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# On a class of critical Riemann–Finsler metrics

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**Abstract.** A generalized Einstein–Hilbert functional in Finsler geometry is defined and its Euler–Lagrange equation is derived, which depends on not only the Ricci scalar but also the mean Landsberg curvature. Such critical metrics include usual Riemann–Einstein metrics. Some non-Riemannian examples of critical metrics are given. Moreover, some rigidity theorems for a Finsler metric to be Riemannian are obtained.

## 0. Introduction

Finsler metrics are just the Riemannian metrics without the quadratic restriction [9]. Finsler manifolds are differentiable manifolds equipped with Finsler metrics. Recently, the study of Finsler geometry has taken on a new look [4], [6], [11], [16].

Let $M$ be an $n$-dimensional compact manifold. As is well known [5], among Riemannian metrics on $M$ there is an important class of metrics called Einstein metrics, which are the critical points of the normalized Einstein–Hilbert functional

$$\frac{1}{\mathrm{Vol}^{1-2/n}(M)}\int_M R\, d\mu_M, \tag{0.1}$$

where $R$ is the scalar curvature of the Riemannian metric, $d\mu_M$ is the volume element of $M$. This motivates us to consider the corresponding functional in Finsler geometry. An attempt in this direction was tried by H. Akbar-Zadeh [1], [2]. Unfortunately, it seems that one could not obtain the tensor characteristic on

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generalized Einstein metrics from the variation calculus in [1] (also cf. D. BAO's comment [8]). It encourages us to look for the Finslerian analogue of critical metrics from the point of view of differential geometry and variational calculus.

By virtue of the Chern connection on a Finsler manifold $(M, F)$ with the Finsler metric $F$, we can define the flag curvature and the Ricci scalar, which are generalizations of the sectional curvature and the Ricci curvature in Riemannian geometry, respectively, [4]. It is natural to define a similar functional in Finsler geometry by using the Ricci scalar and the volume form induced from the projective sphere bundle over $(M, F)$. In fact, this functional can be defined by

$$\mathcal{E}(F) = \frac{1}{\mathrm{Vol}^{1-2/n}(SM)} \int_{SM} \mathrm{Ric}\ d\mu_{SM}, \tag{0.2}$$

where Ric denotes the Ricci scalar and $SM$ is the projective sphere bundle over $M$ with volume element $d\mu_{SM}$.[4] One can check easily (0.2) is just the previous (0.1) if $F$ is Riemannian by means of the integral trace formula (or Lemma 1.4 in [12]).

The purpose of this paper is to derive the Euler–Lagrange equation of the functional (0.2) and to give some examples of Finsler metrics satisfying the Euler–Lagrange equation. Moreover, some rigidity theorems for a Finsler metric to be Riemannian are obtained. We find that the critical points of (0.2) depend on not only the Ricci scalar but also the mean Landsberg curvature. Precisely, we have

**Theorem 0.1.** *The Euler–Lagrange equation of the functional* (0.2) *is*

$$\mathrm{Ric}(x, y) = \frac{2}{n+2}\left(\mathrm{trace}_g \widetilde{\mathcal{R}}c + \mathrm{trace}_g \mathcal{J} - \frac{n-2}{2} r\right), \tag{0.3}$$

*where*

$$\widetilde{\mathcal{R}}c := \frac{1}{2}[F^2\,\mathrm{Ric}]_{y^i y^k} dx^i \otimes dx^k, \quad \mathcal{J} := (J_{i|k} + \dot{J}_{i;k} - J_i J_k) dx^i \otimes dx^k,$$

$r = \int_{SM} \mathrm{Ric}\ d\mu_{SM} / \mathrm{Vol}(SM)$ *is the average of* Ric *on* $SM$, $J = J_i dx^i$ *is the mean Landsberg tensor, "$|$" and ";" denote respectively the horizontal and the vertical covariant derivatives with respect to the Chern connection, and "·" denotes the covariant derivative along the Hilbert form.*

*Definition 0.1.* A Finsler metric which is a critical point of the functional (0.2) is called an *$\mathcal{E}$-critical metric.*

It is easy to show that a Riemannian metric is an $\mathcal{E}$-critical metric iff it is Einstein. On the other hand, we have the following non-Riemannian examples.

*Example 0.1.* Let $\alpha$ be a Ricci-flat Riemannian metric and $\beta$ is parallel with respect to $\alpha$, then the Randers metric $F = \alpha + \beta$ is an $\mathcal{E}$-critical metric. In fact, a non-Riemannian Berwald–Randers metric $F = \alpha + \beta$ is $\mathcal{E}$-critical if and only if $\alpha$ is Ricci-flat. We shall discuss $\mathcal{E}$-critical Randers metrics in the further paper.

*Example 0.2.* Let $(M, g)$ and $(N, h)$ be two Ricci-flat Riemannian manifolds, then the metric

$$F(x, y) := \sqrt{\phi(g(x_M, y_M), h(x_N, y_N))}$$

is $\mathcal{E}$-critical on the product manifold $M \times N$, where the function $\phi(s, t)$ can be defined as

$$\phi(s, t) = s + t + \epsilon \sqrt[k]{s^k + t^k}.$$

Here $\epsilon$ is a nonnegative real number and $k$ is a positive integer.

We also have the following rigidity result.

**Theorem 0.2.** *Let $M$ be a compact closed manifold, and $F$ be a Finsler metric on $M$ with positive constant flag curvature and almost isotropic $S$-curvature. Then $F$ is an $\mathcal{E}$-critical metric if and only if it is Riemann–Einstein.*

From the above theorem we have immediately the following

**Corollary 0.1.** *Any $\mathcal{E}$-critical Randers metric with positive constant flag curvature on a compact closed manifold must be Riemannian.*

The contents of this paper are arranged as follows. In §1, some necessary notations and formulas are given. In §2, the Euler–Lagrange equation (0.3) of the functional (0.2) is derived. In §3, some rigidity theorems are shown, and two non-Riemannian examples of $\mathcal{E}$-critical metrics are given. In the last section §4, some results on surface are considered.

