\star)$ an evolution triple as above (so X embeds into H compactly). Also let Λ be a metric space (the parameter space). We consider the following evolution equation:

$$\left\{ \begin{aligned} \dot{x}(t) + A(t, x(t), \lambda) &= f(t) \\ x(0) = x_0(\lambda) &\in H \end{aligned} \right\} \tag{1}$$

with $f \in L^q(H)$. We will need the following hypothesis:

$H(A)$: $A : T \times X \times \Lambda \to X^\star$ is an operator *s.t.*

(*1*) for all $t, t+\tau \in T$, all $\lambda \in \Lambda$ and all $x \in X$, we have

$$\|A(t,x,\lambda) - A(t+\tau, x, \lambda)\|_\star \leq c(\tau)(1 + \|x\|^{p-1}), \quad 2 \leq p < \infty,$$

with $c(\cdot)$ nondecreasing, $c(\tau) \downarrow 0$ as $\tau \to 0^+$ and is independent of $x \in X$ and $\lambda \in \Lambda$,

(*2*) $x \to A(t, x, \lambda)$ is hemicontinuous (i.e. for all $x, y, z \in X$, $\beta \to \langle A(t, x+\beta y, \lambda), z\rangle$ is continuous from [0,1] into $\mathbb{R}$), and $A(t, 0, \lambda) = 0$ for all $(t, \lambda) \in T \times \Lambda$,

(*3*) if $\lambda_n \to \lambda$ *in* Λ, then $A(t, x, \lambda_n) \xrightarrow{G} A(t, x, \lambda)$ for all $(t, x) \in T \times X$,

(*4*) $c_B\|x-y\|^p \leq \langle A(t,x,\lambda) - A(t,y,\lambda), x-y\rangle$, for all $t \in T$, $x, y \in X$ and all $\lambda \in B \subseteq \Lambda$, B compact and with $c_B > 0$ (strong monotonicity of $A(t, \cdot, \lambda)$),

(*5*) $\|A(t,x,\lambda)\|_\star \leq c_{1B}(1 + \|x\|^{p-1})$ for all $t \in T$, $\lambda \in B \subseteq \Lambda$, B compact and with $c_{1B} > 0$.

Remarks.

(*i*) Since $A(t, \cdot, \lambda)$ is strongly monotone and $A(t, o, \lambda) = 0$, we get that $c_{1B}\|x\|^p \leq \langle A(t,x,\lambda), x\rangle$ for all $t \in T$, $x \in X$ and $\lambda \in B \subseteq \Lambda$ compact (coercivity property).

(*ii*) Since $A(t, \cdot, \lambda)$ is monotone, hemicontinuous, it is demicontinuous; i.e. if $x_n \xrightarrow{s} x$ in X, then $A(t, x_n, \lambda) \xrightarrow{w} A(t, x, \lambda)$ in $X^\star$ (see Zeidler [20], p. 596).

(*iii*) Because $A(t, \cdot, \lambda)$ is a monotone, hemicontinuous, coercive operator, it is surjective (see Zeidler [20], theorem 32.H, p. 887). This combined with the strong monotonocity hypothesis $H(A)$

(*4*), implies that for all $(t,\lambda)\in T\times\Lambda,\ A^{-1}(t,\cdot,\lambda):X^\star\to X$ is well defined. So hypothesis $H(A)$ (*3*) makes sense.

From theorem 30.A, p. 771 of Zeidler [20], we know that for every $\lambda\in\Lambda$, problem (*1*) above has a unique solution $p(f,\lambda)(\cdot)\in W_{pq}(T)$. Recalling that $W_{pq}(T)$ embeds continuously into $C(T,H)$, we see that the initial condition in (*1*) makes sense. For this initial vector, we assume the following:

H_0: $\lambda\to x_0(\lambda)$ is continuous from Λ into H.

The next proposition examines the continuity properties of the solution map $p:L^q(H)\times\Lambda\to C(T,H)$:

Proposition 2.1. *If hypotheses $H(A)$ and H_0 hold, then $p(\cdot,\cdot)$ is continuous.*

Proof. Let $(f_n,\lambda_n)\to(f,\lambda)$ in $L^q(H)\times\Lambda$ and let $x_n=p(f_n,\lambda_n)$ and $x=p(f,\lambda)$. Let $y_n\in W_{pq}(T)\to C(T,H)$ be the unique solution of the evolution equation

$$\left\{\begin{matrix}\dot y_n(t)+A(t,y_n(t),\lambda_n)=f(t)\ a.e.\\ y_n(0)=x_0(\lambda).\end{matrix}\right\}$$

From theorem 1 (see also Lemmata 6 and 7) of Kolpakov [6], we have that $y_n\xrightarrow{w}x$ in $W_{pq}(T)$ and since $W_{pq}(T)$ embeds compactly into $L^p(H)$, we have that $y_n\xrightarrow{s}x$ in $L^p(H)$. Exploiting the monotonicity of the operator $A(t,\cdot,\lambda)$, we have

$$\begin{aligned}&\langle\dot x_n(t)-\dot y_n(t),x_n(t)-y_n(t)\rangle\le(f_n(t)-f(t),x_n(t)-y_n(t))\ a.e.\\ &\Rightarrow\frac12\frac{d}{dt}|x_n(t)-y_n(t)|^2\le|f_n(t)-f(t)|\ .\ |x_n(t)-y_n(t)|\ a.e.\\ &\qquad\Rightarrow\frac12|x_n(t)-y_n(t)|^2\le\frac12|x_0(\lambda_n)-x_0(\lambda_n)|^2\\ &\qquad\quad+\int_0^t|f_n(s)-f(s)|\ .\ |x_n(s)-y_n(s)|ds.\end{aligned}$$

