Publ. Math. Debrecen
72/3-4 (2008), 441–450

On continuous solutions and stability of a conditional Gołąb–Schinzel equation

By NICOLE BRILLOUËT-BELLUOT (Nantes) and JANUSZ BRZDĘK (Kraków)

This paper is dedicated to Professor Zoltán Daróczy on the occasion of his 70th birthday

Abstract. We determine the continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$f(x + f(x)y)f(x)f(y)[f(x + f(x)y) - f(x)f(y)] = 0 \quad (x, y \in \mathbb{R}).$$

We also give some remarks on nonstability of the Gołąb–Schinzel equation.

Let $\mathbb{R}$ denote the set of reals. The Gołąb–Schinzel functional equation

$$f(x + f(x)y) = f(x)f(y)) \tag{1}$$

has been introduced by S. Gołąb and A. Schinzel in [17], for functions $f : \mathbb{R} \to \mathbb{R}$. For continuous functions with more general domains (Hilbert space) the equation has been studied for the first time by Z. Daróczy in [15], where he has obtained a very elegant description of solutions. For further details concerning equation (1) refer e.g. to [1]–[3], [19]–[22] and to the survey paper [5].

This paper has been motivated by the talk of R. Ger at the 44th ISFE (see [16]), in which he stated that the problem of solving the following two conditional

Mathematics Subject Classification: 39B12, 39B22, 39B82.
Key words and phrases: Gołąb–Schinzel functional equation, conditional functional equation, continuous solution; Hyers–Ulam stability, continuous associative operation.

equations

$$f(x+f(x)y)=f(x)f(y) \qquad \text{whenever } f(x+f(x)y)\neq 0, \tag{2}$$

$$f(x+f(x)y)=f(x)f(y) \qquad \text{whenever } f(x)f(y)\neq 0 \tag{3}$$

(e.g. in the class of functions $f:\mathbb{R}\to\mathbb{R}$) is of interest and arises naturally in the study of stability of (1). Namely, investigating Hyers–Ulam stability of (1) in the sense of R. Ger we come across the following two conditional inequalities (see e.g. [11])

$$\left|\frac{f(x)f(y)}{f(x+f(x)y)}-1\right|\leq\varepsilon \qquad \text{whenever } f(x+f(x)y)\neq 0, \tag{4}$$

$$\left|\frac{f(x+f(x)y)}{f(x)f(y)}-1\right|\leq\varepsilon \qquad \text{whenever } f(x)f(y)\neq 0 \tag{5}$$

for a given $\varepsilon>0$ (for the information on the stability of functional equations see [18]; stability of (1) has been investigated in [11], [12], [13]). It is easily seen that those two inequalities are strictly connected with equations (2) and (3): e.g. every solution of (2) ((3), respectively) satisfies (4) ((5), respectively) and one could expect that every $f:\mathbb{R}\to\mathbb{R}$ satisfying (4) ((5), respectively) is close, in some sense, to a solution of (2) ((3), respectively).

Note that every solution $f:\mathbb{R}\to\mathbb{R}$ of either of those two conditional equations is a solution of the functional equation

$$f(x+f(x)y)f(x)f(y)[f(x+f(x)y)-f(x)f(y)]=0. \tag{6}$$

Thus solving (6) we obtain all solutions of (2) and (3) as well; moreover (see Corollary 2), in this way, we also solve the equations

$$f(y)f(x+f(x)y)=0, \tag{7}$$

$$f(x)f(y)f(x+f(x)y)=0. \tag{8}$$

In this paper we determine the continuous solutions $f:\mathbb{R}\to\mathbb{R}$ of (6). Our results correspond to the recent papers [10], [19], [20], [21], [22] on conditional versions of (1). The main tool in our proof is the well known theorem of J. Aczél [1] (see also [14]), concerning the continuous cancellative associative operations on a real interval; or more precisely the following consequence of it.