## 1. Preliminaries

Let $M$ be an $n$-dimensional differentiable manifold with the tangent bundle $TM$. The points in $TM$ are denoted by $(x, y)$, where $x \in M$, $y \in T_xM$, and let $(x^i; y^i)$ be the local coordinates of $TM$ with $y = y^i \partial/\partial x^i$. A Finsler metric on $M$ is a function $F : TM \to [0, +\infty)$ such that (i) $F$ is smooth in $TM \backslash \{0\}$; (ii) $F(x, \lambda y) = \lambda F(x, y)$ for any $\lambda > 0$; and (iii) The fundamental quadratic form

$$g = g_{ik}(x, y) dx^i \otimes dx^k, \quad g_{ik} := \left[\frac{1}{2}F^2\right]_{y^i y^k} \tag{1.1}$$

is positively definite. Here and from now on, the lower index $y^i$ always means partial derivatives, $F_{y^i} := \frac{\partial F}{\partial y^i}$, $[F^2]_{y^i y^k} := \frac{\partial^2 F^2}{\partial y^i \partial y^k}$, etc. We shall use the convention that Latin indices range from 1 to $n(= \dim M)$.

The canonical projection $\pi : TM\backslash\{0\} \to M$ gives rise to a covector bundle $\pi^* T^* M$, on which there exists the Hilbert form $\omega = [F]_{y^i} dx^i$, whose dual is the distinguished section

$$\ell = \ell^i \frac{\partial}{\partial x^i}, \quad \text{with} \quad \ell^i := \frac{y^i}{F}. \tag{1.2}$$

The Cartan tensor and the Cartan form are respectively

$$A = A_{ijk} dx^i \otimes dx^j \otimes dx^k, \quad A_{ijk} := \frac{F}{4} \left[F^2\right]_{y^i y^j y^k}, \tag{1.3}$$

$$I = I_i dx^i, \quad I_i := A_{ijk} g^{jk}, \qquad (g^{jk}) = (g_{ij})^{-1}. \tag{1.4}$$

The nonlinear connection coefficients are given as

$$N^i_k = \gamma^i_{kj} y^j - C^i_{kj} \gamma^j_{pq} y^p y^q, \quad C^i_{jk} := g^{il} C_{ljk}, \quad C_{ijk} := \frac{1}{F} A_{ijk},$$

where $\gamma^j_{pq}$ are the 2nd kind formal Christoffel symbols of $g_{ik}$. Define

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^k_i \frac{\partial}{\partial y^k}, \quad \delta y^i := dy^i + N^i_k dx^k. \tag{1.5}$$

It is well-known that there exists uniquely the Chern connection $\nabla$ on $\pi^* TM$

$$\nabla \frac{\partial}{\partial x^j} = \omega^i_j \frac{\partial}{\partial x^i}, \; \omega^i_j = \Gamma^i_{jk} dx^k, \quad \Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\delta g_{lj}}{\delta x^k} + \frac{\delta g_{lk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^l} \right) \tag{1.6}$$

satisfying

$$dx^j \wedge \omega^i_j = 0, \;\; dg_{ij} - g_{ik} \omega^k_j - g_{kj} \omega^k_i = 2 A_{ijk} \frac{\delta y^k}{F}. \tag{1.7}$$

The spray coefficients are $G^i := \gamma^i_{jk} y^j y^k$, and one can check that

$$G^i = \Gamma^i_{jk} y^j y^k, \quad \frac{1}{2} [G^i]_{y^j} = N^i_j = \Gamma^i_{jk} y^k, \quad \frac{1}{2} [G^i]_{y^j y^k} = \Gamma^i_{jk} + \dot{A}^i_{jk}, \tag{1.8}$$

where "·" denotes the covariant derivative along the Hilbert form.

The curvature 2-forms of the Chern connection have the form

$$\Omega^i_j = d\omega^i_j - \omega^k_j \wedge \omega^i_k = \frac{1}{2} R_j{}^i{}_{kl} dx^k \wedge dx^l + P_j{}^i{}_{kl} dx^k \wedge \frac{\delta y^l}{F} \tag{1.9}$$

which give the $hh$-curvature $R$ and the $hv$-curvature $P$. The flag curvature tensor and the Landsberg tensor are defined by

$$R^i{}_k := \ell^j R_j{}^i{}_{kl}\ell^l, \quad L^i{}_{kl} := -\ell^j P_j{}^i{}_{kl}$$

respectively. By noting $\delta F/\delta x^k = 0$, the following formulae are well known

$$R^i{}_k = \frac{y^j}{F^2}\left[\frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}\right], \quad L_{ijk} := g_{il}L^l{}_{jk} = \dot{A}_{ijk}. \tag{1.10}$$

The mean Landsberg tensor(see §2.1 in [11] or §8 in [16]) is the contraction of $L$

$$J = J_k dx^k, \quad J_k := g^{ij}L_{ijk} = \dot{I}_k. \tag{1.11}$$

The Ricci scalar is defined by

$$\mathrm{Ric} := R^i{}_i = \ell^j R_j{}^i{}_{il}\ell^l = \frac{y^j}{F^2}\left[\frac{\delta N^i_j}{\delta x^i} - \frac{\delta N^i_i}{\delta x^j}\right]. \tag{1.12}$$

On the punctured bundle $TM\backslash\{0\}$, there is the Sasaki type metric $g_{ik}dx^i \otimes dx^k + g_{ik}\frac{\delta y^i}{F} \otimes \frac{\delta y^k}{F}$, which induces a Riemannian metric on the projective sphere bundle $SM$

$$\hat{g} = g_{ik}dx^i \otimes dx^k + F[F]_{y^i y^k}\frac{\delta y^i}{F} \otimes \frac{\delta y^k}{F}.$$

Hence the volume form of $SM$ can be expressed as

$$d\mu_{SM} = \Omega d\eta \wedge dx, \quad \Omega := \det\left(\frac{g_{ik}}{F}\right) \tag{1.13}$$

where

$$d\eta := \sum (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n, \quad dx := dx^1 \wedge \cdots \wedge dx^n.$$

The volume form of $F$ can be defined by

$$d\mu_M = \sigma_F(x)dx, \quad \sigma_F(x) := \frac{1}{\omega_{n-1}}\int_{S_xM}\Omega d\eta, \tag{1.14}$$

where $\omega_{n-1}$ is the volume of the $(n-1)$-dimensional standard sphere. The distorsion $\tau$ and $S$-curvature $S$ are defined as(cf. §7.3 in [17])

$$\tau := \ln\frac{\sqrt{\det(g_{ik})}}{\sigma_F(x)}, \quad S := \tau_{|i}y^i, \tag{1.15}$$

where "$|$" denotes the horizontal covariant derivative.