Invoking lemma A.5, p. 157 of Brezis [23], we get

$$|x_n(t)-y_n(t)|\le|x_0(\lambda_n)-x_0(\lambda)|+\int_0^t|f_n(s)-f(s)|ds,\quad t\in T$$

$$\Rightarrow x_n\xrightarrow{s}x\quad\text{in } C(T,H)\Rightarrow p(\cdot,\cdot)\text{ is continuous.}\qquad\square$$

3. Parametric evolution inclusions

Let $T = [0, r]$ and $(X, H, X^\star)$ an evolution triple as in section 2, with X embedding into H compactly and X being uniformly smooth. We will consider the following evolution inclusion parametrized by elements in Λ:

$$\left\{ \begin{array}{c} \dot{x}(t) + A(t, x(t), \lambda) \in F(t, x(t), \lambda) \ a.e. \\ x(0) = x_0(\lambda). \end{array} \right\} \tag{2}$$

We will need the following hypothesis on the orientor field $F(t, x, \lambda)$:

$H(F) : F : T \times H \times \Lambda \to P_{fc}(H)$ is a multifunction *s.t.*

(*1*) $t \to F(t, x, \lambda)$ is measurable,

(*2*) $h(F(t, x, \lambda), F(t, y, \lambda)) \leq k_B(t)|x - y|$ *a.e.* for all $\lambda \in B \subseteq \Lambda$, B compact and with $k_B \in L^1_+$,

(*3*) $\lambda \to F(t, x, \lambda)$ is d-continuous,

(*4*) $|F(t, x, \lambda)| = \sup\{|y| : y \in F(t, x, \lambda)\} \leq \alpha_B(t) + \beta_B(t)|x|^{2/q}$ *a.e.* for all $\lambda \in B \subseteq \Lambda$, B compact and with $\alpha_B, \beta_B \in L^q_+$.

By a solution of (2), we mean a function $x \in W_{pq}(T) \to C(T, H)$ such that $\dot{x}(t) + A(t, x(t), \lambda) = f(t)$ *a.e.*, $x(0) = x_0(\lambda)$, with $f \in L^q(H)$, $f(t) \in F(t, x(t), \lambda)$ *a.e.*

We will denote the solution set of problem (2) by $S(\lambda) \subseteq W_{pq}(T) \to C(T, H)$. Our goal is to investigate the continuity properties of the multifunction $\lambda \to S(\lambda)$. We already know (see [13]), that $S(\lambda) \in P_k(L^p(H))$ and $S(\lambda) \in P_f(C(T, H))$.

Theorem 3.1. *If hypotheses $H(A)$, $H(F)$ and H_0 hold, then $S : \Lambda \to P_k(L^p(H))$ is Vietoris and Hausdorff continuous.*

Proof. Let $B \subseteq \Lambda$ be compact. We will start by obtaining an a priori bound for the elements in $\bigcup_{\lambda \in B} S(\lambda)$. So let $x(\cdot) \in \bigcup_{\lambda \in B} S(\lambda)$. Then by definition, there exists $f \in S^q_{F(\cdot, x(\cdot), \lambda)}$ *s.t.*

$$\left\{ \begin{array}{c} \dot{x}(t) + A(t, x(t), \lambda) = f(t) \ a.e. \\ x(0) = x_0(\lambda). \end{array} \right\}$$

We have:

$$\begin{aligned} &\langle \dot{x}(t), x(t) \rangle + \langle A(t, x(t), \lambda), x(t) \rangle = (f(t), x(t)) \ a.e. \\ \Rightarrow\ &\frac{1}{2}\frac{d}{dt}|x(t)|^2 + c_B\|x(t)\|^p \leq (f(t), x(t)) = \langle f(t), x(t) \rangle \\ &\qquad \leq \|f(t)\|_\star . \|x(t)\| \ a.e. \end{aligned}$$

Applying Cauchy's inequality with $\varepsilon > 0$ on the rigt-hand side, we get

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 + c_B\|x(t)\|^p \le \frac{\varepsilon^q}{q}\ \|f(t)\|_\star^q + \frac{1}{p\varepsilon^p}\|x(t)\|^p \ a.e. \tag{3}$$

Let $\varepsilon^p = \frac{1}{pc_B}$. Then we get

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 \le M\|f(t)\|_\star^q \qquad (\text{with } M = \frac{1}{q(pc_B)^{q-1}} > 0)$$

$$\Rightarrow |x(t)|^2 \le |x_0(\lambda)|^2 + 2M\int_0^t \|f(s)\|_\star^q ds \le |x_0(\lambda)|^2 + 2M\int_0^t \gamma^q|f(s)|^q ds$$

where $\gamma > 0$ is such that $\|\cdot\|_\star \le \gamma|\cdot|$. It exists since by hypothesis, X embeds into H continuously. So using hypothesis $H(F)$ *(4)*, we get

$$|x(t)|^2 \le |x_0(\lambda)|^2 + 2M\int_0^t \gamma^q(\alpha_B(s) + \beta_B(s)|x(s)|^{2/q})^q ds$$

$$\le |x_0(\lambda)|^2 + 2^{q+1}M\gamma^q\int_0^t (\alpha_B(s)^q + \beta_B(s)^q|x(s)|^2)ds.$$