Theorem 1. *Let L be a nontrivial real interval and let $\circ : L\times L\to L$ be a continuous cancellative associative operation. Then the operation is commutative.*

For some further examples of applications of the Aczél theorem in solving some functional equations of similar type see [4], [6], [7], [8], [9].

We start with some Lemmas. In what follows we assume that $f:\mathbb{R}\to\mathbb{R}$ is a nonconstant continuous solution of the functional equation (6) and $I:=\{x\in\mathbb{R}: f(x)\neq 0\}$. Please note that $I\neq\mathbb{R}$, because otherwise f is a solution of (1) and consequently $f\equiv 1$ (see e.g. [5, Proposition 2]).

Lemma 1. *Let J be a nontrivial interval included in I with $x+f(x)y\in J$ for all $x,y\in J$. Then there is $c\in\mathbb{R}$ with $f(x)=cx+1$ for $x\in J$.*

Proof. Write $A(x,y):=x+f(x)$ for $x,y\in\mathbb{R}$. Then $A:J\times J\to J$ is a continuous and associative operation, cancellative on both sides (cf. e.g. [9, p. 5] or [6]). By Theorem 1, A is a commutative operation, which means that

$$x+f(x)y=y+f(y)x \quad \text{for } x,y\in J. \tag{9}$$

Let $y\in J\setminus\{0\}$ and $c:=(f(y)-1)y^{-1}$. Then (9) implies the statement. □

Lemma 2. *The following two statements hold.*

(i) *The function $g:\mathbb{R}\to\mathbb{R}$, given by: $g(x)=f(-x)$, is a solution of equation (6).*
(ii) *Let $a>0$ and $f(x)<0$ for $x>a$. Then $f(y)\geq 0$ for $y<0$.*

Proof. Since (i) is obvious we prove only (ii). For the proof by contradiction suppose that there is $y<0$ with $f(y)<0$ and take $x>a$. Then $f(x)<0$, which means that $x+f(x)y>x>a$. Hence $f(x+f(x)y)<0$ and $f(x)f(y)>0$. Consequently $f(x)f(y)f(x+f(x)y)\neq 0$ and, by (1), we have $f(x+f(x)y)=f(x)f(y)$, which gives the contradiction. □

Lemma 3. *Let (a,b) be a connected component of I. Then $a=-\infty$ or $b=+\infty$.*

Proof. For the proof by contradiction suppose $-\infty<a<b<+\infty$. The continuity of f implies $f(a)=f(b)=0$.

Suppose that there are $\alpha,\beta\in(a,b)$ with $\alpha+f(\alpha)\beta\leq a$. Since $b+f(b)\beta=b$, by the continuity of f, there is $\gamma\in[\alpha,b)$ such that $\gamma+f(\gamma)\beta=a$. Let $\gamma_0=\sup\{\gamma\in[\alpha,b):\gamma+f(\gamma)\beta=a\}\geq\alpha>a$. Again, the continuity of f implies $\gamma_0+f(\gamma_0)\beta=a$, and therefore $\gamma_0\in(a,b)$.

Let $\gamma_1=\inf\{\gamma\in[\gamma_0,b]:\gamma+f(\gamma)\beta=b\}$. For all $x\in(\gamma_0,\gamma_1)$ we have: $x+f(x)\beta\in(a,b)$ and consequently $f(x)f(\beta)=f(x+f(x)\beta)$. So, as x goes to $\gamma_0 + 0$, we get $0 \neq f(\gamma_0)f(\beta) = f(\gamma_0 + f(\gamma_0)\beta) = f(a) = 0$, which gives the contradiction.