On the Riemannian manifold $(SM, \hat{g})$, we have the following divergence formula

**Lemma 1.1** (see e.g. [12]). *For any 1-form* $\alpha = \alpha_i dx^i + \beta_i \frac{\delta y^i}{F}$ $(\beta_i \ell^i = 0)$ *on* $SM$*, its divergence is*

$$\mathrm{div}_{\hat{g}}\, \alpha = g^{ik}(\alpha_{i|k} - \alpha_i J_k + \beta_{i;k}), \tag{1.16}$$

*where "*$|$*" and "*$;$*" denote the horizontal and the vertical covariant derivatives respectively, and* $J$ *is the mean Landsberg tensor.*

**Lemma 1.2** (see e.g. [12]). *For any function* $f$ *on* $SM$*, we have* $\mathrm{div}_{\hat{g}}(f\omega) = \ell^H f$*, where* $\ell^H = \ell^i \delta/\delta x^i$ *is the horizontal part of* $\ell$*.*

On each punctured tangent fibre $T_xM\backslash 0$, we have the natural Riemannian metric $g_{ik} dy^i \otimes dy^k$, and this metric gives rise to a Riemannian metric $\hat{r}_x = F^{-1} F_{y^i y^k} dy^i \otimes dy^k$ on the projective sphere fibre $S_xM$, where $\{y^i\}$ should be viewed as the homogeneous coordinates on $S_xM$.

**Lemma 1.3.** *Let* $\alpha = \alpha_i dy^i$ $(\alpha_i y^i = 0)$ *be an 1-form on* $(S_xM, \hat{r}_x)$*, then its divergence is*

$$\mathrm{div}_{\hat{r}_x}\, \alpha = F^2 g^{ik} [\alpha_i]_{y^k} - F g^{ik} \alpha_i I_k \tag{1.17}$$

*where* $I$ *is the Cartan form.*

By using (1.17), we have the following Green type formula.

**Lemma 1.4.** *Let* $(M, F)$ *be a Finsler manifold, then the following identity holds for any functions* $\psi$ *and* $\phi$ *defined on* $SM$

$$\int_{S_xM} \psi g^{ij} [F^2\phi]_{y^i y^j} \Omega d\eta = \int_{S_xM} \phi g^{ij} [F^2\psi]_{y^i y^j} \Omega d\eta. \tag{1.18}$$

Proof. The special case of $\psi = 1$ was proved in [12], and our proof is just analogical to that. Set $\sqrt{g} = \sqrt{\det(g_{ik})}$ and $\alpha = \sqrt{g} I_i \frac{dy^i}{F}$. Then according to the above lemma, one can verify that

$$\Delta_{\hat{r}_x}(\sqrt{g}\phi) = g^{ij}[F^2\phi]_{y^i y^j}\sqrt{g} + \mathrm{div}_{\hat{r}_x}(\phi\alpha) - 2n\sqrt{g}\phi,$$

particularly, $\Delta_{\hat{r}_x}(\sqrt{g}) = \mathrm{div}_{\hat{r}_x}(\alpha)$. Then

$$\begin{aligned}
&\psi g^{ij}[F^2\phi]_{y^i y^j}\sqrt{g} - \phi g^{ij}[F^2\psi]_{y^i y^j}\sqrt{g} \\
&\quad = \psi \Delta_{\hat{r}_x}(\sqrt{g}\phi) - \phi\Delta_{\hat{r}_x}(\sqrt{g}\psi) - \langle \psi d\phi, \alpha\rangle + \langle \phi d\psi, \alpha\rangle \\
&\quad = (\psi\sqrt{g})\Delta_{\hat{r}_x}\phi - \phi\Delta_{\hat{r}_x}(\sqrt{g}\psi) + \psi\phi\, \mathrm{div}_{\hat{r}_x}(\alpha) + \langle d(\psi\phi), \alpha\rangle.
\end{aligned}$$

Integrate on $S_xM$ with the volume form $\frac{\sqrt{g}}{F^n} d\eta$, then the classical Green's formula and divergence theorem give the result. $\square$

At the end of this section, we recall some special classes of Finsler metrics [16]. A Finsler metric is *locally Minkowskian* if and only if both its $hh$-curvature and $hv$-curvature vanish. A *Berwald metric* is a Finsler metric with vanishing $hv$-curvature. If the (mean) Landsberg curvature vanishes identically, then the metric is called a *(weakly) Landsberg metric.*

Finally, we remark that for Finsler metrics $F$ on compact manifold $M$ the functional (0.2) is homogenous with respect to $F$, i.e. $\mathcal{E}(\lambda F) = \mathcal{E}(F)$ for any positive number $\lambda$.

## 2. The Euler–Lagrange equation

Let $M$ be an $n$-dimensional differentiable manifold with boundary $\partial M$ (perhaps empty). Let $F$ be a Finsler metric on $M$, and $F(t)$ be a variation of $F$ with $F(0) = F$ and $F(t)|_{\partial(SM)} = F(0)_{\partial(SM)}$. Define the variation function $V$ by

$$\frac{\partial}{\partial t}\bigg|_{t=0} \ln F^2 = V. \tag{2.1}$$

Then $V|_{\partial(SM)} = 0$, and $V$ is a function on the projective sphere bundle $SM$. Since we only consider variations with compact support, then $\nabla V|_{\partial(SM)} = 0$ and the divergence theorem will play its role without boundary value. A moment thought shows $V$ may be an arbitrary function on $SM$ with compact support. In fact, given any $V(x, y)$ on $SM$, we can define a variation of $F(0)$ as $F(t) := F(0)e^{tV/2}$ which are actually Finsler metrics for small $t$. For simplicity, we let all derivatives with respect to $t$ take their values at $t = 0$, and omit the symbol of restriction $t = 0$ after $\partial/\partial t$.