Because of hypothesis H_0, there exists $M' > 0$ *s.t.* $|x_0(\lambda)| \le M'$ for all $\lambda \in B$. Thus invoking Gronwall's inequality, we deduce that there exists $M_1 > 0$ *s.t.*

$$|x(t)| \le M_1 \quad \text{for all } t \in T \text{ and all } x \in \bigcup_{\lambda\in B} S(\lambda). \tag{4}$$

Next in inequality (3) above, we let $\varepsilon^p = \frac{2}{pc_B}$. We get

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 + \frac{c_B}{2}\|x(t)\|^p \le 2M\gamma^q|f(t)|^q \ a.e.$$

$$\Rightarrow |x(t)|^2 + c_B\int_0^t \|x(s)\|^p ds \le$$

$$\le 4M\gamma^q\int_0^t 2^q(\alpha_B(s)^q + \beta_B(s)^q|x(s)|^2)ds + (M')^2$$

$$\Rightarrow c_B\int_0^t \|x(s)\|^p ds \le 2^{q+2}M\gamma^q\int_0^t (\alpha_B(s)^q + \beta_B(s)^q M_1^2)ds + (M')^2$$

$$\Rightarrow \|x\|_{L^p(X)} \leq M_2 \text{ for some } M_2 > 0 \text{ and all } x \in \bigcup_{\lambda \in B} S(\lambda). \tag{5}$$

Finally let $h \in L^p(X)$. Using bounds (4) and (5) above, we get

$$\begin{aligned} \langle \dot{x}(t), h(t) \rangle &\leq \|A(t, x(t), \lambda)\|_\star \,.\, \|h(t)\| + \|f(t)\|_\star \,.\, \|h(t)\| \ a.e. \\ &\leq [c_{1B}(1 + \|x(t)\|^{p-1}) + \gamma(\alpha_B(t) + \beta_B(t) M_1^{2/q})] \,.\, \|h(t)\| \ a.e. \\ \Rightarrow \int_0^r \langle \dot{x}(t), h(t) \rangle dt &= ((\dot{x}, h))_0 \leq M_3 \|h\|_{L^p(X)} \text{ for some } M_3 > 0. \end{aligned}$$

Here by $((\cdot\,,\cdot))_0$ we denote the duality brackets of the pair $(L^q(X^\star),\, L^p(X))$. Since $h \in L^p(X)$ was arbitrary, we conclude that

$$\|\dot{x}\|_{L^q(X^\star)} \leq M_3 \quad \text{for all } x \in \bigcup_{\lambda \in B} S(\lambda). \tag{6}$$

From (5) and (6) above, we get that $\bigcup_{\lambda \in B} S(\lambda)$ is bounded in $W_{pq}(T)$, hence $\overline{\bigcup_{\lambda \in B} S(\lambda)}^{\|\cdot\|_{L^p(H)}}$ is compact (see section 2). Furthermore because of the a priori bound (4) above, we may assume without any loss of generality, that for all $\lambda \in B$, we have

$$|F(t, x, \lambda)| \leq \alpha_B(t) + \beta_B(t) M_1 = \psi_B(t) \ a.e.$$

with $\psi_B(\cdot) \in L^q_+$.

Let $K_B = \{h \in L^1(H) : |h(t)| \leq \psi_B(t) \ a.e.\}$. Clearly $K_B \in P_{wkc}(L^1(H))$. Consider the multifunction $R : K_B \times B \to P_{fc}(L^1(H))$ defined by

$$R(f, \lambda) = S^1_{F(\cdot, p(f,\lambda)(\cdot), \lambda)} \ .$$

Let $\|f\|_B = \int_0^r \exp[-L \int_0^t k_B(s) ds] |f(t)| dt,\ L > 0$. Clearly this is a norm on $L^1(H)$ equivalent to the usual one. We claim that the family $\{R(\cdot\,, \lambda)\}_{\lambda \in B}$ is Hausdorff–Lipschitz continuous with the same constant $\eta_B \in (0, 1)$, provided $L > 1$. To this end, let $f, g \in K_B$ and let $v \in R(g, \lambda)$. Through a straightforward application of Aumann's selection theorem, (see Wagner [19], theorem 5.10), we get $u \in R(f, \lambda)$ *s.t.*

$$d(v(t), F(t, p(f, \lambda)(t), \lambda)) = |v(t) - u(t)| \ a.e.$$

Then we have

$$\begin{aligned} d_B(v, R(f,\lambda)) &\le \|v-u\|_B \\ &= \int_0^r \exp(-L\int_0^t k_B(s)ds)|v(t)-u(t)|dt \\ &= \int_0^r \exp(-L\int_0^t k_B(s)ds)d(v(t), F(t,p(f,\lambda)(t),\lambda))dt \\ &\le \int_0^r \exp(-L\int_0^t k_B(s)ds)h(F(t,p(g,\lambda)(t),\lambda), F(t,p(f,\lambda)(t),\lambda))dt \\ &\le \int_0^r \exp(-L\int_0^t k_B(s)ds)k_B(t)|p(g,\lambda)(t)-p(f,\lambda)(t)|dt. \end{aligned}$$

Exploiting the monotonicity of the operator $A(t,\cdot,\lambda)$, we easily get that

$$|p(g,\lambda)(t)-p(f,\lambda)(t)| \le \int_0^t |g(s)-f(s)|ds.$$

So we have:

$$\begin{aligned} d_B(v, R(f,\lambda)) &\le \int_0^r \exp(-L\int_0^t k_B(s)ds)k_B(t)\int_0^t |g(s)-f(s)|\,ds\,dt \\ &\le -\frac{1}{L}\int_0^r (\int_0^t |g(s)-f(s)|ds)d(\exp(-L\int_0^t k_B(s)ds)) \\ &= \frac{1}{L}\int_0^r \exp(-L\int_0^t k_B(s)ds)|g(t)-f(t)|dt \text{ (integration by parts)} \\ &\le \frac{1}{L}\|g-f\|_B\ . \end{aligned}$$