Thus we have shown $\alpha + f(\alpha)\beta > a$ for $\alpha, \beta \in (a, b)$. Similarly (see Lemma 2(i)), we prove $\alpha + f(\alpha)\beta < b$ for α, $\beta \in (a, b)$. So, by Lemma 1, there is $c \in \mathbb{R}$ with $f(x) = cx + 1$ for $x \in (a, b)$. This is a contradiction, because $f(a) = f(b) = 0$. $\square$

Corollary 1. *$I \in \{(-\infty, b) \cup (a, +\infty), (a, +\infty), (-\infty, a)\}$ for some $a, b \in \mathbb{R}$, $b \leq a$.*

Lemma 4. *Let $a \in \mathbb{R}$, $(a, +\infty) \subseteq I$ and $f(a) = 0$. Then one of the following two statements holds.*

(i) *$a < 0$, $x + f(x)y > a$ for every $x, y > a$ and*

$$f(x) = 1 - \frac{x}{a} \quad \text{for } x > a. \tag{10}$$

(ii) *$a > 0$, $x + f(x)y \leq a$ for every $x, y > a$ and*

$$f(x) \leq 1 - \frac{x}{a} \quad \text{for } x > a. \tag{11}$$

Proof. First we show that either $x+f(x)y > a$ for $x, y > a$ or $x+f(x)y \leq a$ for $x, y > a$. For the proof by contradiction suppose $X_1 := x_1 + f(x_1)y_1 \leq a$ and $X_2 := x_2 + f(x_2)y_2 > a$ for some $x_1, y_1, x_2, y_2 > a$. We consider only the case $x_1 \leq x_2$, $y_1 \leq y_2$; the cases where $x_1 > x_2$ or $y_1 > y_2$ are analogous.

The continuity of f implies that $A([x_1, x_2] \times [y_1, y_2])$ is an interval containing $[X_1, X_2]$. Therefore, there are $\alpha_1 \in [x_1, x_2]$ and $\beta_1 \in [y_1, y_2]$ with

$$A(\alpha_1, \beta_1) = a + \frac{X_2 - a}{2}.$$

Now, $A([x_1, \alpha_1] \times [y_1, \beta_1])$ is an interval containing the interval $[X_1, \frac{a+X_2}{2}]$. So, there exist $\alpha_2 \in [x_1, \alpha_1]$ and $\beta_2 \in [y_1, \beta_1]$ such that

$$A(\alpha_2, \beta_2) = a + \frac{X_2 - a}{2^2}.$$

In this way, we construct two decreasing sequences, $\{\alpha_n\}_{n\in\mathbb{N}}$ in $[x_1, x_2]$ and $\{\beta_n\}_{n\in\mathbb{N}}$ in $[y_1, y_2]$, such that

$$A(\alpha_n, \beta_n) = a + \frac{X_2 - a}{2^n}.$$

We have: $f(A(\alpha_n, \beta_n)) = f(\alpha_n)f(\beta_n)$ for all $n \in \mathbb{N}$. Let $x_0 = \lim_{n\to+\infty} \alpha_n \in [x_1, x_2]$ and $y_0 = \lim_{n\to+\infty} \beta_n \in [y_1, y_2]$. As n goes to $+\infty$, we get: $f(a) = 0 = f(x_0)f(y_0) \neq 0$, which gives the contradiction.

• If $x + f(x)y > a$ for all $x, y > a$, we apply Lemma 1 to $J = (a, +\infty)$. Then the continuity of f at a implies $a \neq 0$ and

$$f(x) = 1 - \frac{x}{a} \quad for \quad x \geq a.$$

Suppose that $a > 0$ and take $x, y > a$. Then $(x - a)(y - a) > 0$, whence $x + y - \frac{xy}{a} < a$. Consequently we have $x + f(x)y < a$. This contradiction proves that $a < 0$.

• Suppose now:

$$x + f(x)y \leq a \quad for \quad x, y > a. \tag{12}$$

Clearly $x = y = 0$ in (12) implies $a \geq 0$. Further, with $a = 0$, (12) gives $x + f(x)y \leq 0$ for $x, y > 0$ and consequently, with $y \to 0+$, we get a contradiction. Therefore $a > 0$. Moreover, as $y \to a + 0$, from (12) we get

$$f(x) \leq 1 - \frac{x}{a} \quad for \quad x \geq a. \qquad \square$$

From Lemmas 2(i) and 4 we easily derive the following result.