Then the variation of the fundamental form (1.1) is

$$\frac{\partial}{\partial t} g_{ik} = v_{ik}, \quad v_{ik} = \frac{1}{2}[F^2V]_{y^iy^k}, \quad v_{k0} = \frac{1}{2F}[F^2V]_{y^k}, \quad v_{00} = V. \tag{2.2}$$

All through this paper, the lower index "0" means taking contraction with the distinguished vector $\ell$, i.e., $v_{0k} = v_{ik}\ell^i$, $v_{00} = v_{0k}\ell^k$. By the above setting, one can easily obtain from (1.13) and (1.3)

$$\frac{\partial}{\partial t} d\mu_{SM} = \frac{1}{2}\left(g^{ik}[F^2V]_{y^iy^k} - nV\right) d\mu_{SM}, \tag{2.3}$$

$$\frac{\partial}{\partial t} g^{il} = -g^{ip}v_{pq}g^{ql} := -v^{il}, \tag{2.4}$$

$$\frac{\partial}{\partial t}C_{ijk} = \frac{1}{2F}v_{ij;k}, \qquad v_{0j;k} = v_{i0;k} = v_{ij;0} = 0. \tag{2.5}$$

On putting

$$H^i_{kj} := \frac{1}{2}g^{il}(v_{lj|k} + v_{lk|j} - v_{jk|l}), \tag{2.6}$$

a direct computation shows

$$2H^i_{kj} = -v^{il}\left(\frac{\delta g_{lj}}{\delta x^k} + \frac{\delta g_{lk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^l}\right) + g^{il}\left(\frac{\delta v_{lj}}{\delta x^k} + \frac{\delta v_{lk}}{\delta x^j} - \frac{\delta v_{jk}}{\delta x^l}\right). \tag{2.7}$$

By (1.5) and (2.7), the variation of the formal Christoffel symbols then is

$$\begin{aligned} 2\frac{\partial}{\partial t}\gamma^p_{kq} &= -v^{pl}\left(\frac{\partial g_{lq}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^q} - \frac{\partial g_{kq}}{\partial x^l}\right) + g^{pl}\left(\frac{\partial v_{lq}}{\partial x^k} + \frac{\partial v_{lk}}{\partial x^q} - \frac{\partial v_{kq}}{\partial x^l}\right) \\ &= 2H^p_{kq} - 2v^{pl}(N^i_k C_{lqi} + N^i_q C_{lki} - N^i_l C_{kqi}) \\ &\quad + \frac{1}{F}g^{pl}(N^i_k v_{lq;i} + N^i_q v_{lk;i} - N^i_l v_{kq;i}). \end{aligned} \tag{2.8}$$

Contracting (2.8) with $y$ twice, we can get the variation of the spray coefficients

$$\frac{\partial}{\partial t}G^p = \frac{1}{2}g^{pl}(v_{lk|q} + v_{lq|k} - v_{kq|l})y^q y^k = F^2 H^p_{00}. \tag{2.9}$$

If we set $T^p_k := \frac{1}{F}\frac{\partial}{\partial t}N^p_k$, we have from (1.8) and (2.9)

$$T^p_k = \frac{1}{F}\frac{\partial}{\partial t}N^p_k = H^p_{k0} - A^p_{kq}H^q_{00}. \tag{2.10}$$

Noting that

$$\left[\frac{\partial}{\partial t}, \frac{\delta}{\delta x^i}\right] = -T^k_i F\frac{\partial}{\partial y^k},$$

we firstly obtain from (1.8), (2.9) and (2.10)

$$\frac{y^j}{F^2}\frac{\partial}{\partial t}\left(\frac{\delta N^i_j}{\delta x^i}\right) = \frac{y^j}{F^2}\frac{\delta}{\delta x^i}\left(\frac{\partial N^i_j}{\partial t}\right) - \frac{y^j}{F}T^k_i\frac{\partial N^i_j}{\partial y^k} = \frac{\delta H^i_{00}}{\delta x^i}, \tag{2.11}$$

$$\frac{y^j}{F^2}\frac{\partial}{\partial t}\left(\frac{\delta N^i_i}{\delta x^j}\right) = \ell^H(T^i_i) - H^k_{00}(\Gamma^i_{ik} + J_k). \tag{2.12}$$

Setting an 1-form

$$\theta := g_{ik}H^i_{00}dx^k - T^k_k\omega,$$

then from (2.11), (2.12) and Lemma 1.1–1.2 we have

$$\frac{\partial}{\partial t}\operatorname{Ric} = -V\operatorname{Ric} + \operatorname{div}_{\hat g}\theta + 2H^i_{00}J_i. \tag{2.13}$$

Next, let's calculate the last term in the RHS of (2.13)

$$2H^i_{00}J_i = 2v_{k0|0}J_i g^{ki} - V_{|k}J_i g^{ki}.$$

One can get from (2.2)

$$\begin{aligned} 2v_{k0|0}J_i g^{ki} &= 2[v_{k0}J_i g^{ki}]_{|0} - 2v_{k0}\dot J_i g^{ki} = 2[v_{k0}J_i g^{ki}]_{|0} - V_{;k}\dot J_i g^{ki} \\ &= \operatorname{div}_{\hat g}(2[v_{k0}J_i g^{ki}]\omega - V\dot J_i\delta y^i) + V\operatorname{div}_{\hat g}(\dot J_i\delta y^i). \end{aligned} \tag{2.14}$$

Note $\dot J_i\ell^i = 0$, hence the 1-forms in the divergences are actually living on $SM$. Moreover, one can immediately have

$$-V_{|k}J_i g^{ki} = -\operatorname{div}_{\hat g}(VJ) + V\operatorname{div}_{\hat g}(J). \tag{2.15}$$