Similarly we can get that if $w \in R(f,\lambda)$, then

$$d_B(w, R(g,\lambda)) \le \frac{1}{L}\|g-f\|_B \Rightarrow h(R(f,\lambda), R(g,\lambda)) \le \frac{1}{L}\|g-f\|_B\ .$$

Thus if $L>1$, we see that $\{R(\cdot\,,\lambda)\}_{\lambda\in B}$ is equi-h-Lipschitz with constant $\eta_B=\frac{1}{L}\in(0,1)$. Next we claim that if $[f_n,\lambda_n]\to[f,\lambda]$ in $(K_B,\|\cdot\|_B)\times B$, then

$$R(f_n,\lambda_n)\xrightarrow{K-M}R(f,\lambda).$$

To this end, let $u\in R(f,\lambda)$. Then by definition, $u(t)\in F(t,p(f,\lambda)(t),\lambda)$ *a.e.* Set

$$\begin{aligned}\theta_n(t)&=d(u(t),F(t,p(f_n,\lambda_n)(t),\lambda_n))\\ &\le d(u(t),F(t,p(f,\lambda)(t),\lambda_n))+h(F(t,p(f,\lambda)(t),\lambda_n),F(t,p(f_n,\lambda_n)(t),\lambda_n))\\ &\le d(u(t),F(t,p(f,\lambda)(t),\lambda_n))+k_B(t)|p(f_n,\lambda_n)(t)-p(f,\lambda)(t)|\ a.e.\end{aligned}$$

But from hypothesis $H(F)(3)$, we have that

$$d(u(t),F(t,p(f,\lambda)(t),\lambda_n))\to 0\quad\text{as } n\to\infty$$

while from proposition 2.1 (note that $f_n\xrightarrow{s}f$ in $L^1(H)$ and $|f_n(t)|\le\psi_B(t)$ *a.e.* imply $f_n\xrightarrow{s}f$ in $L^q(H)$), we have

$$|p(f_n,\lambda_n)(t)-p(f,\lambda)(t)|\to 0\quad\text{as } n\to\infty.$$

Thus $\theta_n(t)\to 0$ *a.e.* as $n\to\infty$. Via Aumann's selection theorem, we get $u_n\in R(f_n,\lambda_n)$ *s.t.*

$$\begin{aligned}&|u(t)-u_n(t)|\le\theta_n(t)+\frac{1}{n},\quad t\in T\\ &\Rightarrow u_n(t)\xrightarrow{s}u(t)\text{ in } H\quad\text{as } n\to\infty\end{aligned}$$

$\Rightarrow u_n\xrightarrow{s}u$ in $L^1(H)$ (dominated convergence theorem).

Since $u_n\in R(f_n,\lambda_n)n\ge 1$, we deduce that

$$R(f,\lambda)\subseteq s\text{-}\underline{\lim}R(f_n,\lambda_n).\tag{7}$$

Next let $u\in w\text{-}\overline{\lim}\,R(f_n,\lambda_n)$. Then by definition and by denoting subsequences with the same index as sequences, we get $u_n\in R(f_n,\lambda_n)$ *s.t.* $u_n\xrightarrow{w}u$ in $L^1(H)$. Then invoking theorem 3.1 of [12], we get

$$u(t)\in\overline{\text{conv}}\,w\text{-}\overline{\lim}\,F(t,p(f_n,\lambda_n)(t),\lambda_n)\ a.e.$$

Note that for every $v\in H$, we have

$$\begin{aligned}d(v,F(t,p(f,\lambda)(t),\lambda_n))&\le d(v,F(t,p(f_n,\lambda_n)(t),\lambda_n))\\ &\quad+h(F(t,p(f,\lambda)(t),\lambda_n),F(t,p(f_n,\lambda_n)(t),\lambda_n))\\ &\le d(v,F(t,p(f_n,\lambda_n)(t),\lambda_n))+k_B(t)|p(f_n,\lambda_n)(t)-p(f,\lambda)(t)|\\ \Rightarrow d(v,F(t,p(f,\lambda)(t),\lambda))&\le\underline{\lim}\,d(v,F(t,p(f_n,\lambda_n)(t),\lambda_n)).\end{aligned}$$

Since X is assumed to be uniformly smooth, it has a Fréchet differentiable norm and so we can apply theorem 2.2 (iv) of Tsukada [17] and get that

$$w\text{-}\overline{\lim} F(t, p(f_n, \lambda_n)(t), \lambda_n) \subseteq F(t, p(f, \lambda)(t), \lambda) \ a.e.$$

$$\Rightarrow u(t) \in F(t, p(f, \lambda)(t), \lambda) \ a.e. \Rightarrow u \in R(f, \lambda).$$

Therefore we deduce that

$$w\text{-}\overline{\lim} R(f_n, \lambda_n) \subseteq R(f, \lambda). \tag{8}$$

From (7) and (8) above, we get that

$$R(f_n, \lambda_n) \xrightarrow{K-M} R(f, \lambda) \quad \text{as } n \to \infty.$$