Lemma 5. *If $b \in \mathbb{R}$, $(-\infty, b) \subseteq I$ and $f(b) = 0$, then one of the following two conditions holds.*

(i) *$b > 0$, $x + f(x)y < b$ for $x, y < b$, and*

$$f(x) = 1 - \frac{x}{b} \quad \text{for } x < b. \tag{13}$$

(ii) *$b < 0$, $x + f(x)y \geq b$ for $x, y < b$, and $f(x) \leq 1 - \dfrac{x}{b}$ for $x < b$.*

Lemma 6. *Let $a, b \in \mathbb{R}$, $0 < b \leq a$, $f(a) = 0$, and (11) and (13) hold. Then $a = b$ and*

$$f(x) = 1 - \frac{x}{a} \quad \text{for } x \in \mathbb{R}. \tag{14}$$

Proof. Take $x, w \in (a, +\infty)$. Then, by (11), $f(x) < 0$ and $f(w) < 0$, whence there exists $y_0 > a$ such that $x + f(x)y, w + f(w)y \in (-\infty, b)$ for $y > y_0$. Consequently, in view of (13), for every $y > y_0$,

$$0 \neq f(x)f(y) = f(x + f(x)y) = 1 - \frac{x + f(x)y}{b}$$

and

$$0 \neq f(w)f(y) = f(w + f(w)y) = 1 - \frac{w + f(w)y}{b},$$

which implies that

$$\frac{b-x}{bf(x)} - \frac{y}{b} = f(y) = \frac{b-w}{bf(w)} - \frac{y}{b}.$$

Thus we have proved that there is $c \in \mathbb{R} \setminus \{0\}$ with $b - x = cf(x)$ for $x \in (a, +\infty)$. Now writing $\alpha = -\frac{1}{c}$ and $\beta = \frac{b}{c}$ we have $f(x) = \alpha x + \beta$ for $x \in (a, +\infty)$. Since f is continuous and $f(a) = 0$, we have $\beta = -a\alpha$. Therefore

$$f(x) = \alpha(x - a) \quad \text{for } x \in (a, +\infty). \tag{15}$$

Let $x \in (a, +\infty)$. There is $\overline{y} \in (a, +\infty)$ with $x + f(x)y < b$ for $y > \overline{y}$. Thus, for $y > \overline{y}$,

$$\alpha^2(x-a)(y-a) = f(x)f(y) = f(x + f(x)y) = 1 - \frac{x + \alpha(x-a)y}{b}$$

and consequently $b\alpha(x-a)\left(\alpha(y-a) + \frac{y}{b}\right) = b - x$, which yields $\alpha = -\frac{1}{b}$.

So we have proved that $-\frac{a}{b}(x-a) = (b-x)$ for $x > a$, whence $a = b$. This, (13), and (15) yield (14). □

Lemma 7. *Let $a, b \in \mathbb{R}$, $b \leq a$, and $I = (-\infty, b) \cup (a, +\infty)$. Then $a = b \neq 0$ and* (14) *holds.*

Proof. In view of Lemma 2(i) it is enough to consider only the case $a \geq 0$. Then, according to Lemma 4, condition (11) holds, which means that $f(x) < 0$ for $x > a$. Hence, in view of Lemma 2(ii), $f(y) \geq 0$ for $y < 0$. On account of Lemma 5, this is possible only if $b > 0$ and (13) is valid. Consequently Lemma 6 yields the statement. □

Now we are in a position to prove the main result.

Theorem 2. *A continuous function $f : \mathbb{R} \to \mathbb{R}$ is a solution of equation* (6) *if and only if one of the following four conditions holds.*

(α) $f \equiv 0$.