Define 1-forms

$$\xi := g_{ik}H^i_{00}dx^k + [2v_{k0}J_i g^{ki} - T^i_i]\omega - V\dot J_i\frac{\delta y^i}{F} - VJ, \tag{2.16}$$

$$\kappa := J_i dx^i + \dot J_i\frac{\delta y^i}{F}. \tag{2.17}$$

It is easily to see from (2.13), (2.14) and (2.15)

$$\frac{\partial}{\partial t}\operatorname{Ric} = \operatorname{div}_{\hat g}\xi - V\operatorname{Ric} + V\operatorname{div}_{\hat g}\kappa. \tag{2.18}$$

Integrating (2.18) gives

$$\begin{aligned} \frac{d}{dt}\int_{SM}\operatorname{Ric} &= \int_{SM}\left(\frac{\partial\operatorname{Ric}}{\partial t}\right) + \int_{SM}\operatorname{Ric}\left(\frac{\partial}{\partial t}d\mu_{SM}\right) \\ &= \int_{SM} V\,(-\operatorname{Ric} + \operatorname{div}_{\hat g}\kappa) + \frac{1}{2}\int_{SM}\operatorname{Ric}\left(g^{ik}[F^2V]_{y^iy^k} - nV\right) \\ &= \frac{1}{2}\int_{SM} V\left(g^{ik}[F^2\operatorname{Ric}]_{y^iy^k} - (n+2)\operatorname{Ric} + 2\operatorname{div}_{\hat g}\kappa\right), \end{aligned} \tag{2.19}$$

where we use $\xi|_{\partial(SM)} = 0$ in the second identity and Lemma 1.4 in the last one. On the other hand, Lemma 1.4 also gives the volume's variation as

$$\frac{d}{dt}\operatorname{Vol}(SM) = \int_{SM}\frac{1}{2}\left(g^{ik}[F^2V]_{y^iy^k} - nV\right) = \frac{n}{2}\int_{SM} V. \tag{2.20}$$

By (2.19) and (2.20), we reach

$$\frac{d}{dt}\mathcal{E}(F) = \frac{1}{2\,\mathrm{Vol}^{1-2/n}\,(SM)}\int_{SM} V\left(F^2 g^{ik}[\mathrm{Ric}]_{y^iy^k} + (n-2)(\mathrm{Ric} - r) + 2\,\mathrm{div}_{\hat{g}}\,\kappa\right),$$

where

$$r = \frac{1}{\mathrm{Vol}\,(SM)}\int_{SM} \mathrm{Ric}\;\, d\mu_{SM}. \tag{2.21}$$

Thus, according to the arbitrariness of $V$ defined by (2.1), we have proved the following

**Theorem 2.1.** *The Euler–Lagrange equation of $\mathcal{E}(F)$ defined by* (0.2) *is*

$$F^2 g^{ik}[\mathrm{Ric}]_{y^iy^k} + (n-2)(\mathrm{Ric} - r) + 2\,\mathrm{div}_{\hat{g}}\,\kappa = 0, \tag{2.22}$$

*where $\kappa = J_i dx^i + \dot{J}_i \frac{\delta y^i}{F}$ and $r$ is given in (2.21).*

*Remark.* Expressing $\mathrm{div}_{\hat{g}}\,\kappa$ explicitly by Lemma 1.1, one can show easily that (2.22) is equivalent to

$$\mathrm{Ric}(x,y) = \frac{2}{n+2}\left(\mathrm{trace}_g\widetilde{\mathcal{R}}c + \mathrm{trace}_g\mathcal{J} - \frac{n-2}{2}r\right), \tag{2.23}$$

where

$$\widetilde{\mathcal{R}}c := \frac{1}{2}[F^2\,\mathrm{Ric}]_{y^iy^k}dx^i\otimes dx^k, \quad \mathcal{J} := (J_{i|k} + \dot{J}_{i;k} - J_iJ_k)dx^i\otimes dx^k. \tag{2.24}$$

Note that if $F$ is weakly Landsberg then $\mathcal{J} = 0$ and the equation becomes simpler. Fortunately, many weakly Landsberg metrics have been constructed in [13] recently. One can see easily that any weakly Landsberg metric with constant Ricci scalar must be $\mathcal{E}$-critical.

On the other hand, by Lemma 1.3, the equation (2.22) can be expressed as

$$\Delta_{\hat{r}_x}\mathrm{Ric} + \langle\nabla_{\hat{r}_x}\mathrm{Ric}, \nabla_{\hat{r}_x}\tau\rangle_{\hat{r}_x} + (n-2)(\mathrm{Ric} - r) + 2\,\mathrm{div}_{\hat{g}}\,\kappa = 0, \tag{2.25}$$

where $\hat{r}_x$ is the induced Riemannian metric on the projective sphere fibre $S_xM$ for each $x\in M$, $\Delta_{\hat{r}_x}$ denotes the Laplacian with respect to $\hat{r}_x$, and $\tau$ is the distorsion of $F$.

From these equations we can find that the critical metrics depend on both Ricci scalar and the mean Landsberg curvature. Taking some attentions to (2.23), one can find the Euler–Lagrange equation is invariant under the homothetic transformation of $F$, i.e., $\lambda F$ is $\mathcal{E}$-critical for any positive constant $\lambda$ if $F$ is $\mathcal{E}$-critical. This is an expected result according to the normalized definition of $\mathcal{E}(F)$.

## 3. Some special critical metrics

**Proposition 3.1.** *A Riemannian metric $F$ is $\mathcal{E}$-critical if and only if it is Riemann–Einstein.*

Proof. If $F$ is Riemannian, then $\tau = 1$, $I = 0$ and hence $\kappa = 0$. By (2.25), we get

$$\Delta_{\hat{r}_x}(\mathrm{Ric} - r) = -(n-2)(\mathrm{Ric} - r). \tag{3.1}$$

In Riemannian case, $(S_xM, \hat{r}_x)$ is a standard $(n-1)$-sphere with the first eigenvalue $(n-1)$. So, (3.1) implies that $\mathrm{Ric} - r = 0$ given $n \geq 3$, while it is trivial for $n = 2$. □

Proposition 3.1 means that Riemann–Einstein metrics remain critical in the much bigger category, i.e. the category of Finsler geometry.