Let $L(\lambda_n) = \{f \in K_B : f \in R(f, \lambda_n)\}$ $n \geq 1$ and $L(\lambda) = \{f \in K_B : f \in R(f, \lambda)\}$. All these are nonempty by Nadler's fixed point theorem [9]. Then from theorem 2.1, we have

$$L(\lambda_n) \xrightarrow{K} L(\lambda) \quad \text{as } n \to \infty$$

in $(L^1(H), \|\cdot\|_B)$, hence in $(L^1(H), \|\cdot\|_1)$ too. Note that if $\{f_n, f\}_{n \geq 1} \subseteq K_B$ and $f_n \xrightarrow{s} f$ in $L^1(H)$, then $f_n \xrightarrow{s} f$ in $L^q(H)$. So using proposition 2.1 and the dominated convergence theorem, we get that $p(L(\lambda_n), \lambda_n) \xrightarrow{K} p(L(\lambda), \lambda)$ as $n \to \infty$ in $L^p(H)$. But note that $p(L(\lambda_n), \lambda_n) = S(\lambda_n)$ and $p(L(\lambda), \lambda) = S(\lambda)$. So finally we have

$$S(\lambda_n) \xrightarrow{K} S(\lambda) \text{ as } n \to \infty \text{ in } L^p(H).$$

Recall that $\overline{\bigcup_{\lambda \in B} S(\lambda)}^{\|\cdot\|_{L^p(H)}}$ is compact. So $S|_B$ is Vietoris continuous, in particular then $u.s.c.$ Since $B \subseteq \Lambda$, B compact was arbitrary, from lemma β of [13], we deduce that $S(\cdot)$ is $u.s.c.$ Also since $S(\lambda_n) \xrightarrow{K} S(\lambda)$ as $n \to \infty$ in $L^p(H)$, from remark 1.7 of DeBlasi–Myjak [2], we have that $S(\cdot)$ is $l.s.c.$ Therefore $S(\cdot)$ is Vietoris continuous. Since $S(\cdot)$ is $P_k(L^p(H))$-valued, we also have that $S(\cdot)$ is Hausdorff continuous.

A useful byproduct of the above proof, is the following convergence result. Note that for all $\lambda \in \Lambda$, $S(\lambda) \in P_f(C(T, H))$.

Theorem 3.2. *If hypotheses $H(A)$, $H(F)$ and H_0 hold, and $\lambda_n \to \lambda$ in Λ, then $S(\lambda_n) \xrightarrow{K} S(\lambda)$ as $n \to \infty$ in $C(T, H)$.*

We can improve the above theorem, if we assume that X is a separable Hilbert space too and as before X embeds compactly in H. Also we take $p = q = 2$. Then from Nagy [10], we know that $W(T)$ imbeds into $C(T, H)$ compactly. Knowing that, we see that proof of theorem 3.1, with $L^p(H)$ replaced by $C(T, H)$, gives us:

Theorem 3.3. *If X is a separable Hilbert space and hypotheses $H(A)$, $H(F)$, H_0 hold with $p = q = 2$, then $S : \Lambda \to P_k(C(T, H))$ is Vietoris and Hausdorff continuous.*

4. Sensitivity analysis of optimal problems

In this section, we use the continuous dependence results obtained earlier, to study the sensitivity of a parametric, nonlinear, parabolic optimal control problem, to changes in the parameter ("variational stability analysis"). The interesting feature of our problem is that the parameter appears in all the data of the problem, including the partial differential operator.

So let $T = [0, r]$ and Z a bounded domain in $\mathbb{R}^N$ with boundary $\Gamma = \partial Z$. Let Λ be a complete metric space (the parameter space). We consider the following parametric nonlinear optimal control problem, with $p \geq 2$:

$$\int_Z \eta(z, x(b, z), \lambda) dz \to \inf = m(\lambda)$$

$$\text{(9)} \quad s.t.\ \frac{\partial x}{\partial t} - \sum_{i=1}^{N} D_i(a\left(z, \lambda)(|D_i x|^{p-2} D_i x)\right) = g(t, z, x(t, z), \lambda) u(t, z)$$

$$x|_{T\times\Gamma} = 0,\ x(0, z) = x_0(z, \lambda),\ |u(t, z)| \leq \theta(t, z, \lambda)\ a.e.$$

$$u(\cdot\,, \cdot) \text{ measurable.}$$

Here $z = (z_k)_{k=1}^N \in Z$ and $D_i = \dfrac{\partial}{\partial z_i}$. We will need the following hypotheses on the data of (9).

$H(a)$: $0 < c_1 \leq a(z, \lambda) \leq c_2$ for all $(z, \lambda) \in Z \times \Lambda$ and if $\lambda_n \to \lambda$ in Λ, then $\frac{1}{a(\cdot, \lambda_n)^{q-1}} \xrightarrow{w} \frac{1}{a(\cdot, \lambda)^{q-1}}$ in $L^1(Z)$.

$H(g)$: $g : T \times Z \times \mathbf{R} \times \Lambda \to \mathbf{R}$ is a function *s.t.*
- *(1)* $(t, z) \to g(t, z, x, \lambda)$ is measurable,
- *(2)* $|g(t, z, x, \lambda) - g(t, z, y, \lambda)| \leq k_B(t, z)|x - y|$ *a.e.*, for all $\lambda \in B \subseteq \Lambda$, B compact and with $k_B \in L^1_+$,
- *(3)* $\lambda \to g(t, z, x, \lambda)$ is continuous,
- *(4)* $|g(t, z, x, \lambda)| \leq \alpha_B(t, z) + \beta_B |x|^{2/q}$ for all $\lambda \in B \subseteq \Lambda$, B compact and with $\alpha_B \in L^2$, $\beta_B \in \mathbb{R}_+$.