(β) *There is $c \in \mathbb{R}$ such that*

$$f(x) = \max\{cx + 1, 0\} \quad \textit{for } x \in \mathbb{R}$$

or

$$f(x) = cx + 1 \quad \textit{for } x \in \mathbb{R}.$$

(γ) *There is* $a \in (0, \infty)$ *such that*

$$f(x) \le 1 - \frac{x}{a} \quad \textit{for } x \in (a, \infty)$$

and

$$f(x) = 0 \quad \textit{for } x \in (-\infty, a].$$

(δ) *There is* $b \in (-\infty, 0)$ *such that*

$$f(x) \le 1 - \frac{x}{b} \quad \textit{for} x \in (-\infty, b)$$

and

$$f(x) = 0 \quad \textit{for } x \in [b, \infty].$$

Proof. First assume that one of conditions (α)–(δ) holds; we show that then f is a solution of equation (6). The cases where (α) or (β) is valid are trivial, because then f is a solution of the Gołąb–Schinzel equation (see e.g. [3], [5], [17]). So, in view of Lemma 2(i), it remains to consider the case of (γ).

Take $x, y \in \mathbb{R}$ with $f(x)f(y) \ne 0$. This means that $x, y \in (a, \infty)$, whence $(x-a)(y-a) > 0$. Consequently $x+(1-\frac{x}{a})y < a$, which implies that $x+f(x)y < a$. Hence $f(x + f(x)y) = 0$ and therefore (6) holds.

Now assume that f is a solution of (6) and $f \not\equiv 0$. If f is constant, i.e., $f \equiv \gamma \ne 0$, then, by (6), $\gamma = \gamma^2$, whence $\gamma = 1$. So (β) holds with $c = 0$.

Now assume that f is not constant; then, in view of [5, Proposition 2], $I \ne \mathbb{R}$. Consequently, by Corollary 1, there exist $a, b \in \mathbb{R}$, $b < a$, such that $I \in \{(a, \infty), (-\infty, b), (-\infty, b) \cup (a, \infty)\}$.

First consider the case where $I = (a, \infty)$. If $a > 0$, then, according to Lemma 4, (γ) holds. If $a < 0$, then Lemma 4 implies (β). The case where $I = (-\infty, b)$ is analogous on account of Lemma 2(i).

In the case where $I = (-\infty, b) \cup (a, \infty)$ it is enough to use Lemma 7. □

Corollary 2. 1° *Every continuous* $f : \mathbb{R} \to \mathbb{R}$, *satisfying* (2) *or* (3), *is a solution of* (1).

2° *Continuous* $f : \mathbb{R} \to \mathbb{R}$ *satisfies* (7) *if and only if* (α) *or* (γ) *or* (δ) *holds.*

3° *Continuous* $f : \mathbb{R} \to \mathbb{R}$ *satisfies* (8) *if and only if* (α) *or* (γ) *or* (δ) *holds.*

Proof. Suppose that (γ) or (δ) holds. Then taking $y = 0$ and $f(x) \ne 0$ we obtain that (2) does not hold. Next, arguing as in the proof of Theorem 2, we show that $f(x + f(x)y) = 0$ for $x, y \in \mathbb{R}$ with $f(x)f(y) \ne 0$. Hence (3) does not hold, either; but equations (7) and (8) are fulfilled. Taking $x = y = 0$ we see that functions given by (β) do not satisfy (7) nor (8). □

Remark 1. It follows from Corollary 2 that a continuous $f : \mathbb{R} \to \mathbb{R}$ is a solution of (6) if and only if it is a solution of (1) or of (8). This is no longer true if f is discontinuous at least at one point. Namely the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0, \end{cases}$$

satisfies (6) and it is neither a solution of (1) (take $x = 2$, $y = -1$) nor of (8) (take $x = y = 1$).