In general, by the Euler–Lagrange equation (2.22), even Finsler metrics with constant flag curvature are not necessarily $\mathcal{E}$-critical. Let $F$ have constant flag curvature $K$. By $F$'s homogeneity of the equation (2.22), we may assume $K = 1, 0, -1$. According to Akbar-Zadeh's rigidity theorem on closed manifolds, we only focus on the positive case $K = 1$. In such a case,

$$\dot{J} = -KI = -I \quad \text{and} \quad \kappa = J_i dx^i - I_i \frac{\delta y^i}{F}. \tag{3.2}$$

Moreover, the equation (2.22) becomes

$$\mathrm{div}_{\hat{g}}\, \kappa = 0. \tag{3.3}$$

Recall the distorsion and $S$-curvature defined in (1.15), we get([16])

$$d\tau = \tau_{|i} dx^i + I_i \frac{\delta y^i}{F} \tag{3.4}$$

and

$$S_{y^i} = \tau_{|i} + \tau_{|k;i}\ell^k = \tau_{|i} + \tau_{;i|k}\ell^k + \tau_{;j} L^j{}_{ki}\ell^k = \tau_{|i} + J_i. \tag{3.5}$$

Combining (3.2),(3.4) and (3.5), we drive directly

$$\kappa + d\tau = (J_i + \tau_{|i})dx^i = S_{y^i} dx^i. \tag{3.6}$$

From (3.3) and (3.6) we see that $F$ is $\mathcal{E}$-critical if and only if

$$\Delta_{\hat{g}} \tau = \mathrm{div}_{\hat{g}}(S_{y^i} dx^i). \tag{3.7}$$

*Definition 3.1* (cf. (5.6) in [11]). A Finsler metric $F$ is said to have *almost isotropic $S$-curvature if* $S = \lambda F + df(y)$, where $\lambda$ and $f$ are two functions defined on $M$. Particularly, when $\lambda$ is constant, we say $F$ has almost constant $S$-curvature.

Now if $F$ has constant flag curvature $K=1$ and almost constant $S$-curvature, then by Lemma 1.2, we have

$$\Delta_{\hat{g}}\tau = \mathrm{div}_{\hat{g}}(\lambda\omega + df) = \Delta_{\hat{g}} f. \tag{3.8}$$

Then Hopf's maximum principal implies $\tau(x,y)-f(x)$ is constant if $M$ is compact. Thus, $\tau$ is independent of $y$ and $F$ is Riemannian.

On the other hand, we have the following

**Lemma 3.1.** *If $F$ has constant flag curvature and almost isotropic $S$-curvature, then it has almost constant $S$-curvature.*

Proof. It is a direct corollary of Theorem 1.1 in [10]. □

By Lemma 3.1 and (3.8), we have proved the following

**Proposition 3.2.** *Let $M$ be a compact closed manifold, and $F$ be a Finsler metric with positive constant flag curvature and almost isotropic $S$-curvature. Then $F$ is $\mathcal{E}$-critical if and only if $F$ is Riemannian.*

Some rigidity theorems on $S$-curvature can be found in [14], [18]. Since any Randers metric with constant flag curvature must have almost constant $S$-curvature(see [19] or §11 of [16]), we have the following

**Corollary 3.1.** *Any $\mathcal{E}$-critical Randers metric with positive constant flag curvature on a compact closed manifold must be Riemannian.*

Similarly, one can see the following

**Proposition 3.3.** *A Finsler metric with zero flag curvature is $\mathcal{E}$-critical if and only if its mean Landsberg tensor is divergence free.*

The above results imply that it is not easy to look for examples of non-Riemannian $\mathcal{E}$-critical metrics. In the following, we give two examples of non-Riemannian $\mathcal{E}$-critical metrics.

*Example 1. Berwald–Randers metrics.*

A Randers metric

$$F(x,y) = \alpha + \beta = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i$$

is a Berwald metric if and only if $b = b_i dx^i$ is parallel with respect to the background Riemannian metric $a = a_{ij}dx^i \otimes dx^j$, and hence the norm of $b$ with respect to $a$ is constant.

For a Berwald metric we have $\kappa = 0$. Thus, (2.22) becomes

$$(n-2)r = g^{ik}[F^2 \,\mathrm{Ric}]_{y^i y^k} - (n+2)\,\mathrm{Ric}\,. \tag{3.9}$$

Denote the Ricci curvature tensor of $a$ by $Rc^a = R_{ik}dx^i \otimes dx^k$, then by the parallelism of $b$ and §2.1.3 of [6], we have

$$Rc^a(b, y) = 0, \quad \mathrm{Ric}(x, y) = R_{ik}y^i y^k / F^2. \tag{3.10}$$

Substituting (3.10) and the expression of $g^{ik}$ into (3.9), one can obtain

$$(n-2)rF^3 = 2\alpha F^2 R^a + (2\alpha\|b\|^2 - (n+2)\alpha - n\beta)R_{ik}y^i y^k, \tag{3.11}$$

where $R^a$ is the scalar curvature of $a$.