$H(\theta)$: $(t, z) \to \theta(t, z, \lambda)$ is measurable, $\lambda \to \theta(t, z, \lambda)$ is continuous and $|\theta(t, z, \lambda)| \leq \xi_B(t, z)$ *a.e.* for all $\lambda \in B \subseteq \Lambda$, B compact and $\xi_B(\cdot\,, \cdot) \in L^\infty_+$.

$H(\eta):$ $\eta : Z \times \mathbb{R} \times \Lambda \to \mathbb{R}$ is an integrand *s.t.*

(1) $z \to \eta(z, x, \lambda)$ is measurable,

(2) $(x, \lambda) \to \eta(z, x, \lambda)$ is continuous,

(3) $|\eta(z, x, \lambda)| \le \psi_{1B}(z) + \psi_{2B}|x|^p$ *a.e.* for all $\lambda \in B \subseteq \Lambda$, B compact and with $\psi_{1B} \in L^1_+,\ \psi_{2B} \in \mathbb{R}_+$.

$H'_0:$ $\lambda \to x_0(\cdot\,, \lambda)$ is continuous from Λ into $L^2(Z)$.

By $Q(\lambda) \subseteq L^p(T, L^2(H))$, we denote the set of all optimal states for problem (9) above.

Theorem 4.1. *If hypotheses $H(a)$, $H(g)$, $H(\theta)$, $H(\eta)$ and H'_0 hold, then for all $\lambda \in \Lambda$, $Q(\lambda) \ne \emptyset$, $m(\cdot)$ is continuous and $\lambda \to Q(\lambda)$ is u.s.c. from Λ into $P_k(L^p(T, L^2(Z)))$.*

Proof. In this case, our evolution triple consists of the spaces $X = W_0^{1,p}(Z)$, $H = L^2(Z)$ and $X^\star = W^{-1,q}(Z)$. From the Sobolev embedding theorem, we know that X embeds into H compactly. Also X is uniformly smooth.

Then consider the Dirichlet form $v_\lambda : W_0^{1,p}(Z) \times W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$v_\lambda(x, y) = \int_Z (a(z, \lambda) \sum_{i=1}^N |D_i x|^{p-2} D_i x D_i y) dz, \quad \lambda \in \Lambda.$$

From Hölder's inequality, we get that

$$\begin{aligned} |v_\lambda(x, y)| &\le c_2 \sum_{i=1}^N \left(\int_Z |D_i x|^p dz \right)^{1/q} \left(\int_Z |D_i y|^p dz \right)^{1/p} \\ &\le c_2 \|x\|^{p-1}\ \|y\| \quad \text{for all } \lambda \in \Lambda. \end{aligned}$$

So we can define $A : X \times \Lambda \to X^\star$ by

$$\langle A(x, \lambda), y \rangle = v_\lambda(x, y).$$

Recall Tartar's inequality, which says that if $a, b \in \mathbb{R}$, then for some fixed $\mu > 0$, we have:

$$\mu |a - b|^p \le (|a|^{p-2} a - |b|^{p-2} b)(a - b).$$

Using this inequality, we easily check that

$$c_3 \|x - y\|^p \le \langle A(x, \lambda) - A(y, \lambda), x - y \rangle$$

for some $c_3 > 0$ and all $\lambda \in B$, $x, y \in W_0^{1,p}(Z) = X$, while from above we have

$$\|A(x, \lambda)\|_\star \le c_2\ \|x\|^{p-1}$$

for all $\lambda \in \Lambda$ and all $x \in X$. Also note that $A(0, \lambda) = 0$.

Next we will show the continuity of $A(\cdot\,, \lambda)$. To this end, let $x_n \xrightarrow{s} x$ in X. Also let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|^{p-2}x$. Then by Krasnosel'ski's theorem, the Nemitsky (superposition) operator $\widehat{f} : L^p(Z) \to L^q(Z)$ corresponding to $f(\cdot)$ and defined by $\widehat{f}(x)(\cdot) = |x(\cdot)|^{p-2}x(\cdot)$ is continuous. So $\widehat{f}(D_i x_n) \xrightarrow{s} \widehat{f}(D_i x)$ in $L^q(Z)$. Then using Hölder's inequality, we get that

$$\langle A(x_n, \lambda) - A(x, \lambda) y\rangle \le c_4 \left(\sum_{i=1}^{N} \|\widehat{f}(D_i x_n) - \widehat{f}(D_i x)\|_q \right) \|y\|$$

$$\Rightarrow \|A(x_n, \lambda) - A(x, \lambda)\|_\star \le c_4 \cdot \sum_{i=1}^{N} \|\widehat{f}(D_i x_n) - \widehat{f}(D_i x)\|_q \to 0$$

$\Rightarrow A(\cdot\,, \lambda)$ is indeed continuous, in particular then hemicontinuous.