Remark 2. J. Chudziak [11, Theorem 1] has proved that every, continuous at 0, function $f : \mathbb{R} \to \mathbb{R}$ that satisfies (4) and (5) is a solution of (1) or it is bounded. This is not the case if we replace conditions (4) and (5) by the following weaker one

$$\left| \frac{f_0(x)f_0(y)}{f_0(x+f_0(x)y)} - 1 \right| \leq \varepsilon \quad \text{and} \quad \left| \frac{f_0(x+f_0(x)y)}{f_0(x)f_0(y)} - 1 \right| \leq \varepsilon \\ \text{for } x, y \in \mathbb{R},\ f_0(x+f_0(x)y)f_0(x)f_0(y) \neq 0. \tag{16}$$

The following two examples show this.

Example 1. Let $f_0 : \mathbb{R} \to \mathbb{R}$ be given by

$$f_0(x) = \begin{cases} x+1, & \text{if } x \geq 1; \\ 0, & \text{if } x < 1. \end{cases}$$

It is easy to check that $f_0(x + f_0(x)y) = f_0(x)f_0(y)$ for $x, y \in [1, \infty)$, which means that (16) holds with any $\varepsilon > 0$, but f_0 is not a solution of (1) (take e.g. $x = 2$, $y = -\frac{1}{4}$) nor it is bounded. Moreover, for every solution $f : \mathbb{R} \to \mathbb{R}$ of the Gołąb–Schinzel equation (1), we have

$$\sup_{x \in \mathbb{R}} |f(x) - f_0(x)| \geq 1,$$

because either $f(\mathbb{R}) = \{0\}$ or $f(0) = 1$.

However, f_0 satisfies (6).

Example 2. Let $\varepsilon \in (0, 1)$ and $f_0 : \mathbb{R} \to \mathbb{R}$ be given by

$$f_0(x) = \begin{cases} x, & \text{if } x \geq \dfrac{1}{\varepsilon}; \\[2ex] 0, & \text{if } x < \dfrac{1}{\varepsilon}. \end{cases}$$

It is easy to check that (16) holds, but f_0 is not a solution of (1) or of (6) (take e.g. $x = y = \frac{1}{\varepsilon}$), nor it is bounded. Moreover, analogously as in Example 2, for every solution $f : \mathbb{R} \to \mathbb{R}$ of equation (1), we obtain

$$\sup_{x\in\mathbb{R}} |f(x) - f_0(x)| \geq 1.$$

We also have the following example.

Example 3. Let $\varepsilon \in (0, 1)$, $k \in \left[\frac{1}{1+\varepsilon}, 1 + \varepsilon\right]$, $k \neq 1$, and $f_0 : \mathbb{R} \to \mathbb{R}$ be given by: $f_0(x) = k$ for $x \in \mathbb{R}$. Then

$$1 - \varepsilon < \frac{1}{1+\varepsilon} \leq k \leq \varepsilon + 1 < \frac{1}{1-\varepsilon},$$

which means that f_0 is a continuous solution of (16), but it is not a solution of (6). We do not know if there exist an unbounded continuous solution $f_0 : \mathbb{R} \to \mathbb{R}$ of (16) that do not satisfy equation (6).

References

[1] J. Aczél, Sur les opérations définies pour nombres réels, *Bull. Soc. Math. France* **76** (1949), 59–64.

[2] J. Aczél and J. Schwaiger, Continuous solutions of the Gołąb–Schinzel equation on the nonnegative reals and on related domains, *Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II* **208** (1999, 2000), 171–177.

[3] N. Brillouët and J. Dhombres, Équations fonctionnelles et recherche de sous-groupes, *Aequationes Math.* **31** (1986), 253–293.

[4] N. Brillouët-Belluot and B. Ebanks, Localizable composable measures of fuzziness – II, *Aequationes Math.* **60** (2000), 233–242.