Since $\alpha(-y) = \alpha(y)$ and $\beta(-y) = -\beta(y)$, the equation (3.11) is equivalent to the system

$$\begin{cases} (n-2)r(\alpha^2 + 3\beta^2) = 2(\alpha^2+\beta^2)R^a + (2\|b\|^2 - n - 2)R_{ik}y^i y^k, \\ (n-2)r(3\alpha^2\beta + \beta^3) = 4\alpha^2\beta R^a - n\beta R_{ik}y^i y^k. \end{cases} \tag{3.12}$$

Setting $y = b$ in (3.12), one will reach $r = R^a = 0$ by noting $0 < \|b\| < 1$. Then, using (3.12) again, we have $\mathrm{Ric}(x, y) = R_{ik}y^i y^k/F^2 = 0$. Therefore, we have the following

**Proposition 3.4.** *A non-Riemannian Berwald–Randers metric $F = \alpha + \beta$ is $\mathcal{E}$-critical if and only if $\alpha$ is Ricci-flat.*

*Example 2. Product manifolds.*

Let $(M, g)$ and $(N, h)$ be two Riemannian manifolds, where

$$g = g_{i_1 j_1} y^{i_1} y^{j_1}, \qquad h = h_{i_2 j_2} y^{i_2} y^{j_2}.$$

Consider the product manifold $L = M \times N$. Let $\phi : [0,\infty) \times [0,\infty) \to [0,\infty)$ be a smooth function satisfying

(a) $\phi(\lambda s, \lambda t) = \lambda\phi(s, t)$, $(\lambda > 0)$ and $\phi(s, t) = 0 \Leftrightarrow (s, t) = 0$;

(b) $\phi_s > 0$, $\phi_t > 0$, $\phi_s + 2s\phi_{ss} > 0$, $\phi_t + 2t\phi_{tt} > 0$;

(c) $\phi_s\phi_t - 2\phi\phi_{st} + 4st[\phi_{ss}\phi_{tt} - \phi_{st}\phi_{st}] > 0$.

A typical example is $\phi_{\epsilon,k} = s+t+\epsilon\sqrt[k]{s^k+t^k}$ where $\epsilon$ is a nonnegative real number and $k$ is a positive integer.

Now set

$$F(x,y) := \sqrt{\phi(g(x_1,y_1),h(x_2,y_2))}, \tag{3.13}$$

where $x=(x_1,x_2)\in L$ and $y=y_1\oplus y_2\in T_xL$. Then $F$ is a Berwald metric (see §5 and §14 in [16]) on $L$ with

$$\mathrm{Ric}(F) = \frac{g}{F^2}\,\mathrm{Ric}(g) + \frac{h}{F^2}\,\mathrm{Ric}(h). \tag{3.14}$$

If $M$ and $N$ be two Ricci-flat Riemannian manifolds, then $F$ defined as above is $\mathcal{E}$-critical. As is well known, Calabi–Yau manifolds are Ricci-flat. So, we can construct many non-Riemannian $\mathcal{E}$-critical Finsler metrics.

Before the end of this section, let's define a new quantity $\rho(x)$

$$\rho(x) := \frac{\int_{S_xM}\mathrm{Ric}\ \Omega d\eta}{\int_{S_xM}\Omega d\eta}. \tag{3.15}$$

For Riemannian metric it is just the scalar curvature $R$ up to a constant. In fact, by Lemma 1.4, one can easily drive $\rho = \frac{1}{n}R$. So, we give the following

*Definition 3.2.* The function $\rho$ defined by (3.15) is called the *normalized Ricci curvature* of the Finsler metric $F$.

We now have the following

**Proposition 3.5.** *For $n\geq 3$, any weakly Landsberg $\mathcal{E}$-critical metric must has constant normalized Ricci curvature.*

Proof. Since $J=0$, then (2.22) becomes

$$g^{ik}[F^2\,\mathrm{Ric}]_{y^iy^k} - 2n\,\mathrm{Ric} + (n-2)(\mathrm{Ric} - r) = 0. \tag{3.16}$$

Integrating on $S_xM$ with $\Omega d\eta$, Lemma 1.4 gives the result. □

## 4. Critical metrics on surfaces

Now let $M$ be a connected orientable surface. By Z. Szabó's rigidity theorem for surfaces, any Berwald surface is $\mathcal{E}$-critical. A Finsler metric $F$ is said to have *isotropic flag curvature* if its flag curvature depends only on $x$. On surfaces, the flag curvature is just the Ricci scalar. Then it is equivalent to saying to have isotropic Ricci scalar. Theorem 7.2.4 in [11] implies

**Proposition 4.1.** *Any compact closed surface with nonpositive isotropic flag curvature is $\mathcal{E}$-critical.*

Proof. In this case, the flag curvature $K = K(x) \leq 0$, then Theorem 7.2.4 in [11] means $F$ is weakly Landsberg ($J = 0$). Therefore, we can get the result from $\mathrm{Ric}(x, y) = K(x)$, $\kappa = 0$ and (2.22). □

Now, we want to know under what conditions an $\mathcal{E}$-critical metric is of isotropic flag curvature or even Riemannian. It is well-known that there is the Berwald frames on $p^*TM$ where $p$ is the canonical projection $p : SM \to M$, and hence on $SM$ we have

$$e_1 = \frac{F_{y^2}}{\sqrt{g}}\frac{\delta}{\delta x^1} - \frac{F_{y^1}}{\sqrt{g}}\frac{\delta}{\delta x^2}, \quad e_2 = \ell^1\frac{\delta}{\delta x^1} + \ell^2\frac{\delta}{\delta x^2}, \quad e_3 = \frac{F_{y^2}}{\sqrt{g}}F\frac{\partial}{\partial y^1} - \frac{F_{y^1}}{\sqrt{g}}F\frac{\partial}{\partial y^2},$$

where $g = \det(g_{ik})$. The first two are horizontal, while the third is vertical. The dual frames are denoted by $\omega^1$, $\omega^2$, $\omega^3$.

It can be easily found that the Cartan form has only one component

$$I = I\omega^1, \tag{4.1}$$

where the left $I$ is the Cartan form while the right one is called the Cartan scalar. Since there is no confusion, we use the same latter $I$.

The Chern connection 1-forms are given by (see [4])

$$\begin{pmatrix} \omega_1^1 & \omega_1^2 \\ \omega_2^1 & \omega_2^2 \end{pmatrix} = \begin{pmatrix} -I\omega^3 & -\omega^3 \\ \omega^3 & 0 \end{pmatrix}.$$

By using the connection 1-forms, one can have the mean Landsberg tensor

$$J = \nabla_{e_2}(I\omega^1) = J\omega^1, \tag{4.2}$$

where again we use the same latter, and the right $J = I_2 := e_2(I)$ is called the Landsberg scalar. There is an elegant formula for $I$, $J$ and the flag curvature $K$, which reads

$$K_3 + KI + J_2 = 0, \tag{4.3}$$

where $K_3 := e_3(K)$ and $J_2 := e_2(J)$. So, using the connection 1-forms, we can express the Euler–Lagrange equation in another way.