Finally let $\varphi : Z \times \mathbb{R}^N \times \Lambda \to \mathbb{R}$ be defined by

$$\varphi(z, v, \lambda) = a(z, \lambda) \frac{1}{p} \sum_{i=1}^{N} |v_i|^p = a(z, \lambda) \frac{1}{p} \|v\|^p$$

$$\Rightarrow \varphi(z, \cdot\,, \lambda) \text{ is convex and } \varphi^\star(z, v^\star, \lambda) = \frac{1}{q a(z, \lambda)^{q-1}} \|v^\star\|^q.$$

Let $\Phi : W_0^{1,p}(Z) \times \Lambda \to \mathbb{R}$ be defined by

$$\Phi(x, \lambda) = \int_Z a(z, \lambda) \frac{1}{p} \sum_{i=1}^{N} |D_i x|^p dz$$

$$= \int_Z a(z, \lambda) \frac{1}{p} \|Dx\|^p dz \ (D = grad).$$

Because of hypothesis $H(A)$ (*3*), we have that if $\lambda_n \to \lambda$ in Λ, then $\varphi^\star(\cdot\,, v^\star, \lambda_n) \xrightarrow{w} \varphi^\star(\cdot\,, v^\star, \lambda)$ in $L^1(Z)$ and so by theorem 3.4 of Marcellini–Sbordone [8] we have that $\Phi(\cdot\,, \lambda) = \Gamma_{\text{seq}}(w) - \Phi(\cdot\,, \lambda_n)$, where $\Gamma_{\text{seq}}(w)$ denotes the sequential Γ-limit on the space $(W_0^{1,p}(Z), w)$ (see for example Buttazzo [1]). Then from theorems 3.3 and 2.17 of Defianceschi [3], we get that if $\lambda_n \to \lambda$ in Λ, $A(\cdot\,, \lambda_n) \xrightarrow{G} A(\cdot\,, \lambda)$. So we have satisfied hypothesis $H(A)$.

Let $\widehat{g} : T \times H \times \Lambda \to H$ be defined by:

$$\widehat{g}(t, x, \lambda)(\cdot) = g(t, \cdot\,, x(\cdot), \lambda);$$

i.e. $\widehat{g}$ is the Nemitsky (superposition) operator corresponding to g. Also let $\widehat{U}(t,\lambda)=\{u\in L^2(Z):|u(z)|\le\theta(t,z,\lambda)\ a.e.\}$. Define the multifunction $F:T\times H\times\Lambda\to P_{wkc}(H)$ by $F(t,x,\lambda)=\widehat{g}(t,x,\lambda)\widehat{U}(t,\lambda)=\{\widehat{g}(t,x,\lambda)u: u\in\widehat{U}(t,\lambda)\}\in P_{wkc}(H)$. We will check that $F(\cdot,\cdot,\cdot)$ satisfies hypothesis $H(F)$. To this end let $w\in H$ be given. Then we have:

$$
\begin{aligned}
d(w,F(t,x,\lambda)) &= \inf\{\|w-\widehat{g}(t,x,\lambda)u\|_2 : u\in\widehat{U}(t,\lambda)\}\\
&= \inf\left[\int_Z |w(z)-g(t,z,x(z),\lambda)u(z)|^2dz : u\in\widehat{U}(t,\lambda)\right]^{1/2}\\
&= \left[\inf\int_Z |w(z)-g(t,z,x(z),\lambda)u(z)|^2dz : u\in\widehat{U}(t,\lambda)\right]^{1/2}\\
&= \left(\int_Z \inf[\,|w(z)-g(t,z,x(z),\lambda)u|^2 : |u|\le\theta(t,z,\lambda)]dz\right)^{1/2}
\end{aligned}
$$

(the last equality follows from theorem 2.2 of Hiai–Umegaki [4])

$$
\Rightarrow d(w,F(t,x,\lambda))=\left(\int_Z d(w(z),G(t,z,\lambda))^2dz\right)^{1/2}
$$

where $G(t,z,\lambda)=\{g(t,z,x(z),\lambda)u:|u|\le\theta(t,z,\lambda)\}$. But from hypotheses $H(g)$ and $H(\theta)$ it is clear that $(t,z)\to G(t,z,\lambda)$ is a measurable multifunction. So

$$
\begin{aligned}
&t\to\left(\int_Z d(w(z),G(t,z,\lambda))^2dz\right)^{1/2}\quad\text{is measurable}\\
&\Rightarrow t\to d(w,F(t,x,\lambda))\quad\text{is measurable}\\
&\Rightarrow t\to F(t,x,\lambda)\quad\text{is measurable.}
\end{aligned}
$$

Next using hypothesis $H(g)(2)$, we have that

$$
\begin{aligned}
h(F(t,x,\lambda),F(t,y,\lambda)) &\le \|\widehat{g}(t,x,\lambda)-\widehat{g}(t,y,\lambda)\|_2\|\theta\|_\infty r\\
&\le \widehat{k}\|x-y\|_2\quad\text{with } \widehat{k}>0.
\end{aligned}
$$

Finally let $\lambda_n\to\lambda$ in Λ and let $u\in\widehat{U}(t,\lambda)$. Because of hypothesis $H(\theta)$, clearly $\widehat{U}(t,\cdot)$ is continuous and so we can find $u_n\in\widehat{U}(t,\lambda_n)$ $s.t.$

$u_n \xrightarrow{s} u$ in $L^2(Z)$. We then have:

$$\begin{aligned} d(w, F(t,x,\lambda_n)) &\leq \|w - \widehat{g}(t,x,\lambda_n)u_n\|_2 \\ \Rightarrow \overline{\lim}\, d(w, F(t,x,\lambda_n)) &\leq \|w - \widehat{g}(t,x,\lambda)u\|_2. \end{aligned}$$