[5] J. Brzdęk, Gołąb–Schinzel functional equation and its generalizations, *Aequationes Math.* **70** (2005), 14–24.

[6] J. Brzdęk, On continuous solutions of some functional equations, *Glasnik Mat.* **30(50)** (1995), 261–267.

[7] J. Brzdęk, On the Baxter functional equation, *Aequationes Math.* **52** (1996), 105–111.

[8] J. Brzdęk, On some conditional functional equations of Gołąb–Schinzel type, *Ann. Math. Siles.* **9** (1995), 65–80.

[9] J. Brzdęk, On continuous solutions of a conditional Gołąb–Schinzel equation, *Anz. Österreich. Akad. Wiss. Math.-Natur. Kl.* **138** (2001, 2002), 3–6.

[10] J. Brzdęk and A. Mureńko, On a conditional Gołąb–Schinzel equation, *Arch. Mat. (Basel)* **84** (2005), 503–511.

[11] J. Chudziak, Approximate solutions of the Gołąb–Schinzel functional equation, *J. Approx. Theory* **136** (2005), 21–25.

[12] J. Chudziak, Stability of the generalized Gołąb–Schinzel equation, *Acta Math. Hung.* **113** (2006), 133–144.

[13] J. Chudziak and J. Tabor, On the stability of the Gołąb–Schinzel functional equation, *J. Math. Anal. Appl.* **302** (2005), 196–200.

[14] R. Craigen and Z. Páles, The associativity equation revisited, *Aequationes Math.* **37** (1989), 306–312.

[15] Z. Daróczy, Az $f[x+yf(x)] = f(x)f(y)$ függvenyegyenlet folytones megoldasairol Hilbert – terekben, *Matematikai Lapok* **17** (1966), 339–343.

[16] R. Ger, Gołąb–Schinzel equation almost everywhere, Report of Meeting (The Forty-forth International Symposium on Functional Equations, May 14–20, 2006, Louisville, Kentucky, USA, *Aequationes Math.* **73** (2007), 178.

[17] S. Gołąb and A. Schinzel, Sur l'équation fonctionnelle $f(x + yf(x)) = f(x)f(y)$, *Publ. Math. Debrecen* **6** (1959), 113–125.

[18] D. H. Hyers, G. Isac and T. M. Rassias, Stability of Functional Equations in Several Variables, *Birkhäuser*, *Boston, Basel, Berlin*, 1998.

[19] P. Kahlig and J. Matkowski, A modified Gołąb–Schinzel equation on a restricted domain (with applications to meteorology and fluid mechanics), *Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II* **211** (2002, 2003), 117–136.

[20] M. Sablik, A conditional Gołąb–Schinzel equation, *Anz. Österreich. Akad. Wiss. Math.-Natur. Kl.* **137** (2000, 2001), 11–15.

[21] L. Reich, Über die stetigen Lösungen der Gołąb–Schinzel-Gleichung auf $\mathbb{R}_{\geq 0}$, *Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II* **208** (1999, 2000), 165–170.

[22] L. Reich, Über die stetigen Lösungen der Gołąb–Schinzel-Gleichung auf $\mathbb{R}$ und auf $\mathbb{R}_{\geq 0}$, *Anz. Österreich. Akad. Wiss. Math.-Natur. Kl.* **138** (2001, 2002), 7–12.

NICOLE BRILLOUËT-BELLUOT
ECOLE CENTRALE DE NANTES
DÉPARTEMENT D'INFORMATIQUE ET DE MATHÉMATIQUES
1 RUE DE LA NOË, B.P. 92101
44321 NANTES CEDEX 3
FRANCE

E-mail: Nicole.Belluot@ec-nantes.fr

JANUSZ BRZDĘK
DEPARTMENT OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
PODCHORĄŻYCH 2
30-084 KRAKÓW
POLAND

E-mail: jbrzdek@ap.krakow.pl

(Received July 24, 2007)