**Proposition 4.2.** *A Finsler metric $F$ on the surface $M$ is $\mathcal{E}$-critical if and only if*

$$K_{33} + IK_3 + 2IJ_2 + 2J_{23} + 2J_1 = 2J^2, \tag{4.4}$$

*where $K_{33} := e_3e_3(K)$, etc.*

**Corollary 4.1.** *Let $F$ be an $\mathcal{E}$-critical metric. If $J$ is horizontal constant, then $F$ is a Landsberg metric with isotropic flag curvature.*

Proof. Since $J$ is horizontal constant, then $J_1 = J_2 = 0$, hence

$$K_{33} + IK_3 = 2J^2 \geq 0. \tag{4.5}$$

The maximum principal on $S_xM$ shows $K_3 = 0$, then $K = K(x)$ and $J = 0$. $\square$

*Remark.* If $M$ is closed, then $\nabla J = 0 \Leftrightarrow \nabla_{\mathrm{horiz}} J = 0 \Leftrightarrow J = 0$. In fact, since $\mathrm{div}_{SM}(J) = \mathrm{trace}\nabla_{\mathrm{horiz}} J - \|J\|^2_{\hat{g}}$, the integration on $SM$ gives the result.

Similarly, we have

**Corollary 4.2.** *Let $M$ be an oriented closed surface and $F$ be an $\mathcal{E}$-critical metric. If $J$ is vertical constant, then $F$ is a Landsberg metric with isotropic flag curvature.*

Proof. Now write $J = J(x)$. Let $J(x_1) = \max J$, $J(x_2) = \min J$. Therefore, at $x_i$ $(i = 1, 2)$, we have

$$K_{33}(x_i, y) + I(x_i, y)K_3(x_i, y) = 2J^2(x_i) \geq 0, \tag{4.6}$$

and hence $K_3(x_i, y) = 0$, $J(x_i) = 0$, and then $J = 0$, $K_3 = 0$, $K = K(x)$. $\square$

By (4.3), Corollary 4.1 and Deicke's theorem , we immediately have

**Corollary 4.3.** *Let $F$ be an $\mathcal{E}$-critical metric with nonzero flag curvature. If its Landsberg scalar is horizontal constant, then $F$ must be Riemannian.*

*Definition 4.1* (see e.g. [16]). A Finsler metric $F$ is said to have *relatively constant Landsberg curvature* if the Landsberg curvature $L$ satisfies $L = \lambda A$ where $\lambda$ is a constant and $A$ denotes the Cartan tensor.

For surfaces, it reduces to $J = \lambda I$.

**Lemma 4.1** (see e.g. [10]). *Let $M^2$ be a surface, and $F$ be a Finsler metric with relatively constant Landsberg curvature, ie. $J = \lambda I$ for some constant $\lambda$. Then the flag curvature has the form*

$$K(x, y) = -\lambda^2 + \sigma(x)e^{-\tau(x,y)} \tag{4.7}$$

*where $\sigma(x)$ is a function on $M$, $\tau$ is the distorsion.*

The theorem in [10] is more general, and we only consider the case of surfaces. By this lemma, we have

**Proposition 4.3.** *Let $F$ be an $\mathcal{E}$-critical metric with constant relative Landsberg curvature $\lambda \neq 0$. If $I_1 = 0$, then $F$ is Riemannian.*

Proof. By (5.3) and Lemma 5.1, we have

$$K_3 = -\lambda^2 I - KI = -\sigma e^{-\tau} I,$$

$$K_{33} = \lambda^2 I^2 + KI^2 - \lambda^2 I_3 - KI_3 = \sigma e^{-\tau} I^2 - \sigma e^{-\tau} I_3.$$

Substituted into the Euler–Lagrange equation, it turns out

$$2I_3\lambda^2 + 2I_1\lambda - \sigma e^{-\tau} I_3 = 0.$$

Note $\tau_3 = I$, then

$$2((Ie^{\tau})_3 - I^2 e^{\tau})\lambda^2 + 2e^{\tau} I_1 \lambda - \sigma I_3 = 0.$$

Integrating on $S_xM$, we obtain

$$\int_{S_xM} I^2 e^{\tau} = 0,$$

where we use $\int_{S_xM} I_3 = \int_{S_xM} (Ie^{\tau})_3 = 0$, $I_1 = 0$ and $\lambda \neq 0$. Now, $I^2 e^{\tau} = 0$ means $F$ is Riemannian. $\square$

Before finishing this section, we give two non-Riemannian examples with $\mathcal{E}$-critical metrics. However, the metrics given below may be not globally smooth, ie. they have singular directions.

*Example 3. Berwald–Rund surface.*

The Berwald–Rund surface is a Finsler surface with $I = 3/\sqrt{2}$ and $J = 0$, but is only $y$-local(see §10.3 of [4]). Since its Gaussian curvature is not smooth, we change (5.4) into another form. By (5.3), (5.4) can be rewritten as $-KI_3 + 2IJ_2 + J_{23} + 2J_1 = 2J^2$. Then we see that the Berwald–Rund surface satisfies this equation.

*Example 4. Asanov metric.*

In [3], G. S. Asanov defined $\mathcal{FF}_g^{PD}$-spaces which are certain almost regular Finsler manifolds. When the parameter $g$ is not zero, the metric is non-Riemannian with Cartan scalar $I = |g|$. Then for any nonzero number $g$, the $\mathcal{FF}_g^{PD}$ metric satisfies the equation $-KI_3 + 2IJ_2 + J_{23} + 2J_1 = 2J^2$. Under this condition, its curvature doesn't vanish, and hence the metric is not locally Minkwski. The details can be found in §5 of [3].

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