Since $u \in \widehat{U}(t,\lambda)$ was arbitrary, we get that

$$\overline{\lim}\, d(w, F(t,x,\lambda_n)) \leq d(w, F(t,x,\lambda)). \tag{10}$$

On the other hand, let $u_n \in \widehat{U}(t,\lambda_n)$ *s.t.*

$$d(w, F(t,x,\lambda_n)) = \|w - \widehat{g}(t,x,\lambda_n)u_n\|_2.$$

Their existence is guaranteed by the fact that $\widehat{U}(\cdot\,,\cdot)$ is $P_{wkc}(L^2(Z))$-valued and the $L^2(Z)$-norm $\|\cdot\|_2$ is *w-l.s.c.* Because of hypothesis $H(\theta)$ and by passing to a subsequence if necessary, we may assume that $u_n \xrightarrow{w^\star} u$ in $L^\infty(Z)$. So for every $h \in L^2(Z)$ we have

$$\begin{aligned} (\widehat{g}(t,x,\lambda_n)u_n, h)_{L^2(Z)} &= \int_Z g(t,z,x(z),\lambda_n)u_n(z)h(z)dz \\ \to (\widehat{g}(t,x,\lambda)u, h)_{L^2(Z)} &= \int_Z g(t,z,x(z),\lambda)u(z)h(z)dz \quad \text{as } n \to \infty, \\ \Rightarrow \widehat{g}(t,x,\lambda_n)u_n &\xrightarrow{w} \widehat{g}(t,x,\lambda)u \quad \text{in } H = L^2(Z). \end{aligned}$$

Hence we have

$$\begin{aligned} \|w - \widehat{g}(t,x,\lambda)u\|_2 &\leq \underline{\lim}\, \|w - \widehat{g}(t,x,\lambda_n)u_n\|_2 \\ \Rightarrow d(w, F(t,x,\lambda)) &\leq \underline{\lim}\, d(w, F(t,x,\lambda_n)). \end{aligned} \tag{11}$$

From (10) and (11) above, we deduce that

$$\begin{aligned} d(w, F(t,x,\lambda_n)) &\to d(w, F(t,x,\lambda)) \quad \text{as } n \to \infty \\ \Rightarrow \lambda &\to F(t,x,\lambda) \quad \text{is } d\text{-continuous.} \end{aligned}$$

Note that because of hypothesis $H(g)$ *(4)*, we have

$$|F(t,x,\lambda)| \leq \widehat{\alpha}_B(t) + \widehat{\beta}_B|x|^{2/q}$$

with $\widehat{\alpha}_B(t) = 2\|\xi_B\|_\infty \|\alpha_B(t,\cdot)\|_2 \in L^2_+$ and $\widehat{\beta}_B = 2\|\xi_B\|_\infty \|\beta_B\|_\infty > 0$. So we have satisfied hypethesis $H(F)$. Also $\lambda \to \widehat{x}_0(\lambda) = x_0(\cdot\,,\lambda)$ is continuous from Λ into H (see hypothesis H_0').

Rewrite the dynamics of (9) as the following equivalent abstract evolution inclusion:

$$(9)'_d \qquad \left\{ \begin{array}{c} \dot{x}(t) + A(t, x(t), \lambda) \in F(t, x(t), \lambda) \ a.e. \\ x(0) = \widehat{x}_0(\lambda). \end{array} \right\}$$

As always, the equivalence can be established through an easy application of Aumann's selection theorem. Let $S(\lambda) \in P_k(L^p(T, H))$ be the solution set of $(9)'_d$. From theorem 3.1 we know that $S : \Lambda \to P_k(L^p(T, H))$ is Vietoris and Hausdorff continuous.

Now let $\widehat{\eta} : GrS \to \mathbb{R}$ be defined by $\widehat{\eta}(x, \lambda) = \int\limits_Z \eta(z, x(b, z), \lambda) dz$ (the pointwise evaluation $x(b, \cdot)$, makes sense since $S(\lambda) \subseteq W_{pq}(T) \to C(T, H)$). Using hypothesis $H(\eta)$, we can easily check that $\widehat{\eta}(\cdot, \cdot)$ is in fact continuous. So $Q(\lambda) = \{x \in S(\lambda) : \widehat{\eta}(x, \lambda) = m(\lambda)\} \neq \emptyset$.

Next let $\lambda_n \to \lambda$ in Λ and pick $x \in S(\lambda)$ $s.t.$ $\widehat{\eta}(x, \lambda) = m(\lambda)$. Then since $S(\cdot)$ is Vietoris continuous, we can find $x_n \in S(\lambda_n)$ $s.t.$ $x_n \xrightarrow{s} x$ in $L^p(T, H)$. Then $\widehat{\eta}(x_n, \lambda_n) \to \widehat{\eta}(x, \lambda) \Rightarrow \overline{\lim}\, m(\lambda_n) \leq m(\lambda)$. On the other hand, let $x_n \in S(\lambda_n)$ $s.t.$ $\widehat{\eta}(x_n, \lambda_n) = m(\lambda_n)$. Since $\overline{\bigcup\limits_{n \geq 1} S(\lambda_n)}^{\|\cdot\|_{L^p(T,H)}}$ is compact (see the proof of theorem 3.1), by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{s} x$ in $L^p(T, H)$. Then theorem 3.1 tells us that $x \in S(\lambda)$. Also $\widehat{\eta}(x_n, \lambda_n) \to \widehat{\eta}(x, \lambda) \Rightarrow \underline{\lim}\, m(\lambda_n) \geq m(\lambda)$. Therefore, $m(\lambda_n) \to m(\lambda) \Rightarrow m(\cdot)$ is continuous.

Using the continuity of $m(\cdot)$, we can easily check that

$$\begin{aligned} s\text{-}\overline{\lim}\, Q(\lambda_n) &\subseteq Q(\lambda) \\ \Rightarrow Q|_B &\quad \text{is } u.s.c. \end{aligned}$$

and so by lemma β of [13], we conclude that $Q(\cdot)$ is $u.s.c.$ □

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